

Pacific Journal of Mathematics

BONDED QUADRATIC DIVISION ALGEBRAS

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Osborn has shown that any quadratic algebra over a field of characteristic not two can be decomposed into a copy of the field and a skew-commutative algebra with a bilinear form. For any nonassociative algebra G over a field of characteristic not two, Albert and Oehmke have defined an algebra over the same vector space, which is bonded to G by a linear transformation T . In this paper this process is specialized to the class \mathcal{A} of finite dimensional quadratic algebras A over fields of characteristic not two, which define a symmetric, nondegenerate bilinear form, to obtain quadratic algebras $B(A, T)$ bonded to A . In the main results T will be defined as a linear transformation on the skew-commutative algebra V defined by Osborn's decomposition of A . An algebra in \mathcal{A} is called a division algebra if $A \neq 0$ and the equations $ax = b$ and $ya = b$, where $a \neq 0$ and b are elements in A , have unique solutions for x and y in A . Consequently, a finite dimensional algebra $A \neq 0$ is a division algebra if and only if A has no divisors of zero. A basis for V is said to be orthogonal, if it is orthogonal with respect to the above mentioned bilinear form. An algebra in \mathcal{A} is weakly flexible if the i th component of the skew-commutative product of the i th and j th members of each orthogonal basis of V is 0. If $\mathcal{D}(\mathcal{A})$ denotes the class of division algebras in \mathcal{A} and I denotes the identity transformation on V , then the main results are: (1) $A \in \mathcal{D}(\mathcal{A})$, T nonsingular and $B(A, T)$ flexible imply $B(A, T) \in \mathcal{D}(\mathcal{A})$, (2) if $A \in \mathcal{D}(\mathcal{A})$ and A is weakly flexible, then $B(A, T)$ is weakly flexible if and only if $T = \delta I$ for δ a scalar, and (3) if A is a Cayley-Dickson algebra in $\mathcal{D}(\mathcal{A})$, then $B(A, T)$ is a Cayley-Dickson algebra in $\mathcal{D}(\mathcal{A})$ if and only if $T = \pm I$. Finally, a class of nonflexible quadratic division algebras bonded to Cayley-Dickson division algebras will be exhibited.

1. Introduction. A finite dimensional algebra A with identity element 1 over a field F of characteristic not 2 is called a quadratic algebra in case 1, a , and a^2 are linearly dependent over F for all $a \in A$. Following the conventions used by Osborn [6] we shall identify the field F with the subalgebra $F1$ and refer to an element in $F1$ as a scalar. Furthermore, if an element $x \in A$ squares to a scalar but x is not a scalar, x is called a vector. If V is the set of all vectors in A , then A is a vector space direct sum of F and V . For x and $y \in A$, let (x, y) denote the scalar component of xy . Clearly (x, y) is a bilinear form from $A \times A$

to F . If x and $y \in V$, we define " \times " by $x \times y = xy - (x, y)$, V is closed under this product and Osborn [6, p. 203] shows it is skew-commutative. If $\alpha + x$ and $\beta + y \in A = F + V$, where α and $\beta \in F$ and x and $y \in V$, then

$$(\alpha + x)(\beta + y) = [\alpha\beta + (x, y)] + [\alpha y + \beta x + x \times y] \in F + V.$$

This decomposition of A into a copy of the field and a skew-commutative algebra with bilinear form makes it possible to restate questions about quadratic algebras in terms of questions about bilinear forms and skew-commutative algebras. For example, it is easy to show that A satisfies the flexible law if and only if the bilinear form (x, y) is symmetric and $(x, x \times y) = 0$ for all x and y in V , and that A is alternative if and only if A is flexible and $(y, x)x - (x, x)y + (y \times x) \times x = 0$ for all x and y in V .

