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GEOMETRY AND THE RADON-NIKODYM THEOREM IN STRICT MACKEY CONVERGENCE SPACES

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The main purpose of this paper is to indicate a technique for extending certain types of results which are known for Banach spaces to the corresponding results in more general locally convex topological vector spaces. We shall extend these results to the class of locally convex spaces possessing the strict Mackey convergence [SMC] property. The technique involves a natural embedding into a Banach space and an application of the Banach space result. Loosely speaking, we have that those properties of closed bounded sets in Banach spaces that do not depend upon any open set will yield analogous results for closed bounded sets in quasicomplete locally convex spaces with the SMC property.

Peck [16] and Saab [20] have used a constructive embedding technique for extending results to Fréchet spaces and we shall see that for spaces with the SMC property the embedding used is obtained in a very natural manner without necessity of construction.

In the first section we shall be concerned with vector integration, the Radon-Nikodym Theorem [RNT], and the Radon-Nikodym Property [RNP] for spaces with the SMC property. Rieffel [18, 19], Maynard [13], and others [1, 5] have proved a RNT for the Bochner and the Dunford second integral in Banach spaces. By modifying their techniques, Chi [3] has extended this theorem to Fréchet spaces. We shall extend the Dunford second integral to spaces with the SMC property and obtain the RNT directly from the Banach space result.

Banach spaces with the RNP have been characterized [5, 9, 10, 13, 17] in terms of the geometric and extremal structure of their closed bounded convex sets. Saab [20] and Chi [3] have extended some of these results to Fréchet spaces for a Bochner type integral. We shall obtain these results for our integration theory in spaces with the SMC property.

The second section deals with the separation of closed bounded convex sets, denseness of support points for closed bounded convex sets and denseness of support functionals for closed bounded convex sets. We shall give two results on weakly compact subsets of spaces with the SMC property. Bishop and Phelps [1] gave several results on support points of convex sets in Banach spaces and Peck [16] has given analogs of some of them in Fréchet spaces. We shall show that many of the Bishop and Phelps results extend naturally to spaces with the SMC property. Finally, in [4] it was shown that a weakly compact subset of a Banach space is affinely homeomorphic to a subset of some reflexive Banach space and that a weakly compact operator from a locally convex space to a Banach space factors through a reflexive Banach space. In the same paper it was indicated how these results extend to Fréchet spaces. We demonstrate that these results hold for spaces with the SMC property, the results following directly from the Banach space results.

1. Definitions and notations. For a locally convex topological vector space E over the reals, $\boldsymbol{R}$, we denote by $\{\rho\}$ a fundamental family of continuous seminorms which determine the topology of E, by E^* the algebraic dual of E and by E' the continuous dual of E. If A is a bounded disked set, then we shall denote by E_A the normed linear space consisting of the linear span of A in E with the norm $\|x\|_A = \inf\{\lambda: \lambda > 0,\ x \in \lambda A\}$. We shall say that E has the strict Mackey convergence property [SMC] if for every bounded disked set A in E there is a bounded disked set B containing A such that the topologies of E and E_B agree on A. We observe that the uniform structures on A induced by E and E_B also coincide [7, page 75]. If E is also quasicomplete (every closed bounded set is complete), then E_B is a Banach space. We shall say that a space with the SMC property is a SMC space, and a quasicomplete space with the SMC property a QSMC space.

The SMC property was introduced by Grothendieck [6], who showed that metrizable spaces are SMC spaces. We say a locally convex space E satisfies the Mackey convergence [MC] condition if for every sequence $\{x_n\}$ converging to zero in E, there is a bounded disked set B in E containing $\{x_n\}$ such that $\{x_n\}$ converges to zero in E_B. Grothendieck showed that a locally convex space with a fundamental sequence of bounded sets (a sequence of bounded sets in E such that every bounded set in E is contained in a member of the sequence) has the SMC property if and only if it has the MC property and the topology of E is metrizable on bounded sets. This gives that strict inductive limits of Banach spaces and quasi-complete (LF)-space which have a fundamental sequence of bounded sets, are QSMC spaces. We also note that the countable product of spaces with SMC property has the SMC property, subspaces of spaces with the SMC property have the SMC property and any topological direct sum of a family of spaces with the SMC property has the SMC property.

