CHAINED RINGS

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All rings considered are commutative with identity. A chained ring is any ring whose set of ideals is totally ordered by inclusion. The main object of this paper is stating conditions in which every valuation overring of a given ring is a chained ring. It is shown that every valuation overring of a ring R is a chained ring if and only if the ideal of zero divisors of $T(R)$, the total quotient ring of R, is the conductor of $\bar{R}$, the integral closure of R, in $T(R)$. An example is provided of a valuation ring which is not a chained ring even though its total quotient ring is a chained ring.

1. **Preliminaries.** In this short section we set notation and establish some elementary results.

A *total quotient ring* is a ring consisting entirely of zero divisors and units. If R is a ring, then let $T(R)$ denote the smallest total quotient ring containing R. We will call $T(R)$ *the total quotient ring of R*.

Let G be a totally ordered abelian group. A *valuation* on a ring T is a function v from T onto $G \cup \{\infty\}$, such that for all x and y in T:

(1) $v(xy) = v(x) + v(y)$;

(2) $v(x + y) \geqq \min \{v(x), v(y)\}$;

(3) $v(1) = 0$ and $v(0) = \infty$.

Let V be a ring, let P be a prime ideal of V, and let T be a ring containing V. Manis [11] defined the pair (V, P) to be a *valuation pair* of T if the following equivalent conditions are satisfied:

(a) If S is a subring of T containing V and if M is a prime ideal of S such that $M \cap V = P$, then $V = S$;

(b) For each x in $T \backslash V$, there exists y in P such that xy is in $V \backslash P$;

(c) There is a valuation (v, G) on T such that

$$V = \{x \in T \,|\, v(x) \geqq 0\} \quad \text{and} \quad P = \{x \in T \,|\, v(x) > 0\} \,.$$

If (V, P) is a valuation pair of T, then V is called a *valuation ring*. We will sometimes represent V by T_v. In this paper we always assume that T is $T(V)$. Chained rings are valuation rings, and in the domain case the concepts are equivalent to the concept of a valuation domain.

When we deal with rings containing zero divisors, we say that

an ideal is *regular* provided it contains a regular element, a nonzero divisor. We therefore define a *Prüfer ring* as a ring in which every finitely generated regular ideal is invertible. Fourteen equivalent definitions for a Prüfer ring were given by Griffin [7]. Moreover, examples by Gilmer [4] and Boisen and Larsen [1] show that a valuation ring need not be a Prüfer ring. Consequently, we define a *Prüfer valuation ring* as a valuation ring that is also a Prüfer ring.

A ring R has *few zero divisors* if the set of zero divisors of R is a finite union of prime ideals. A ring R is *additively regular* if for each x in $T(R)$ there exists a u in R such that $x + u$ is a regular element of $T(R)$. Davis [3] showed that a ring with few zero divisors is additively regular. Gilmer and Huckaba [6] have shown that the converse is not true. In his paper Davis defined what he called a quasi-valuation ring as a ring R with few zero divisors such that if x and y are in R, both regular, then x divides y or y divides x in R. However, Griffin [7] showed that if R is an additively regular ring, then R is a valuation ring if and only if R is a quasi-valuation ring. But then Huckaba [8] established that an additively regular valuation ring is a Prüfer ring. Consequently, we define R to be a *quasi-valuation ring* if x and y in R, both regular, implies that x divides y or y divides x in R.