Let \mathcal{A} denote the class of algebras satisfying: A is a finite dimensional quadratic algebra over a field F of characteristic not two and A defines a symmetric, nondegenerate bilinear form (x, y) . We call an algebra in \mathcal{A} a division algebra if $A \neq 0$ and the equations $ax = b$ and $ya = b$, where $a \neq 0$ and b are elements in A , have unique solutions for x and y in A . Consequently, a finite dimensional algebra $A \neq 0$ is a division algebra if and only if A has no divisors of zero. Let $\mathcal{D}(\mathcal{A})$ denote the class of division algebras in \mathcal{A} . In the case that (x, y) is defined by a division algebra it will be nondegenerate, since otherwise there exists an element $\alpha + x \in A$ such that $(y, \alpha + x) = 0$ for all $y \in A$. But then $0 = (x, \alpha + x) = (x, \alpha) + (x, x) = x^2$, which contradicts the division property of A .

The assumptions of finite dimensionality of A and symmetry of (x, y) are sufficient to prove V has a basis u_1, u_2, \dots, u_n of mutually orthogonal vectors with respect to (x, y) . Henceforth, when we speak of an orthogonal basis for V , we shall always mean orthogonal with respect to the bilinear form (x, y) . Moreover, we will let $u_i^2 = \alpha_i \in F$ for $i = 1, \dots, n$; and for $i \neq j$, let $u_i u_j = \sum_{k=1}^n \xi_{ijk} u_k$, so that the ξ_{ijk} 's are the multiplication constants of an orthogonal basis of V : Note that

$$\sum_{k=1}^n \xi_{ijk} u_k = u_i u_j = u_i \times u_j = -u_j \times u_i = -u_j u_i = -\sum_{k=1}^n \xi_{jik} u_k$$

for all $i, j, k = 1, \dots, n$ and $i \neq j$. So

$$(1.1) \quad \xi_{ijk} = -\xi_{jik} \quad \text{for all nonzero } i, j, \text{ and } k.$$

If (x, y) is nondegenerate, then $\alpha_i \neq 0$ for $i = 1, \dots, n$, since $\alpha_i = 0$ implies $0 = (u_i, u_i)$ which would imply $(u_i, y) = 0$ for all $y \in A$.

For $A \in \mathcal{A}$ let U be the subspace of A consisting of all finite linear combinations of vectors of the form $xy - yx$ for x and $y \in A$. Let T be a linear mapping from the subspace U into A and let $B(A, T)$ be an algebra with the same vector space as A and multiplication defined by

$$(1.2) \quad x \cdot y = \frac{1}{2}(xy + yx) + \frac{1}{2}(xy - yx)^T,$$

where xy denotes multiplication in A . T will be called a bonding mapping ([2] and [5]) of A and $B(A, T)$ will be said to be bonded to A . Using (1.2) it is seen that powers in $B(A, T)$ agree with those in A and that the identity of A is also the identity in $B(A, T)$. Thus $B(A, T)$ is also a quadratic algebra and we will let $(x, y)_T$ denote the bilinear form defined as the scalar component of $x \cdot y$ in $B(A, T)$ and let $x \times_T y = x \cdot y - (x, y)_T$, for all x and $y \in V$. V is closed under this skew-commutative product. Since (x, y) is assumed to be symmetric and $x \times y$ is skew-commutative, we have for all x and $y \in V$:

$$(1.3) \quad \begin{aligned} \frac{1}{2}(xy + yx) &= (x, y) \quad \text{and} \\ \frac{1}{2}(xy - yx) &= x \times y. \end{aligned}$$

So for all x and $y \in V$:

$$(1.4) \quad x \cdot y = (x, y)_T + x \times_T y = (x, y) + (x \times y)^T.$$

Clearly, for any basis u_1, \dots, u_n of V , the set of vectors $\{u_i \times u_j \mid i, j = 1, \dots, n\}$ spans the space $U \subseteq V$. Since most of our knowledge is obtained under the assumption that T is a mapping into V , we will henceforth make the restriction

$$(1.5) \quad (u_i \times u_j)^T = \sum_{k=1}^n \beta_{ijk} u_k.$$

The β_{ijk} 's for $i, j, k = 1, \dots, n$ are then the corresponding multiplication constants for V in $B(A, T)$ and $(x, y)_T = (x, y)$.