Throughout, we denote by (T, Σ, μ) a positive finite measure space over Σ, a sigma algebra of subsets of T and let $\Sigma^+ = \{S \in \Sigma: \mu(S) > 0\}$. A set property P is said to be local in Σ^+ if for every S in Σ^+, there is a $S_1 \subset S$, S_1 in Σ^+ such that S_1 has property P.

DEFINITION 1. For a set $K \subset E$, we define $d(K) =$ closure $\{\sum_{i \in I} \alpha_i x_i : \{x_i\} \subset K,\ \Sigma |\alpha_i| \leqq 1,$ I finite$\}$. We call $d(K)$ the closed disked hull of K.

DEFINITION 2. If $m: \Sigma \to E$ is a countably additive vector measure, then the average range of m on S in Σ^+ is $A_S(m) = \{m(S')/\mu(S'): S' \subset S,\ S' \in \Sigma^+\}$.

DEFINITION 3. If $m: \Sigma \to E$ is a countably additive vector measure, then m has locally small average range if for every $\rho \in \{\rho\}$, $\varepsilon > 0$ and $S \in \Sigma^+$, there is an $S_1 \subset S$, $S_1 \in \Sigma^+$ such that ρ-diam $(A_{S_1}(m))$ is less than ε, i.e.; $\rho(x - y) < \varepsilon$, $\forall x, y \in A_{S_1}(m)$.

DEFINITION 4. A set K in E is dentable if and only if for every $\rho \in \{\rho\}$ and $\varepsilon > 0$, there is an $x \in K$ such that $x \notin \bar{c}(K \backslash S_\rho^\varepsilon(x))$. Here $\bar{c}(\cdot)$ denotes the closed convex hull and $S_\varepsilon^\rho(x) = \{y \in E: \rho(x - y) < \varepsilon\}$.

DEFINITION 5. Suppose E is a QSMC space. A function $f: T \to E$ is integrable if and only if (i) there is a sequence of simple functions $\{f_n\}$ which converge to f almost uniformly on T; (ii) $\left\{\int_S f_n d\mu\right\}_n$ converges for each $S \in \Sigma$. Then we define $\int_S f d\mu = \lim_n \int_S f_n d\mu$, for every $S \in \Sigma$. Note that for Banach spaces this is the Dunford second integral or equivalently the Pettis integral for strongly measurable functions.

We note that many of the following results are ture for more general locally convex spaces with a slight modification of the above definition of integrable functions, but to preserve the unity and purpose of the paper we only indicate the results for QSMC spaces.

THEOREM. [RNT] *Suppose E is a* QSMC *space and (T, Σ, μ) is a positive finite measure space. If $m: \Sigma \to E$ is a countably additive vector measure, $m \ll \mu$, and m has locally bounded average range, then the following are equivalent*:

(1) *There is an integrable function $f: T \to E$ such that $m(S) = \int_S f d\mu$, $\forall S \in \Sigma$.*

(2) *m has locally relatively compact average range.*

(3) *m has locally relatively weakly compact average range.*

(4) *m has locally dentable average range.*

(5) *m has locally small average range.*

Proof. We show (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2) and where the proof is only a slight modification of the Banach space case the proof is omitted.

(1) $\Rightarrow$ (2): The proof is exactly that given by Rieffel [19, Page 472] using an essential range argument.

(2) $\Rightarrow$ (1): Using the exhaustion principle of Maynard [14, Page 450] we decompose T into a countable disjoint collection of sets $\{T_i\} \subset \Sigma^+$ such that $\forall i$, $A_{T_i}(m)$ is relatively compact. Let $D_n = \mu(T_n) \cdot d(A_{T_n}(m))$ where $d(A_{T_n}(m))$ is the closed disked hull of $A_{T_n}(m)$. Hence the range of $m_n \equiv m|_{T_n}: T_n \cap \Sigma \to E$ is contained in D_n. Now by the QSMC property there is a collection of subsets of E, $\{\hat{D}_n\}$, $D_n \subset \hat{D}_n$, which are closed bounded and disked and such that the topologies of E and $E_{\hat{D}_n}$ agree on D_n. Thus m_n is a countable additive vector measure into $E_{\hat{D}_n}$ with relatively compact average range and hence has a Bochner integrable density, $f_n: T_n \to E_{\hat{D}_n}$. By a standard piecing argument [14, Page 455; 3, Page 14] we obtain a density for m.

