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In this paper we show that there exist functions $f \in C[-1, +1]$ with all $(r+1)$ -st order divided differences uniformly bounded away from zero for r fixed ($f[x_0, x_1, \dots, x_{r+1}] \geq \delta > 0$ for fixed δ and all sets $x_0 < \dots < x_{r+1}$ in $[-1, +1]$), for which infinitely many of the polynomials of best approximation to f do not have nonnegative $(r+1)$ -st derivatives on $[-1, +1]$.

1. Introduction. In [6]-[10] there appear many examples of functions f in $C[a, b]$ with nonnegative $(r+1)$ -st divided differences there for which infinitely many of the polynomials of best approximation to f fail to have nonnegative $(r+1)$ st derivatives. None of these examples has the $(r+1)$ st divided differences uniformly bounded away from zero. In [11] Roulier shows that if $f \in C^{2r+2}[-1, +1]$ and if $f^{(r+1)}(x) \geq \delta > 0$ on $[-1, 1]$ then for n sufficiently large the polynomial of best approximation of degree less than or equal to n has a positive $(r+1)$ st derivative on $[-1, +1]$.

On the other hand for the case $r = 0$ Roulier in [12] shows that first divided differences of f uniformly bounded away from zero is not sufficient to insure that for n sufficiently large the polynomial of best approximation to f is increasing on $[-1, 1]$.

In this paper we extend the results of [12] to the case when $r \geq 0$. The proofs are similar to those in [12] but make use of higher order divided differences and their properties.

2. Notation and preliminary concepts. For $n = 0, 1, 2, \dots$ define H_n to be the set of all algebraic polynomials of degree less than or equal to n . For $f \in C[a, b]$, let

$$\|f\| = \sup \{|f(x)|: a \leq x \leq b\}.$$

We define the degree of approximation to f to be

$$E_n(f) = \inf \{\|f - p\|: p \in H_n\},$$

$n = 0, 1, 2, \dots$. It is well-known that there is a unique $p_n \in H_n$ for which $\|f - p_n\| = E_n(f)$. This p_n is called the *polynomial of best approximation to f on $[a, b]$ from H_n* . Unless specifically stated otherwise we will restrict ourselves to the interval $[-1, +1]$.

Define C^* to be the class of continuous 2π -periodic functions and H_n^* the trigonometric polynomials of degree n or less. Then

$E_n^*(f)$ is defined for $f \in C^*$ as the degree of approximation to f by trigonometric polynomials from H_n^* . That is,

$$E_n^*(f) = \inf \{ \|f - T\| : T \in H_n^* \}$$

where

$$\|f\|^* = \sup \{ |f(x)| : -\pi \leq x \leq \pi \}.$$

If $I = [-1, 1]$ or $I = [-\pi, \pi]$ and $f \in C[-1, +1]$ or $f \in C^*$ we define the r -th modulus of smoothness $\omega_r(f, h) = \sup \{ |\Delta_t^r f(x)| : |t| \leq h \text{ and } rh \leq |I| \}$, where $\Delta_t^1 f(x) = f(x+t) - f(x)$ and $\Delta_t^i f(x) = \Delta_t^i(\Delta_t^{r-1} f(x))$, and $|I|$ is the length of I .

If $r = 1$ then $\omega_r(f, h)$ is called the modulus of continuity of f and is written $\omega(f, h)$.

Estimates for $E_n(f)$ are intimately related to $\omega_r(f, h)$ by the theorems of D. Jackson. These theorems are well-known and will not be given here. See [5].

As in [3] let $f[x_0, \dots, x_r]$ denote the r th order divided difference of f . It is well-known that if $f \in C^r[x_0, x_r]$ and $x_0 < x_1 < \dots < x_r$ then there is ξ in (x_0, x_r) for which

$$f^{(r)}(\xi) = r! f[x_0, \dots, x_r].$$

It is also well-known that if all $(r+1)$ st order divided differences of f are nonnegative in $[-1, +1]$ then $f \in C^{r-1}(-1, +1)$. See [2].

In the following sections, p_n will always denote the polynomial from H_n of best approximation to f on the appropriate interval.

3. The main theorems. The following theorems treat the situations where all $(r+1)$ st order divided differences of f are bounded away from zero on $[-1, +1]$ and $f \in C^{r-1}[-1, +1]$ or $f \in C^r[-1, +1]$. The first two theorems and their corollaries show that for all functions with nonnegative $(r+1)$ st order divided differences for which $E_n(f)$ does not get small too fast there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \geq 0$ on $[-1, +1]$. The last two theorems show that this will also occur for some functions with $(r+1)$ st order divided differences bounded away from zero even if $E_n(f)$ does get small faster than allowed in the first two theorems.