2. LEMMA 2.1. *Let $A \in \mathcal{A}$ and let u_1, \dots, u_n be any orthogonal basis of V . Then A is flexible if and only if*

- (a) $\xi_{iji} = 0$ for all $i, j = 1, \dots, n$ and
- (b) $\xi_{ijk}\alpha_k = \xi_{kij}\alpha_j = \xi_{jki}\alpha_i$ for all i, j, k distinct in $\{1, \dots, n\}$.

Proof. By assumption (x, y) is symmetric, so it suffices to show that the condition $0 = (x, x \times y)$ for all x and y in V is equivalent to conditions (a) and (b). The condition $0 = (x, x \times y)$ is equivalent to the

linearization $0 = (x, z \times y) - (z, x \times y)$, and by the linearity of this relation it is equivalent to the set of equations

$$\begin{aligned} 0 &= (u_i, u_j \times u_k) + (u_j, u_i \times u_k) \\ &= (u_i, \xi_{jki} u_i) + (u_j, \xi_{ikj} u_j) \\ &= \xi_{jki} \alpha_i + \xi_{ikj} \alpha_j, \end{aligned}$$

for all $i, j, k \in \{1, \dots, n\}$. The latter conditions are condition (a) of the theorem when $k = i$ or j , and condition (b) when i, j , and k are distinct.

We shall call $A \in \mathcal{A}$ *weakly flexible* if property (a) in Lemma 2.1 is satisfied for each orthogonal basis of V . Osborn [6, pp. 204–206] calls a skew-commutative algebra V *division-like* if there do not exist linearly independent u and $v \in V$ such that $u \times v = 0$ or $u \times v = u$ and he shows that $A = F + V$ is a division algebra if and only if V is division-like and a certain condition is satisfied by its bilinear form.

LEMMA 2.2. *Let $A \in \mathcal{D}(\mathcal{A})$. A is weakly flexible if and only if for x and $y \in V$ such that $(x, y) = 0$, there exists $z \in V$ such that $x = y \times z$.*

Proof. Suppose first that A is weakly flexible. Since $(x, y) = 0$, there exists an orthogonal basis $u_1 = x$, $u_2 = y$, u_3, \dots, u_n for V . Since $A \in \mathcal{D}(\mathcal{A})$, there exists $\alpha + z \in A$ such that

$$u_1 = u_2(\alpha + z) = \alpha u_2 + (u_2, z)u_2 + u_2 \times z = \alpha u_2 + u_2 \times z.$$

Let $z = \sum_{j=1}^n \gamma_j u_j$. Then

$$u_2 \times z = \sum_{j=1}^n \gamma_j (u_2 \times u_j) = \sum_{k=1}^n \sum_{j=1}^n \gamma_j \xi_{2jk} u_k.$$

Since A is weakly flexible, $\xi_{2j2} = 0$ for all $j = 1, \dots, n$, so the coefficient of u_2 in $u_2 \times z$ is 0, which then implies $\alpha = 0$. Thus $x = y \times z$.

Conversely, let u_1, \dots, u_n be an orthogonal basis of V . Fix i and let $z^L = u_i \times z$ for $z \in V$. The assumption implies u_k is in the image of L for all $k \neq i$. V is division-like, so $u_i \neq u_i \times z$ for any $z \in V$, which implies the set of vectors $\{u_k \mid k \neq i\}$ spans the image of L . Hence $u_j^L = u_i \times u_j = \sum_{k \neq i} \xi_{ijk} u_k$, which implies $\xi_{iji} = 0$ for all $j = 1, \dots, n$. The arbitrariness of i gives the desired conclusion.