(2) $\Rightarrow$ (3): If $K \subset E$ is relatively compact, then K is relatively weakly compact.

(3) $\Rightarrow$ (4): Claim: If $K \subset E$ is relatively weakly compact, then K is dentable.

Proof of the claim: If $K \subset E$ is relatively weakly compact, then by Krein's theorem, we have $d(K)$ is weakly compact. Take $\widehat{d(K)} \supset d(K)$ by the QSMC property so that the topologies of E and $E_{\widehat{d(K)}}$ agree on $d(K)$; then using an argument of Grothendeick [7, Page 61] we have that K is relatively weakly compact in $E_{\widehat{d(K)}}$. This implies that K is dentable in $E_{\widehat{d(K)}}$ and because the topologies and uniform structures agree on $d(K)$, it is easily checked that K is dentable in E.

(4) $\Rightarrow$ (5): Take $S \in \Sigma^+$, $\rho \in \{\rho\}$ and $\varepsilon > 0$, then by (4) there is an $S_1 \subset S$, $S_1 \in \Sigma^+$, such that $A_{S_1}(m)$ is dentable and bounded in E. Define $K = \mu(S_1) \cdot d(A_{S_1}(m))$ and take $\hat{K} \supset K$ by the QSMC property as above; then $m_1 \equiv m|_{S_1}: \Sigma \cap S_1 \to E_{\hat{K}}$ is a countably additive vector measure with range contained in K. Now m_1 has locally dentable average range in $E_{\hat{K}}$ and hence it has locally small average range in $E_{\hat{K}}$ by [14]. By our selection of $\hat{K}$ we have that $m|_{S_1}$ has locally small average range in E.

(5) $\Rightarrow$ (2): This follows just as in (4) $\Rightarrow$ (5).

DEFINITION 6. If E is a QSMC space, then E has the Radon-Nikodym property [RNP] if and only for every positive finite measure space (T, Σ, μ) and any $m: \Sigma \to E$, m a countably additive vector measure with locally bounded average range such that $m \ll \mu$, there is an integrable $f: T \to E$ such that $m(S) = \int_S f d\mu$, $S \in \Sigma$.

In what follows we give several characterizations of QSMC spaces possessing the RNP. The next theorem was proved for Banach spaces by Davis and Phelps [5] and Huff [9].

THEOREM. *If E is a* QSMC *space, the following are equivalent:*

(1) *E has the* RNP.

(2) *Every bounded set in E is dentable.*

(3) *Every bounded closed convex set in E is dentable.*

Proof.

(2) $\Rightarrow$ (1): Follows from the RNT.

(3) $\Leftrightarrow$ (2): This follows from the extension of a result of Reiffel [18] to QSMC spaces, if $K \subset E$ and $\bar{c}(K)$ is dentable, then K is dentable.

(1) $\Rightarrow$ (2): Assume C' is a nondentable bounded set in E, then $C \equiv \bar{c}(C')$ is nondentable. Let K be the disked hull, $d(C)$, of C and take $\hat{K} \supset K$ by the QSMC property. Then C is not dentable in $E_{\hat{K}}$, thus, as in Huff [9], there exists a positive finite measure space (T, Σ, μ) and a countably additive vector measure $m: \Sigma \to E_{\hat{K}}$ with finite total variation such that $A_T(m) \subset C$ and such that m has no density in $E_{\hat{K}}$. Then $m: \Sigma \to E$ is a countably additive vector measure and by assumption has a density in E. Thus we can assume m has locally relatively compact average range in $C \subset E$ and hence in E_K, but this contradicts the RNT for Banach spaces.

Next we give an extension of the strongly exposed point characterization of Banach spaces with the RNP given in [17] and show that for some more general spaces the result must be modified slightly.

DEFINITION 7. If $C \subset E$, then $(E^*)_C$ is the collection of linear functionals on E whose restrictions to C are continuous.

