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SOME CONVERGENCE PROPERTIES OF THE BUBNOV-GALERKIN METHOD

S. R. SINGH

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We generalize the Bubnov-Galerkin method to approximate the resolvent of the m -sectorial operator associated with a densely defined, closed, sectorial form in a Hilbert space. Some special cases of interest are also discussed.

1. Introduction. The Bubnov-Galerkin method [3] was originally devised to approximate the solutions of the equations of the form

$$(1) \quad (z - A)f = g$$

where A is an operator in a Hilbert space, \mathcal{H} , g is a vector in \mathcal{H} and z is a complex number. The method proceeds with solving the following set of equations:

$$(2) \quad \sum_{j=1}^n \alpha_j (\phi_i | (z - A)\phi_j) = (\phi_i | g) \quad i = 1, \dots, n;$$

where $(\cdot | \cdot)$ denotes the scalar product in \mathcal{H} and $\{\phi_i\} \subset \mathcal{D}(A)$ is some linearly independent (l.i.) set in \mathcal{H} . $\mathcal{D}(\cdot)$ denotes the domain. The questions of interest are the existence and the convergence of the solutions of equation (2). Until recently, the only cases that received a detailed treatment have been when A is compact, bounded or essentially self-adjoint [3, 6]. However, recently the following result was proven by Masson and Thewarapperuma [2]:

R.1. Let A be symmetric, bounded below by b , z be at a non-zero distance from $[b, \infty)$ and $\{\phi_i\}$ be the orthonormal set formed from $\{A^i h\}$ where h is in $\mathcal{D}(A^i)$ for each i . Then $\lim_{n \rightarrow \infty} \|\sum_{j=1}^n \alpha_j \phi_j - (z - A_F)^{-1}g\| = 0$, where $\|\cdot\|$ denotes the norm in \mathcal{H} and A_F is the Friedrichs extension of A .

Consider the following set of equations:

$$(3) \quad \sum_{j=1}^n \alpha_j [z(\phi_i | \phi_j) - t(\phi_i, \phi_j)] = (\phi_i | g) \quad i = 1, \dots, n;$$

where t is a densely defined, closable, sectorial, sesquilinear form in \mathcal{H} . The sector of t will be denoted by S and since it causes no loss of generality, the vertex will be taken to be one. In the present note we determine the limit of $f_n = \sum_{j=1}^n \alpha_j \phi_j$ as n becomes large.

R.1. and some other generalizations of it, will follow from our main result (Theorem 1).

2. Results. Define a new scalar product $(\cdot | \cdot)_t$ on $\mathcal{D}(t)$ by $(u | v)_t = \text{Re. } t(u, v)$, [1, pp. 309-10] and complete $\mathcal{D}(t)$ in the new metric to a Hilbert space \mathcal{H}_t . Let the closure of t be \bar{t} . We have that $\mathcal{D}(t) \subset \mathcal{D}(\bar{t}) = \mathcal{H}_t \subset \mathcal{H}$. The norm in \mathcal{H}_t will be denoted by $\|\cdot\|_t$. Also $\mathcal{B}(X, Y)$ will denote the space of bounded operators with $\mathcal{D}(\cdot) \subset X$ and range $\mathcal{R}(\cdot) \subset Y$, and $\mathcal{B}(X) = \mathcal{B}(X, X)$.

LEMMA 1. Let t be as in equation (3), $\{\phi_i\} \subset \mathcal{D}(t)$ and $g \in \mathcal{H}$. Equation (3) is equivalent to

$$(4) \quad \sum_{j=1}^n \alpha_j (\phi_i | [1 - T(z)] \phi_j)_t = -(\phi_i | Bg)_t, \quad i = 1, \dots, n;$$

where $B \in \mathcal{B}(\mathcal{H}, \mathcal{H}_t)$, $T(z) = (zB_t - C) \in \mathcal{B}(\mathcal{H}_t)$ and B_t is the restriction of B to $\mathcal{D}(t)$.