We note that if $A \in \mathcal{A}$ is weakly flexible and $x, y \in V$ are such that $(x, y) \neq 0$, then $x = y \times z$ for $z \in V$ is impossible. There exists an orthogonal basis $y = u_1, u_2, \dots, u_n$ of V and $x = \gamma u_1 + w$ for w in the span of $\{u_2, \dots, u_n\}$ and $\gamma \neq 0$. Since A is weakly flexible, for any $z \in V$,

$y \times z$ is in the span of $\{u_2, \dots, u_n\}$. Thus $x = \gamma u_1 + w = y \times z$ is not possible.

THEOREM 2.1. *Let $A \in \mathcal{D}(\mathcal{A})$, T nonsingular on U , and $B(A, T)$ flexible. Then $B(A, T) \in \mathcal{D}(\mathcal{A})$.*

Proof. Since $(x, y)_T = (x, y)$, $B(A, T)$ will be a division algebra, if V is division-like with respect to “ \times_T ”. Suppose there exist linearly independent x and y in V such that $x \times_T y = x$. The flexibility of $B(A, T)$ implies $(x, x \times_T y)_T = 0$. Now

$$x^2 = x \cdot x = x \cdot (x \times_T y) = (x, x \times_T y)_T + x \times_T (x \times_T y) = 0 + x \times_T x = 0,$$

which contradicts the assumption that $A \in \mathcal{D}(\mathcal{A})$. Suppose there exist linearly independent x and y in V such that $x \times_T y = 0$. Then by (1.4), $0 = x \times_T y = (x \times y)^T$. But T is nonsingular, so $x \times y = 0$ which also contradicts $A \in \mathcal{D}(\mathcal{A})$.

If 1, u_1, u_2, \dots, u_n is an orthogonal basis of $A \in \mathcal{D}(\mathcal{A})$, then $u_1 \times x \neq 0$ for x in the span of $\{u_2, \dots, u_n\}$. Thus the $n - 1$ vectors $u_1 \times u_2, u_1 \times u_3, \dots, u_1 \times u_n$ are linearly independent. Moreover, since V is division-like, we cannot have

$$u_1 = \sum_{i=2}^n \beta_i (u_1 \times u_i) = u_1 \times \sum_{i=2}^n \beta_i u_i,$$

so the n vectors $u_1, u_1 \times u_2, u_1 \times u_3, \dots, u_1 \times u_n$ are linearly independent. Let v be any vector such that $(u_1, v) = 0$. If A were weakly flexible, then by Lemma 2.2 there exists $z \in V$ such that $u_1 = v \times z$ which puts $u_1 \in U$. Thus U is a n -dimensional space contained in V , if $n > 1$. Hence it is plausible to assume T is a linear transformation from V into V .

COROLLARY 2.1. *Let A be flexible and in \mathcal{A} but not in $\mathcal{D}(\mathcal{A})$. Then $B(A, T)$ is not in $\mathcal{D}(\mathcal{A})$ for any nonsingular $T: V \rightarrow V$.*

Proof. Since T is nonsingular on V , $T^{-1}: V \rightarrow V$ exists and it is easily checked that $A = B(B(A, T), T^{-1})$. So by Theorem 2.1, if $B(A, T)$ were in $\mathcal{D}(\mathcal{A})$, then A would have to also be in $\mathcal{D}(\mathcal{A})$.

THEOREM 2.2. *Let $A \in \mathcal{A}$ and suppose that for all $x \in U$, there exist y and $z \in V$ such that $x = y \times z$. If T is singular on U , then $B(A, T)$ is not a division algebra.*

Proof. T singular implies there exists $x \neq 0$ in U such that $x^T = 0$. Choose y and $z \in V$ such that $x = y \times z$. Then $0 = x^T =$

$(y \times z)^T = y \times_T z$, which implies $B(A, T)$ is not a division algebra since V is not division-like with respect to " \times_T ".