DEFINITION 8. If $C \subset E$, then $x \in C$ is an $(E^*)_C$-strongly exposed point of C if there is an $x^* \in (E^*)_C$ such that $x^*(x) > x^*(C \backslash \{x\})$ and $x^*(x_\alpha) \to x^*(x)$, where $\{x_\alpha\}$ is a net in C, implies $x_\alpha \to x$ in E.

The next result was obtained by Saab [20] for Fréchet spaces where the integral involved was a Bochner type integral. Our proof is accomplished for QSMC spaces by modifying only slightly the proofs of lemmas given by Phelps [17], so the proofs of the lemmas are omitted. We will state the lemmas as the statements must be altered slightly as well. We also point out that the first three lemmas hold for general quasicomplete locally convex topological vector spaces and the SMC property is mainly used thereafter to insure we can take a sequence of nested slices which intersect in a point.

THEOREM. *E has the RNP $\Leftrightarrow$ Every closed bounded convex set, C, is the closed convex hull of its $(E^*)_C$ strongly exposed points.*

DEFINITION 9. If $K \subset E$, $x' \in E'$, $\alpha > 0$ then define $S(x', \alpha, K) = \{x \in K: x'(x) \geqq \sup_{y \in K} x'(y) - \alpha\}$. Such a set is called a slice of K.

LEMMA. *A subset K of E is dentable if and only if for every $\rho \in \{\rho\}$ and $\varepsilon > 0$, there is a slice $S(x', \alpha, K)$ with $x' \in E$, $\alpha > 0$ such that the ρ-diam of $S(x', \alpha, K)$, denoted $\delta_\rho(S(x', \alpha, K))$, is less than ε $(\delta_\rho(S(x', \alpha, K)) < \varepsilon)$.*

LEMMA. *If $T: E \rightarrow E$ is an isomorphism of E and $S(x', \alpha, C)$ is a slice of C such that for some $\rho \in \{\rho\}$ and $\varepsilon > 0$, we have $\delta_\rho(S(x', \alpha, C)) < \varepsilon$, then $T(S(x', \alpha, C))$ is a slice of $T(C)$ and for $x \in S(x', \alpha, C)$ we have $T(S(x', \alpha, C)) \subset T(S_\varepsilon^\rho(x))$.*

LEMMA. *Suppose every closed bounded set in E is dentable, C is closed bounded convex, $x' \in E'$ and $C \backslash N_{x'} \neq \emptyset$, where $N_{x'} = \{x: x'(x) = 0\}$. Then there is a $y' \in E'$ and $\beta > 0$ such that $\delta_\rho(S(y', \beta, C)) < \varepsilon$ and $S(y', \beta, C) \cap N_{x'} = \emptyset$.*

LEMMA. *Suppose every closed bounded set in E is dentable and C is closed bounded and convex. Then C is the closed convex hull of its E'-denting points.*

A point $x \in C$ is a E'-denting point of C if for each $\varepsilon > 0$ and $\rho \in \{\rho\}$ there is a slice $S(x', \alpha, C)$ of ρ-diameter less than ε which contains x.

LEMMA. *Suppose every closed bounded set in E is dentable and $S(x', \alpha, C)$ is a slice of the closed bounded convex set C, $\|x'\|_K = 1$ $(K = 2d(C)$, $\|x'\|_K = \sup_{k \in K} |x'(k)|)$. Let $1 > \varepsilon > 0$, $\rho \in \{\rho\}$ and $\delta > 0$, then there is a slice $S(y', \beta, C)$ such that $\delta_\rho(S(y', \beta, C)) < \delta$, $\|x' - y'\|_K < \varepsilon$ and $\|x'\|_K = \|y'\|_K = 1$, $S(y', \beta, C) \subset S(x', \alpha, C)$.*

REMARK. Here we take $C_1 = \bar{c}(S(x', \alpha, C) \cup [\lambda K \cap N_{x'}])$, $\lambda > 2/\varepsilon$, and proceed as in Phelps [17]. We use K instead of the unit ball in the Banach space case.