2. Structure theorems for chained rings. If R is a chained ring, then obviously any homomorphic image of R is also a chained ring. Consequently, the homomorphic image of a valuation domain is a chained ring. By making use of recent results of Hungerford [9] and Boisen and Larsen [2], we are able to characterize a chained ring with Noetherian total quotient ring:

THEOREM 1. *Let R be a chained ring whose total quotient ring is a Noetherian ring. Then R is the homomorphic image of a valuation domain.*

Proof. Since R is a chained ring with $T(R)$ a Noetherian ring, then it is easy to see that $T(R)$ is a principal ideal ring and satisfies the descending chain condition. Hungerford [9] has shown that $T(R)$ is the homomorphic image of a valuation domain. Indeed, $T(R)$ is the homomorphic image of a Prüfer domain. However, since R is a Prüfer ring, Boisen and Larsen [2] have shown that R is the homomorphic image of a Prüfer domain, say D. Let π be the desired homomorphism and let M be the unique maximal ideal of R. There is an ideal M' of D such that $\pi(M') = M$. If $b \notin M'$, then $\pi(b) \notin M$, and hence $\pi(b)$ is a unit of R. The function defined by

$\pi^*: D_{M'} \to R$ such that $\pi^*(a/b) = \pi(a)\pi(b)^{-1}$ is an epimorphism. Therefore, R is the homomorphic image of the valuation domain $D_{M'}$.

The question remains open as to whether every chained ring can be realized as the homomorphic image of a valuation domain.

Let V be an arbitrary valuation ring with corresponding valuation v. If z is a zero divisor of V, then nothing can be determined about the image of z under v. This is not the case in chained rings, as our next result shows. It is basic to what follows.

THEOREM 2. (1) *Every chained ring is a valuation ring of its total quotient ring.*

(2) *If v is a valuation on T and T_v is chained, then $Z(T)$ is a maximal ideal of T and there is a valuation v' on the field $T/Z(T)$ such that v is given by $v(x) = v'(x + Z(T))$ for all x in T. Moreover, in this case $Z(T_v) = \{x \in T \mid v(x) = \infty\}$.*

Proof. For (1) let v' be the valuation of R_P/P_P whose valuation ring is R/P, where P is the prime ideal of zero divisors of the chained ring R. One sees easily that $R = T(R)_u$, where u is given by the formula $u(a/s) = v'(a + P) - v'(s + P)$, where a and s are in R and s is regular.

For the proof of (2), suppose $R = T_v$ is chained and P the prime ideal of zero divisors of R, so that $T = R_P$. Then every element of R_P either belongs to R or has the form $1/s$ for some s in $R \backslash P$. If $R = T(R)$, the result is straightforward. Otherwise, we have $v(1/s) < 0$ for some regular s in R. But then if r is an element of $Z(R)$ and $v(r) = 0$, we have $rR \subseteq sR$ so that $v(s) = 0$ for *all* regular s, a contradiction. Hence, if r is an element of $Z(R)$, $v(r) > 0$. If $v(r) < \infty$, there exists s regular in R such that

$$v(1/s) = -v(r), \text{ i.e. } v(r/s) = 0 \ .$$

But then $r = r's$ and r' is a zero divisor for which $v(r') = 0$, a contradiction. This proves the "moreover" of (2). Now define a valuation of $T/Z(T)$ by $v'(x + Z(T)) = v(x)$. This is easily seen to be well defined, etc., and $Z(T) = P_P$ is maximal in T; clearly v is equivalent to u, where u is defined as in (1).

We now prove our first structure theorem:

THEOREM 3. *Let T be a chained total quotient ring and V a valuation ring of T. Then V is a chained ring iff $Z(T) \subseteq V$, in which case V is quasilocal. Otherwise, V has exactly two maximal ideals, namely $Z(V)$ and a maximal regular ideal.*

Proof. If V is chained, then x is an element of $Z(T)$ if and only if $x = a/s$ with a and s elements of V and s regular, a an element of $Z(V)$. But clearly $aV \subsetneqq sV$, so a/s is an element of V. Conversely, if $Z(T) \subsetneqq V$, then for all a in $Z(V)$ and regular s in V, a/s is in V. Now let x and y be in V, say $xT \subsetneqq yT$, i.e. $x = y(a/s)$ for a and s in V, s regular. If a is regular, then $v(a/s)$ or $v(s/a)$ is ≥ 0, i.e. a/s or s/a is in V. Hence, $xV \subsetneqq yV$ or $yV \subsetneqq xV$. If a is a zero divisor, then x is in yV. In either case, xV and yV are comparable.