By Lemma 2.2 the condition on V in Theorem 2.2 holds in particular if $A \in \mathcal{D}(\mathcal{A})$ is weakly flexible and $n > 1$. If $n = 1$, then $U = 0$. Thus for $A \in \mathcal{D}(\mathcal{A})$ weakly flexible and $B(A, T)$ flexible we have, by Theorems 2.1 and 2.2, $B(A, T) \in \mathcal{D}(\mathcal{A})$ if and only if T is nonsingular on V .

If we assume $A \in \mathcal{D}(\mathcal{A})$ is flexible and that T is a scalar δ times the identity transformation I on V , then for any orthogonal basis u_1, \dots, u_n of V we have

$$\sum_{k=1}^n \beta_{ijk} u_k = u_i \times_T u_j = (u_i \times u_j)^T = \sum_{k=1}^n \xi_{ijk} u_k^T = \sum_{k=1}^n \xi_{ijk} \delta u_k.$$

So $\beta_{ijk} = \xi_{ijk} \delta$ for all $i, j, k = 1, \dots, n$. Since A is flexible, the β_{ijk} clearly satisfy the conditions in Lemma 2.1 which make $B(A, T)$ flexible and then by Theorem 2.1 $B(A, T) \in \mathcal{D}(\mathcal{A})$.

THEOREM 2.3. *Let A be weakly flexible in $\mathcal{D}(\mathcal{A})$ and let $T: V \rightarrow V$ be such that $B(A, T)$ is weakly flexible. Then T is a scalar multiple of I .*

Proof. Let u_1, \dots, u_n be an orthogonal basis of V . Pick u_r and u_s such that $r \neq s$. A is weakly flexible, so by Lemma 2.2 there exists $z \in V$ such that $u_s = u_r \times z$. Suppose $u_s^T = \sum_{i=1}^n {}_s\delta_i u_i$ and $z = \sum_{j=1}^n \gamma_j u_j$. Then

$$\sum_{i=1}^n {}_s\delta_i u_i = (u_r \times z)^T = \sum_{j=1}^n \gamma_j (u_r \times u_j)^T = \sum_{i=1}^n \sum_{j=1}^n \gamma_j \beta_{rji} u_i.$$

So ${}_s\delta_r = \sum_{j=1}^n \gamma_j \beta_{rjr} = 0$, since $\beta_{rjr} = 0$ for all r and j . Thus $u_s^T = {}_s\delta_s u_s = \delta_s u_s$ for each $s = 1, \dots, n$. The extra subscript is now dropped for the sake of simplicity. To show T is a scalar multiple of the identity let u_1 be any nonzero element of V . Then u_1 may be embedded in a basis u_1, \dots, u_n of V and we have $u_1^T = \delta_1 u_1$ for some $\delta_1 \in F$. Then also for any $v \in V$, $v^T = \delta_2 v$ for some $\delta_2 \in F$, and $\delta_1 u_1 + \delta_2 v = u_1^T + v^T = (u_1 + v)^T = \delta_3(u_1 + v) = \delta_3 u_1 + \delta_3 v$ for some $\delta_3 \in F$. Hence $\delta_1 = \delta_2 = \delta_3$ and $T = \delta_1 I$.

Since the Cayley-Dickson algebras are alternative, they are flexible. So for A a Cayley-Dickson algebra in $\mathcal{D}(\mathcal{A})$, $B(A, T)$ is flexible if and only if T is a scalar times I , the identity transformation on V .

COROLLARY 2.2. *Let A be a Cayley-Dickson algebra in $\mathcal{D}(\mathcal{A})$. $B(A, T)$ is a Cayley-Dickson algebra in $\mathcal{D}(\mathcal{A})$ if and only if $T = \pm I$.*

Proof. It is easily checked that a quadratic algebra is alternative if and only if it is flexible and $(y, x)x - (x, x)y + (y \times x) \times x = 0$ for all vectors x and y in the algebra. All that remains to be shown is that in $B(A, T)$, $(y, x)_T x - (x, x)_T y + (y \times_T x) \times_T x = 0$ if and only if $T = \pm I$. By (1.4) and Theorem 2.3

$$\begin{aligned} (y, x)_T x - (x, x)_T y + (y \times_T x) \times_T x &= (y, x)x - (x, x)y + [(y \times x)^T \times x]^T \\ &= (y, x)x - (x, x)y + \delta^2(y \times x) \times x \end{aligned}$$

for some scalar δ . Since A is alternative, this expression is 0 if and only if $\delta^2 = 1$.