LEMMA. *Let C be a closed bounded convex set in E. Suppose for every slice $S(x', \alpha, C)$, $\rho \in \{\rho\}$, $\delta > 0$ and $\varepsilon > 0$, there is a slice $S(y', \beta, C)$ such that $\delta_\rho(S(y', \beta, C)) < \delta$, $S(y', \beta, C) \subset S(x', \alpha, C)$ and $\|x' - y'\|_K < \varepsilon$ where $K = 2d(C)$. Then every slice $S(x', \alpha, C)$ of C contains an $(E^*)_K$ strongly exposed point.*

REMARK. (1) The above lemmas show that the RNP implies that every closed bounded convex set C is the closed convex hull of its $(E^*)_C$-strongly exposed points. The other direction is trivial since $(E^*)_C$-strongly exposed points are denting points.

(2) The functionals giving the slices in the proof of the last lemma are only converging to some function in sup norm on K. This function will be linear in K and we can extend it to a function which is continuous on K and linear on E, i.e.; in $(E^*)_K$.

(3) Even in Banach spaces there are linear functionals which are continuous on a closed bounded convex set and which are not continuous on the disked hull of the set. Hence our result is actually stronger than stated above since the functionals we obtain are continuous on the disked hull of the sets. In fact they can be chosen to be continuous on any preassigned boundeds et.

EXAMPLE. Let $\{e_n\}$ be the standard unit vector basis for c_0 and define $\delta_n = \sum_{i=1}^n e_i$. Let $K = \bar{c}(\{\delta_n\}_{n=1}^\infty) \subset c_0$, then we have $K = \{\{\alpha_i\}_1^\infty \in C_0 \colon \alpha_1 = 1 \text{ and } \alpha_i \geqq \alpha_{i+1}\}$. Now define $f(\{\alpha_i\}_1^\infty) = \sum_{i=1}^\infty (-1)^i \alpha_i$ for all $\{\alpha_i\} \in K$ and extend linearly to c_0. Then f is continuous on K.

Claim. f is not continuous on $d(K)$.

Proof. Let $x_n = \sum_{k=1}^n ((-1)^k/2n)\delta_k = (\underbrace{0, 1/2n, 0, 1/2n, 0, \cdots, 1/2n,}_{2n}$ $0, 0, \cdots)$ then $x_n \in d(K)$ and $x_n \to 0$ yet $f(x_n) = 1/2n - 0 + 1/2n - 0 + \cdots + 1/2n = 1/2$.

We now show that we cannot necessarily expect continuity of the linear functionals on the whole space.

THEOREM. *Let $\{B_i\}$ be sequence of Banach spaces and $E = \times_i B_i$, then E has the* RNP *if and only if B_i has the* RNP *for all i.*

Proof. ($\Rightarrow$) If there is a B_i which doesn't have the RNP, then take $K_i \subset B_i$ closed bounded convex and not dentable in B_i. Then define $K = K_i \times \times_{j \neq i} \{0\}$ and it is easily verified that K is not dentable in E.

($\Leftarrow$) If (T, Σ, μ) is a countably additive vector measure with locally bounded average range, then $\pi_i \circ m \colon \Sigma \to B_i$ is a countably additive vector measure in B_i. $\pi_i \circ m$ has locally compact average range by the RNP and RNT in B_i. Take a set $S \in \Sigma^+$ and construct a sequence $\{S_i\} \subset \Sigma^+$, $S_{i+1} \subset S_i \subset S$, for all i such that $A_{S_i}(\pi_i(m))$ is relatively compact in B_i and such that $S' = \bigcap_i S_i \in \Sigma^+$. This can be done as in Maynard [14, Page 457]. Then we have $A_{S'}(m)$ is relatively compact in E. Hence apply RNT for QSMC spaces to obtain density for m.

REMARK. Peck [16, Page 275] showed that if $\{B_i\}_{i=1}^{\infty}$ is a collection of nonreflexive Banach spaces and $E = X_i B_i$, then there is a $C \subset E$ closed bounded convex with no E'-support points. A point $x \in C$ is an E'-support point of C if there is an $x' \in E'$ such that $x'(x) = \sup_{c \in C} x'(c)$. Thus if $B_i = l^1$ for all i, we see that E has the RNP; thus there is a closed bounded convex set which is the closed convex hull of its $(E^*)_C$-strongly exposed points and yet has no E'-support points.