Proof. Since $t_t = (t - \text{Re. } t)$ is a bounded form on \mathcal{H}_t [1, p. 314], there is a $C \in \mathcal{B}(\mathcal{H}_t)$ such that

$$t_t(u, v) = (u | Cv)_t; \quad u, v \in \mathcal{D}(t).$$

Also from Ref. [4] pp. 332-3, it follows that there is a unique $B \in \mathcal{B}(\mathcal{H}, \mathcal{H}_t)$ such that $\mathcal{D}(B) = \mathcal{H}$ and for $u \in \mathcal{H}_t$, $w \in \mathcal{H}$,

$$(5) \quad (u | \omega) = (u | B\omega)_t.$$

In particular, in equation (3), $(\phi_i | g) = (\phi_i | Bg)_t$ and $(\phi_i | \phi_j) = (\phi_i | B\phi_j)_t = (\phi_i | B_t \phi_j)_t$.

The assertion now follows from direct substitution.

LEMMA 2. In the notation of Lemma 1, we have that B_t , C are closable, B is closed and invertible and $B^{-1}(1 + \bar{C}) = A_t$ where A_t is the unique m -sectorial operator associated with \bar{t} .

Proof. Since B_t and C are bounded and densely defined, they are closable. Since B is bounded and $\mathcal{D}(B) = \mathcal{H}$, it is closed. Invertibility of B has been proven in Reference [4] p. 333.

Now, $\mathcal{D}([B^{-1}(1 + \bar{C})]) \subset \mathcal{H}_t = \mathcal{D}(\bar{t})$ and for $u, v \in \mathcal{D}(t)$,

$$\begin{aligned} (u | B^{-1}(1 + \bar{C})v) &= (u | B^{-1}(1 + C)v) \\ &= (u | (1 + C)v)_t \quad (\text{equation (5)}) \\ &= t(u, v) \end{aligned}$$

From the closability of t , this result extends for $u, v \in \mathcal{H}_t$. The

result now follows from Theorem 2.1, Chapter 6, Reference [1].

THEOREM 1. *In addition to the assumptions of Lemma 1 and 2, let $\{\phi_i\}$ be l.i. and complete in \mathcal{H}_t , and z be at a nonzero distance from S . $f_n = \sum_{j=1}^n \alpha_j \phi_j$ of equation (3) is then defined for each n and $\lim_{n \rightarrow \infty} \|f_n - (z - A_t)^{-1}g\| = 0$.*

Proof. From Lemma 1, equation (3) is equivalent to equation (4). Also without loss of generality, we may assume $\{\phi_i\}$ to be an orthonormal basis in \mathcal{H}_t . It is straightforward to check that (4) is equivalent to

$$(1 - T_n(z))f_n = -P_n Bg$$

where $T_n(z) = P_n T(z) P_n$, and P_n is the ortho-projection on the n -dimensional subspace of \mathcal{H}_t determined by $\{\phi_i\}$, $i = 1$ to n . It follows, for $h \in \mathcal{H}_t$, that

$$\lim_{n \rightarrow \infty} \|(T_n(z) - \bar{T}(z))h\|_t = 0.$$

Also, since z is at a nonzero distance from S , $\text{dist.}(1, W(\bar{T}(z))) = d' > 0$, where $W(\cdot)$ denotes the numerical range. Further, since the spectrum of T_n , $\sigma(T_n) \subset (W(\bar{T}(z)) \cup \{0\})$, for each n , $(1 - T_n(z))^{-1} \in \mathcal{B}(\mathcal{H}_t)$ with $\|(1 - T_n(z))^{-1}\|_t \leq 1/d$ where $d = \min.(1, d')$. Also $(1 - \bar{T}(z))^{-1} \in \mathcal{B}(\mathcal{H}_t)$.