3. In this section the bonding mapping process is applied to a class of Cayley-Dickson division algebras over formally real fields to obtain nonflexible quadratic division algebras of dimension 8. We use the definition, as given by Kleinfeld [4], of a Cayley-Dickson algebra in terms of its multiplication table with respect to a basis $1, u_1, \dots, u_7$ and parameters α, β , and γ . Exact conditions on α, β, γ , and the field F which make the algebra a division algebra are given by Schafer [7]. We consider only the Cayley-Dickson division algebras over formally real fields with $\alpha = \beta = \gamma = -1$. (The Cayley numbers are in this class.) The multiplication table for the nonidentity basis elements in such an algebra $A = F + V$ is given in Table I. It is clear by Table I that for such a

TABLE I

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	-1	u_3	$-u_2$	u_5	$-u_4$	$-u_7$	u_6
u_2	$-u_3$	-1	u_1	u_6	u_7	$-u_4$	$-u_5$
u_3	u_2	$-u_1$	-1	u_7	$-u_6$	u_5	$-u_4$
u_4	$-u_5$	$-u_6$	$-u_7$	-1	u_1	u_2	u_3
u_5	u_4	$-u_7$	u_6	$-u_1$	-1	$-u_3$	u_2
u_6	u_7	u_4	$-u_5$	$-u_2$	u_3	-1	$-u_1$
u_7	$-u_6$	u_5	u_4	$-u_3$	$-u_2$	u_1	-1

Cayley-Dickson algebra u_1, \dots, u_7 is an orthogonal basis for V , and that each u_i for $i = 1, \dots, 7$ is equal to $u_j \times u_k$ for some $j, k \in \{1, \dots, 7\}$, so that the subspace U as defined in §1 is equal to V . Moreover, $\alpha_i = (u_i, u_i) = -1$ for $i = 1, \dots, 7$. A is the special case $\tau = 0$ of the class of division

algebras we are about to define. Let T be a bonding mapping from V to V with matrix representation

(3.1)
$$\begin{bmatrix} 1 & \tau & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & & 1 & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, \text{ where } \tau \neq 0$$

is in F , with respect to the basis u_1, \cdots, u_7 . By (1.4) the multiplication in $B(A, T)$ of two nonidentity basis elements is given by $u_i \cdot u_j = (u_i, u_j) + (u_i \times u_j)^T$, where (x, y) is the bilinear form determined by A and “ \times ” is the skew-commutative multiplication in V determined by A . So

(3.2)
$$\begin{aligned} u_i \cdot u_i &= (u_i, u_i) + (u_i \times u_i)^T \\ &= (u_i, u_i) = -1 \quad \text{for } i = 1, \cdots, 7 \\ u_i \cdot u_j &= (u_i, u_j) + (u_i \times u_j)^T \\ &= (u_i \times u_j)^T \quad \text{for } i \neq j; \quad i, j = 1, \cdots, 7. \end{aligned}$$

Using (3.2) one obtains the multiplication table for $B(A, T)$ given in Table II.