COROLLARY. *Assume F is a Fréchet space, let F_n be the completion of $F/\ker \rho_n$, and assume F_n has the* RNP *for all n, then F has the* RNP.

COROLLARY. *Assume E is* QSMC *space. If for each bounded disk B, there is a $\hat{B} \supset B$ bounded closed disked set such that $E_{\hat{B}}$ has the* RNP, *then E has the* RNP.

These corollaries follow immediately from the above theorems.

II. In this section we extend a few results of Bishop and Phelps [1] and Davis, Figiel, Johnson, Pelczynski [4] to QSMC spaces. Some analogous results along these lines for Fréchet spaces are found in Peck [16]. The proofs here are straightforward consequences of the corresponding results in Banach spaces.

THEOREM. *Assume E is a* QSMC *space, C is a closed convex set in E, and C_1 is any other closed bounded convex set containing C. Then the $(E^*)_{C_1}$-support points are dense in the boundary of C.* (Here $x \in C$ is an $(E^*)_{C_1}$-support point of C if there is an $x^* \in (E^*)_{C_1}$ such that $x^*(x) = \sup_{c \in C} x^*(C)$. We also point out that if E is not a normed space then the support points are dense in the whole set C and since the Banach space theorem is already known we assume in the following proof that E is not a normed space.)

Proof. Take any $z \in C$, $\rho \in \{\rho\}$ and $\delta > 0$, then we want $z_1 \in S_\delta^\rho(z) \cap C$ and $x^* \in (E^*)_{C_1}$ such that $x^*(z_1) = \sup x^*(C)$. Take

$\omega \in S^{\rho}_{\delta}(z)\backslash C$, let $K = d(C_1 \cup \{\omega\})$ and take $\hat{K} \supset K$ by the QSMC property. Now there is some u in C on the line joining z and ω which is an $E_{\hat{K}}$ boundary point of C. Take $\varepsilon > 0$ such that for all $x, y \in K$ such that $x - y \in S^{\hat{K}}_{\varepsilon}(0)$ implies $x - y \in S^{\rho}_{\delta/2}(0)$. Then by [1, Page 29] there exists a z_1 in the boundary of C in $E_{\hat{K}}$ intersect the $S^{\hat{K}}_{\varepsilon}(u)$ and an $x' \in (E_{\hat{K}})$ such that $x'(z_1) = \sup x'(C)$. Now z_1 is contained in $C \cap S^{\rho}_{\delta}(z)$ and x' can be extended linearly to all of E, say to x^*. Then $x^* \in (E^*)_{C_1}$ and $x^*(z_1) = \sup x^*(C)$.

THEOREM. *Suppose C_1, $C_2 \subset E$, C_1 closed bounded convex, C_2 is bounded $D = d(C_1 \cup C_2)$, and D_1 is any closed bounded disked set containing D. If there is an $x^* \in (E^*)_{D_1}$ such that $\sup x^*(C_1) < \inf x^*(C_2)$, then for every $\varepsilon > 0$, there is a $y^* \in (E^*)_{D_1}$ and $y \in C_1$ such that $y^*(y) = \sup y^*(C_1) < \inf y^*(C_2)$ and the $\sup_{x \in D_1} |\langle x^* - y^*, x\rangle| < \varepsilon$.*

Proof. Choose real numbers a, b, c, d such that $\sup x^*(C_1) < a < b < c < d = \inf x^*(C_2)$ and take $\varepsilon < \inf [b - a/2, d - c/2]$, then by Horvath [8, Page 248] there is an $x' \in E'$ such that $\sup_{x \in D_1} |\langle x^* - x', x\rangle| < \varepsilon/6$, hence $\sup x'(C_1) < b < c < \inf x'(C_2)$. Take $\hat{D} \supset D_1$ by the QSMC property. Since $x' \in E'$, it is also in $(E_{\hat{D}})'$; thus by Bishop and Phelps [1, Page 30] there is a $y' \in (E_{\hat{D}})'$ and $y \in C_1$ such that $y'(y) = \sup y'(C_1) < \inf y'(C_2)$ and $\sup_{x \in D_1} |\langle y' - x', x\rangle| < \varepsilon/3$. We now have $|\langle y' - x^*, x\rangle| \leqq |\langle y' - x', x\rangle| + |\langle x^* - x', x\rangle| < \varepsilon/2$, for all $x \in D_1$. Now extend y' to be linear on E, call the extension y^*. Then $y^* \in (E^*)_{D_1}$ $y^*(y) = \sup y^*(C_1) < \inf y^*(C_2)$ and $\sup_{x \in D_1} |\langle y^* - x^*, x\rangle| < \varepsilon$.