Finally, if $Z(T) \nsubseteq V$, then there is a regular nonunit s in V and zero divisor such that a/s is not in V, i.e. $aV \nsubseteq sV$. So let M be the *unique* maximal ideal containing sV. [Regular primes are comparable by an argument of Davis [3]: if M and N are incomparable such ideals, then $M \nsubseteq N \cup Z(V)$, since $Z(V) = Z(T) \cap V$ is a prime ideal, whence there is a regular element x in $M \backslash N$, and symmetrically, regular y in $N \backslash M$. Since x and y are comparable, this is impossible.] and N a maximal ideal containing aV. Since $N \neq M$, N consists of zero divisors, so $N = Z(V)$, as required.

This theorem tells us precisely when a valuation ring with chained total quotient ring is or is not a chained ring. It rests completely on whether the zero divisors of the total quotient ring are or are not contained in the ring. Any homomorphic image of a valuation domain satisfies the first part of Theorem 3. We now construct an example which show the second part of Theorem 3 is realizable; that is, we construct a chained total quotient ring T that admits a valuation ring V such that $T(V) = T$, but V is not a chained ring.

EXAMPLE 1. Let K be a Prufer 2^∞-group [2-primary part of the group Q/Z]. Then K is a $Z_{(2)}$-module in a natural way. Let $T = K + Z_{(2)}$ be the idealization of the $Z_{(2)}$-module K [addition is coordinate-wise and multiplication is $(k_1 + z_1)(k_2 + z_2) = (k_1 z_2 + k_2 z_1) + z_1 z_2$]. The divisibility of K shows that the ideals of T are exactly the subgroups of K and ideals of the form $K + J$, J an ideal of $Z_{(2)}$. Hence T is a chained ring and in fact $Z(T) = K + (2)$, so T is a total quotient ring. Define a valuation on T by $v(k + z) = v'(z)$, where v' is the 3-adic valuation of Q. Then $T_v = K + (Z_{(2)} \cap Z_{(3)})$, clearly, and $Z_{(2)} \cap Z_{(3)} = Z_{(2) \cup (3)}$ is not a quasilocal ring. Since it is a homomorphic image of T_v, T_v is not a chained ring, yet the total quotient ring of T_v is T itself. Incidentally, in this case $Z(T) = K + (2)$ and $K + (3)$ are the two maximal ideals of T_v.

We now list three immediate corollaries to Theorem 3.

COROLLARY 1. *Let T be a chained total quotient ring and V a proper valuation ring of T such that $T = T(V)$.*

(a) *If V has exactly two maximal ideals, then every overring of V, other that T, is a valuation ring with exactly two maximal ideals;*

(b) *If V is a chained ring, then every overring of V is chained ring.*

COROLLARY 2. *Let R be a ring such that $T(R)$ is a chained ring. Let $\{V_a\}$ be a set of valuation rings, totally ordered by inclusion, between R and $T(R)$. If one member of $\{V_a\}$ is a chained ring, then every member of $\{V_a\}$ is a chained ring. If one member of $\{V_a\}$ has exactly two maximal ideals, then every member of $\{V_a\}$ has exactly two maximal ideals.*

COROLLARY 3. *Every overring of a chained ring is a chained ring.*

We have seen by Example 1 that there exist chained total quotient rings T that admit valuation rings which are not chained rings. The content of the next theorem is that this situation cannot arise when the total quotient ring is a 0-dimensional ring.

THEOREM 4. *Let R be a ring with 0-dimensional total quotient ring, then the following are equivalent:*
(a) *R admits at least one chained overring;*
(b) *$T(R)$ is a chained ring; and*
(c) *Every valuation overring of R is a chained ring.*

Proof. (a) $\Rightarrow$ (b): Let V be a chained overring of R, then $T(V) = T(R)$ and by Corollary 3 $T(R)$ is a chained ring.