TABLE II

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	-1	u_3	$-u_2$	u_5	$-u_4$	$-u_7$	u_6
u_2	$-u_3$	-1	$u_1 + \tau u_2$	u_6	u_7	$-u_4$	$-u_5$
u_3	u_2	$-u_1 - \tau u_2$	-1	u_7	$-u_6$	u_5	$-u_4$
u_4	$-u_5$	$-u_6$	$-u_7$	-1	$u_1 + \tau u_2$	u_2	u_3
u_5	u_4	$-u_7$	u_6	$-u_1 - \tau u_2$	-1	$-u_3$	u_2
u_6	u_7	u_4	$-u_5$	$-u_2$	u_3	-1	$-u_1 - \tau u_2$
u_7	$-u_6$	u_5	u_4	$-u_3$	$-u_2$	$u_1 + \tau u_2$	-1

We shall prove that $B(A, T)$ is a division algebra for any T as in (3.1) such that $|\tau| < 2$, and we shall give examples of zero divisors when $\tau = 2$. We take T in (3.1) such that $|\tau| < 2$ and let

$$x = au_0 + bu_1 + cu_2 + du_3 + fu_4 + gu_5 + hu_6 + ku_7$$

and

$$y = a'u_0 + b'u_1 + c'u_2 + d'u_3 + f'u_4 + g'u_5 + h'u_6 + k'u_7,$$

where $a, \dots, k, a', \dots, k' \in F$ be two arbitrary elements of $B(A, T)$. Using Table II we can express the relation $x \cdot y = 0$ in terms of the basis elements $1, u_1, \dots, u_7$ for $B(A, T)$ in the following way:

$$\begin{aligned} 0 = & (aa' - bb' - cc' - dd' - ff' - gg' - hh' - kk')1 \\ & + (ab' + ba' + cd' - dc' + fg' - gf' + kh' - hk')u_1 \\ & + (ac' + ca' + db' - bd' + \tau cd' - \tau dc' + \tau fg' \\ & - \tau gf' + fh' - hf' + gk' - kg' + \tau kh' - \tau hk')u_2 \\ & + (ad' + da' + bc' - cb' + fk' - kf' + hg' - gh')u_3 \\ & + (af' + fa' + gb' - bg' + hc' - ch' + kd' - dk')u_4 \\ & + (ag' + ga' + bf' - fb' + kc' - ck' + dh' - hd')u_5 \\ & + (ah' + ha' + bk' - kb' + cf' - fc' + gd' - dg')u_6 \\ & + (ak' + ka' + hb' - bh' + cg' - gc' + df' - fd')u_7. \end{aligned}$$

This gives eight homogeneous bilinear equations in the elements $a, \dots, k, a', \dots, k'$. The equation $x \cdot y = 0$ has a solution in $B(A, T)$ if and only if these eight equations can be made to equal zero simultaneously. We may think of the primed letters a', \dots, k' as variables and consider the coefficient matrix M_T of the set of eight equations. We have

$$M_T = \begin{bmatrix} a & -b & -c & -d & -f & -g & -h & -k \\ b & a & -d & c & -g & f & k & -h \\ c & d & a - \tau d & -b + \tau c & -h - \tau g & -k + \tau f & f + \tau k & g - \tau h \\ d & -c & b & a & -k & h & -g & f \\ f & g & h & k & a & -b & -c & -d \\ g & -f & k & -h & b & a & d & -c \\ h & -k & -f & g & c & -d & a & b \\ k & h & -g & -f & d & c & -b & a \end{bmatrix}.$$

It suffices to show this matrix is nonsingular for all choices of a, \dots, k not all zero. To show M_T is nonsingular for $|\tau| < 2$ we utilize a technique found in [6]. Let

$$M'_T = \begin{bmatrix} a & b & c & d & f & g & h & k \\ -b & a & d & -c & g & -f & -k & h \\ -c & -d & a & b & h & k & -f & -g \\ -d & c & -b & a & k & -h & g & -f \\ -f & -g & -h & -k & a & b & c & d \\ -g & f & -k & h & -b & a & -d & c \\ -h & k & f & -g & -c & d & a & -b \\ -k & -h & g & f & -d & -c & b & a \end{bmatrix}.$$

If we set $\Gamma = a^2 + b^2 + c^2 + d^2 + f^2 + g^2 + h^2 + k^2$, then

$$M_T M'_T =$$

$$\begin{bmatrix} \Gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau(\Gamma - a^2 - b^2) & \Gamma - \tau(ad + bc) & \tau(ac - bd) & -\tau(ag + bf) & \tau(af - bg) & \tau(ak - bh) & \tau(ah - bh) \\ 0 & 0 & 0 & \Gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma \end{bmatrix}$$