THEOREM. *If $K \subset E$ is a closed bounded disked set and $K_1 \supset K$ is any closed bounded disked set, then for every $x^* \in (E^*)_{K_1}$ and $\varepsilon > 0$, there exists a $y^* \in (E^*)_{K_1}$ such that y^* is a support functional of K and $\sup_{x \in K_1} |\langle x^* - y^*, x\rangle| \leqq \varepsilon$.*

Proof. Take $x^* \in (E^*)_{K_1}$ and $\hat{K} \supset K_1$ by QSMC property, then $x^* \in (E^*_{\hat{K}})_{K_1}$ (i.e.: x^* is linear on $E_{\hat{K}}$ and continuous on K_1 since the topologies of E and $E_{\hat{K}}$ agree on K_1). Now K_1 is symmetric, hence by Horvath [8, Page 248] there exists a $z' \in (E_{\hat{K}})'$ such that $\sup_{x \in K_1} |\langle x^* - z', x\rangle| < \varepsilon/2$. Therefore we can apply a result of Bishop and Phelps [1, Page 31] to obtain a $y' \in (E_{\hat{K}})'$ and a $y \in K$ such that $y'(y) = \sup y'(K)$ and $\sup_{x \in K_1} |\langle z' - y', x\rangle| < \varepsilon/2$. Now extend y' to $y^* \in (E^*)_{K_1}$ and we have that $\sup_{x \in K_1} |\langle x^* - y^*, x\rangle| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ and y^* is a support functional for K.

The next two results are extensions of results from [4] and require some additional notation and a technical result from their paper which we state without proof. Assume E is a QSMC space and $W \subset E$ is a convex, symmetric bounded set. Let $K = d(W)$ and take $\hat{K} \supset K$ by the QSMC property. For every n, define $U_n = 2^n W + 2^{-n}\hat{K}$. Then U_n defines an equivalent norm on $E_{\hat{K}}$, call the norm $\|\cdot\|_n$. For each $x \in E_{\hat{K}}$ define $\|x\| = (\sum_1^\infty \|x\|_n^2)^{1/2}$. Let $Y = \{x \in E_{\hat{K}}: \|x\| < \infty\}$ and denote by C the set of $x \in Y$ such that $\|x\| \leqq 1$ and by $j: Y \to E_{\hat{K}}$ the identity embedding.

LEMMA. (1) $W \subseteqq C$.

(2) $(Y, \|\cdot\|)$ *is a Banach space and* j *is continuous.*

(3) $j''(Y'') \to E''_{\hat{K}}$ *is* $1-1$ *and* $(j'')^{-1}(E_{\hat{K}}) = Y$.

(4) Y *is reflexive if and only if* W *is relatively weakly compact. The proof is found in* [4].

THEOREM. *If* L *is a locally convex topological vector space,* E *is a* QSMC *space;* $T: L \to E$ *is a weakly compact operator, then there exists a reflexive Banach space* R *and continuous linear operators* T_1 *and* T_2 *such that* $T = T_2 \circ T_1$, $T_1: L \to R$, $T_2: R \to E$.

Proof. Take U an absolutely convex balanced neighborhood of 0 in L such that $T(U)$ is relatively weakly compact in E. Let $K = \overline{T(U)}$ and take $\hat{K} \supset K$ by the QSMC property. Denote by i the continuous embedding of $E_{\hat{K}}$ into E. Now $i^{-1} \circ T: L \to E_{\hat{K}}$ is a continuous linear transformation since a net convergent to zero in L is eventually in U, and the topologies of E and $E_{\hat{K}}$ agree on K. Thus the map $i^{-1} \circ T: L \to E_{\hat{K}}$ is a weakly compact operator into a Banach space, which implies by [4, Page 314] that there exists a reflexive Banach space R and operator F_1, F_2 such that $i^{-1} \circ T = F_2 \circ F_1$ where $F_1: L \to R$, $F_2: R \to E_{\hat{K}}$. Then let $T_1 = F_1$, $T_2 = i \circ F_2$ and we have the appropriate factorization.