(b) $\Rightarrow$ (c): If V is a valuation overring of R, let $T = T(V) = T(R)$. Since T is 0-dimensional, we have that $Z(T) = N(T)$ [the nil radical of T]. Since z in $Z(T)$ implies that there is an integer $n > 1$ such that $z^n = 0$, then z is integral over V, and therefore z is in V since V is integrally closed. Thus $Z(T) \subseteq V$, and we have that V is a chained ring.

(c) $\Rightarrow$ (a): Obvious.

A ring R is a *von Neumann regular ring* if for each r in R, there is a b in R such that $r^2 b = r$. von Neumann regular rings are 0-dimensional rings and have no nonzero nilpotents.

COROLLARY 4. *If R admits a chained overring, then $T(R)$ is*

von Neumann regular if and only if R is an integral domain.

We characterize rings in which every valuation overring is a chained ring.

THEOREM 5. *Let R be ring with total quotient ring T. Let $\bar{R}$ be the integral closure of R in T. Every valuation overring of R is a chained ring if and only if $Z(T)$ is the conductor of $\bar{R}$ in T.*

Proof. ($\Rightarrow$): Let V be an arbitrary valuation overring of R, then V is a chained ring. Clearly, $Z(T) = Z(V)$ is the conductor of V in T. Therefore, $Z(R) = Z(T) \cap R$, and R is an additively regular ring. This implies that $\bar{R}$ is the intersection of a family of valuation rings $\{V_a\}$. So, $Z(T) \subseteq \bigcap V_a = \bar{R}$.

($\Leftarrow$): If $Z(T)$ is the conductor of $\bar{R}$ in T, then $Z(T)$ is the unique maximal ideal of T and thus is a prime ideal. Hence, $Z(R) = Z(T) \cap R$ is a prime ideal of R, and thus R is additively regular. Whence $\bar{R} = \bigcap V_a$ where $\{V_a\}$ is the set of valuation rings between R and $T(R)$. Therefore, $Z(T) \subseteq V_a$ for all a and hence each V_a is a chained ring.

Our final result is of interest in the light that it extends the theory of chained rings although it is not directly concerned with overrings. It gives necessary and sufficient conditions for a Prüfer valuation ring to be a chained ring.

Let P be a prime ideal of a ring R. The *core* of P, denoted by $C(P)$, is the set of all elements b in R such that for each regular r in R, there exists an element e in $R \backslash P$ such that be/r is in R. For a valuation ring V with $M = \{x \in T(V) \,|\, v(x) > 0\}$, the core of M is the prime ideal of zero for which the valuation, v, is infinite.

THEOREM 6. *Let V be a valuation ring with a unique maximal regular ideal M. Then V is a chained ring if and only if the following three conditions hold:*
  (a) *V is a Prüfer ring;*
  (b) *$C(M) = Z(V)$;*
  (c) *$a, b \in C(M)$ implies that a divides b or b divides a in V.*

Proof. ($\Rightarrow$): The fact that chained rings are Prüfer is trivial, since finitely generated regular ideals are principal ideals generated by regular elements. Theorem 2 tells us that

$$C(M) = \{x \in V \,|\, v(x) = \infty\} = Z(V) \,.$$

And certainly (c) is true.

($\Leftarrow$): If a and b are in V, we must show that a divides b or b divides a in V. This follows from the hypotheses, if a and b are in $C(M)$; and by the quasi-valuation property, if a and b are regular. Assume that a is regular and that b is a zero divisor. Then b/a is in $T(V)$ and $v(b/a) = v(b) - v(a)$. But $v(b) = \infty$ and $v(a) < \infty$. Thus, $v(b/a) = \infty$ which implies that b/a is in V, or that a divides b in V.

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