Any choice of τ which makes $M_T M'_T$ nonsingular will clearly make M_T nonsingular. We have

$$(3.3) \quad \det M_T M'_T = \Gamma^7 [\Gamma - \tau(ad + bc)].$$

Since F is a formally real field, $\Gamma > 0$ in F unless $a = b = c = d = f = g = h = k = 0$. We expand the other factor of $\det M_T M'_T$ to obtain

$$(3.4) \quad \begin{aligned} \Gamma - \tau(ad + bc) &= a^2 - \tau ad + d^2 + b^2 - \tau bc + c^2 \\ &\quad + f^2 + g^2 + h^2 + k^2. \end{aligned}$$

We want to show the expression in (3.4) is nonzero for any $\tau \in F$ such that $|\tau| < 2$. Consider the quadratic form $q = \lambda_1^2 + \tau \lambda_1 \lambda_2 + \lambda_2^2$ and the nonsingular linear transformation given by $\lambda_1 = \mu_1 - \mu_2$ and $\lambda_2 = \mu_1 + \mu_2$. This transformation applied to q gives a new quadratic form $p = (2 + \tau)\mu_1^2 + (2 - \tau)\mu_2^2$. Since the transformation connecting them is nonsingular, q and p are congruent. Therefore, they have the same range of

values when λ_1, λ_2 and μ_1, μ_2 assume all values in the formally real field F . But $|\tau| < 2$ implies $2 + \tau > 0$ and $2 - \tau > 0$. So $p > 0$ which implies $q > 0$ for $|\tau| < 2$. Applying this conclusion to (3.4) shows $\Gamma - \tau(ad + bc) > 0$ for $|\tau| < 2$. Thus $M_T M'_T$ and M_T are nonsingular and $B(A, T)$ has no nontrivial zero divisors.

Let T_0 be the nonsingular linear transformation obtained by setting $\tau = 2$ in (3.1). $B(A, T_0)$ will have divisors of zero. The multiplication table for $B(A, T_0)$ is Table II with $\tau = 2$. Let M_{T_0} be the matrix obtained from M_T by setting $\tau = 2$. It is easily seen that $\det M_{T_0} = 0$ for $a = d$, $b = c$, and $f = g = h = k = 0$, so that nontrivial solutions to $x \cdot y = 0$ do exist in $B(A, T_0)$. (E.g. $x = 1 + u_1 + u_2 + u_3$ and $y = 1 + u_1 + u_2 - u_3$ have product 0 in $B(A, T_0)$.)

Albert [1], Bruck [3], and Osborn [6] have constructed classes of quadratic division algebras. A full determination of quadratic division algebras obtainable by this bonding mapping process has not been made even when A is taken to be a Cayley-Dickson algebra. The class of division algebras obtained above with $\tau \neq 0$ does not contain any flexible algebras, since $u_2 \cdot u_3 = u_1 + \tau u_2$ with $\tau \neq 0$ violates condition (a) of Lemma 2.1. Moreover, for T as in (3.1) with $\tau \neq 0$ one obtains $u_1 \times_T u_4 = u_5$, $u_4 \times_T u_5 = u_1 + \tau u_2$, $u_1 \times_T u_2 = u_3$, $u_2 \times_T u_4 = u_6$, and $u_2 \times_T u_5 = u_7$, so that the skew-commutative algebra generated in V by u_1 and u_4 is V itself. This shows that no $B(A, T)$ obtained as above with $\tau \neq 0$ is a division algebra of dimension 8 in the class discovered by Osborn [6], since in his class of examples every two independent elements in V generate a subalgebra in V of dimension 3.

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Received November 5, 1975. A portion of the results herein were submitted to the Graduate School of the University of Iowa as part of the author's dissertation supervised by Professor R. H. Oehmke.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

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