THEOREM. *If* $K \subset E$ *is a weakly compact set, then* K *is affinely homeomorphic to a subset of a reflexive Banach space.*

Proof. Let $K_1 = d(K)$ and take $\hat{K}_1 \supset K_1$ by the QSMC property. Then by the lemma, Y is reflexive. Therefore $K' = j^{-1}(K_1)$ is weakly compact, $j|_{K'}$ is the homeomorphism for K between $E_{\hat{K}_1}$ and Y and we need only compose this map with i^{-1} of the preceding proof.

REMARKS. It follows from Grothendieck [6, Ex. 2, p. 61] that weakly compact subsets of QSMC spaces are Eberlein compacts (i.e., a compact hausdorff space which is homeomorphic to a weakly compact subset of a Banach space); thus one can obtain results along the lines of Lindenstrauss [12] such as: If $K \subset E$ is a weakly compact subset of a QSMC space then K is affinely homeomorphic to a weakly compact subset of a $c_0(\Gamma)$ space for some Γ.

In closing we shall discuss some examples and nonexamples of QSMC spaces and the RNP.

(1) An obvious collection of spaces one would hope to be QSMC spaces, in light of a result mentioned earlier, are not. Take any separable infinite dimensional Banach space X and consider $E=(X', \sigma(X', X))$. Then E has a fundamental sequence of bounded sets and the closed bounded sets are metrizable and complete. Thus if E has the MC property Grothendieck's result would say that E has the QSMC property. To see that E is not a QSMC space, let $U_{X'}$ be the closed unit ball in X'. Then $U_{X'}$ is closed bounded (in fact compact) in E. If K is a closed bounded disked set which contains $U_{X'}$, then there is a $\lambda>0$ such that $U_{X'}\subset K\subset\lambda U_{X'}$ and hence E_K is isomorphic to X'. But $U_{X'}$ is compact in E hence the topologies of E_K and E cannot agree on $U_{X'}$.

It is an easy matter to define our integral and RNP for quasi-complete locally convex spaces with bounded sets metrizable. In which case it follows immediately that dual spaces of separable Banach spaces with the weak-* topology have the RNP (since bounded sets are relatively compact).

(2) If X is a separable Banach space, consider $E=B_S(X)$, the space of bounded operators on X with the strong operator topology. E is a quasicomplete locally convex space with a fundamental sequence of bounded sets and bounded sets are metrizable. Since subspaces of spaces with the SMC property must have the SMC property, we shall show that E has a closed subspace without the SMC property. Take $x_1\in X$, $\|x_1\|=1$ and for every $x'\in X'$ define $T_{x'}: X\to X$ by $T_{x'}(x)=x'(x)x_1$. Let $S=\{T_{x'}: x'\in X'\}$, then S is a closed subspace of E (in fact S is a closed subspace of the compact operators on X with the strong operator topology). By the argument given in (1) above we see that S doesn't have the SMC property.

(3) One can see immediately from the definition [7, Page 176] that if E is a quasi-barreled locally convex topological vector space, then E is quasi-normable if and only if the strong dual of E is a SMC space. We note that if E is barreled then the strong dual is not necessary.

(4) A complete (DF) space is bornological if and only if it is an (LF) space [7, Page 172]. Now (DF) spaces have a fundamental sequence of bounded sets and hence bornological (DF) spaces are SMC spaces.

(5) As indicated earlier arbitrary topological direct sums of

spaces with the SMC property have the SMC property. Hence it is easy to see that a topological direct sum of a collection of Banach spaces with the RNP has the RNP.

(6) Many spaces of distributions and other inductive limit spaces are SMC spaces and are also Montel spaces. These spaces by the RNT all possess the RNP.

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