Hence for $h \in \mathcal{H}_t$

$$\begin{aligned} & \|[(1 - T_n(z))^{-1} - (1 - \bar{T}(z))^{-1}]h\|_t \\ &= \|(1 - T_n(z))^{-1}(T_n(z) - \bar{T}(z))(1 - \bar{T}(z))^{-1}h\|_t \\ &\leq \|(1 - T_n(z))^{-1}\|_t \|(T_n(z) - \bar{T}(z))(1 - \bar{T}(z))^{-1}h\|_t \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Further, for $g \in \mathcal{H}$,

$$\lim_{n \rightarrow \infty} \|(P_n B - B)g\|_t = 0$$

and hence

$$\lim_{n \rightarrow \infty} \|f_n - f\|_t = 0$$

where

$$\begin{aligned} f &= -(1 - \bar{T}(z))^{-1}Bg = -(1 - z\bar{B}_t + \bar{C})^{-1}Bg \\ &= (z - B^{-1}(1 + \bar{C}))^{-1}g \\ &= (z - A_t)^{-1}g \quad (\text{Lemma 2}). \end{aligned}$$

Assertion of the theorem follows by observing that $\|\cdot\|_t \geq \|\cdot\|$. For a symmetric t , $s = [b, \infty)$ with some $b > -\infty$, $\bar{C} = 0$ and $A_t = B^{-1}$ is self-adjoint.

In the following, f_n will stand for $\sum_{j=1}^n \alpha_j \phi_j$ as defined by equation (2).

COROLLARY 1. *Let A be densely defined sectorial operator and z be at a nonzero distance from its sector, $\{\phi_i\}$ be a l.i. basis in $\mathcal{D}(A)$. We have that $\lim_{n \rightarrow \infty} \|f_n - (z - A_F)^{-1}g\| = 0$.*

Proof. Define t of Theorem 1 by $t(u, v) = (u | Av)$, $u, v \in \mathcal{D}(A)$. t is closable from Theorem 1.27, Chapter 6 of [1]. Since $\{\phi_i\}$ is a l.i. basis in $\mathcal{D}(A)$ and $\mathcal{D}(A)$ is dense in $\mathcal{D}(\bar{t}) = \mathcal{H}_t$, it is a l.i. basis in \mathcal{H}_t . The result now follows from the fact that A_t of Theorem 1 now becomes A_F [1, pp. 325-6].

COROLLARY 2. *Let A be symmetric, bounded below by b , z be at a nonzero distance from $[b, \infty)$ and $\{\phi_i\}$ be a l.i. basis in $\mathcal{D}(A)$. Then $\lim_{n \rightarrow \infty} \|f_n - (z - A_F)^{-1}g\| = 0$.*

Proof. The result follows from Corollary 1, by noticing that the sector of A is $[b, \infty)$.

If the set $\{\phi_i\}$ is taken to be $\{A^i h\}$ for some $h \in \mathcal{D}(A^i)$ for $i = 0, 1, 2, \dots$; the Bubnov-Galerkin method is called the method of moments [7]. Since $\{A^i h\}$ satisfies the conditions of Corollaries 1 and 2, the convergence of the method of moments also is established by these results. The result R.1 [2] thus is a special case of Corollary 2.

In Corollaries 1 and 2 we have considered the case of a densely defined A . In these results one can replace this condition by requiring that the form domain of A be dense. However since the Friedrichs extension is defined only for a densely defined A , the limit operator A_t may not be A_F . This situation is of a particular interest in Physics which we describe in brief.

Let A be given, formally, by $A = A_1 + A_2$, where A_1 and A_2 are symmetric but $\mathcal{D}(A) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2)$ is not dense. However if the form domain of A is dense, the self-adjoint operator A_t associated with the form $t(u, v) = (u | (A_1 + A_2)v)$ is a legitimate operator to describe a physical system [5]. This construction enables one to include a larger class of interactions in the treatment than the requirement that A be densely defined [5]. It is obvious that the Bubnov-Galerkin method enables one to compute the resolvent of A_t in this case also, which is of prime importance in Physics.

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