THEOREM 3.1. *Let $f \in C[-1, 1]$ have bounded r th order divided differences (if $f \in C^r[-1, 1]$, then this happens) and nonnegative $(r+1)$ st order divided differences on $[-1, +1]$. Assume that there is no $C > 0$ for which*

$$E_n(f) \leq C/(n+1)^{r+1} \text{ for } n = 0, 1, \dots$$

Then there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \geq 0$ on $[-1, +1]$.

COROLLARY 3.1(a). Let $f \in C^r[-1, +1]$ and assume that f has nonnegative $(r+1)$ st order divided differences on $[-1, +1]$. Define $g(t) = f(\cos t)$. Assume that

$$(1) \quad \limsup_{k \rightarrow \infty} k^{r+1} \omega_{r+1}\left(g, \frac{1}{k}\right) / \log k = +\infty.$$

Then there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \geq 0$ on $[-1, +1]$.

COROLLARY 3.1(b). If f has nonnegative $(r+1)$ st order divided differences on $(-1-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$ and if there is no $C > 0$ for which

$$E_n(f) \leq C/(n+1)^{r+1} \quad \text{for } n = 0, 1, \dots$$

then there are infinitely many n for which we do not have

$$p_n^{(r+1)}(x) \geq 0 \quad \text{on } [-1, +1].$$

THEOREM 3.2. Let $f \in C^{r-1}[-1, +1]$ and assume that f has nonnegative $(r+1)$ st order divided differences. Assume that there is no $C > 0$ for which

$$E_n(f) \leq C/(n+1)^r \quad \text{for } n = 0, 1, \dots$$

Then there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \geq 0$ on $[-1, +1]$.

COROLLARY 3.2. Let $f \in C^{r-1}[-1, +1]$ and assume that f has nonnegative $(r+1)$ st order divided differences. Define

$$g(t) = f(\cos t).$$

Assume that

$$(2) \quad \limsup_{k \rightarrow \infty} k^r \omega_r\left(g, \frac{1}{k}\right) / \log k = +\infty.$$

Then there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \geq 0$ on $[-1, +1]$.

THEOREM 3.3. For each integer $r \geq 0$ and modulus of continuity ω there exists $f \in C^r[-1, +1]$ with

$$(3) \quad f[x_0, \dots, x_{r+1}] \geq \delta > 0 \text{ for all } x_0 < \dots < x_{r+1}$$

in $[-1, +1]$ and with

$$(4) \quad \omega(h) \leq \omega(f^{(r)}, h) \leq K\omega(h)$$

and yet there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \geq 0$.

THEOREM 3.4. *For each integer $r \geq 1$ and modulus of continuity ω there exists $f \in C^{r-1}[-1, +1]$ with*

$$(5) \quad f[x_0, \dots, x_{r+1}] \geq \delta > 0 \text{ for all } x_0 < \dots < x_{r+1}$$

in $[-1, +1]$ and with

$$\omega(h) \leq \omega(f^{(r-1)}, h) \leq K\omega(h)$$

and yet there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \geq 0$.

4. Proofs of the main theorems. We first state some known lemmas. The first lemma is due to Steckin [13] and is found in [5] page 59.

LEMMA 4.1. *There exist constants $M_p, p = 1, 2, \dots$, such that for each $f \in C^*$*

$$(6) \quad \omega_p(f, h) \leq M_p h^p \sum_{0 \leq n \leq h^{-1}} (n+1)^{p-1} E_n^*(f).$$

LEMMA 4.2. *Let $f \in C[-1, +1]$ and define $g \in C^*$ by $g(t) = f(\cos t)$. If*

$$(7) \quad \limsup_{k \rightarrow \infty} k^{r+1} \omega_{r+1}\left(g, \frac{1}{k}\right) / \log k = +\infty,$$

then there does not exist $M > 0$ for which

$$E_n(f) \leq M/(n+1)^{r+1}, \text{ for } n = 0, 1, 2, \dots.$$

Proof. Assume such a constant M exists. Then $E_n^*(g) = E_n(f) \leq M/(n+1)^{r+1}$ for $n = 0, 1, \dots$. Now use Lemma 4.1 with $h = 1/N$. This gives

$$\omega_{r+1}(g, 1/N) \leq \frac{A_r}{N^{r+1}} \sum_{n=0}^N \frac{1}{n+1} \leq \frac{K_r \log N}{N^{r+1}}.$$

Hence

$$N^{r+1}\omega_{r+1}(g, 1/N)/\log N \leq K_r .$$

This is a contradiction.

The next lemma is stated in [12] and is a simple consequence of a theorem of Kadec [4].

LEMMA 4.3. *Let $f \in C[-1, +1]$ and for each $n = 0, 1, 2, \dots$ let $x_{0,n} < \dots < x_{n+1,n}$ be a Chebyshev alternation for f .*

Let $\delta_n = \max_{0 \leq k \leq n+1} |x_{k,n} - \cos(k\pi/(n+1))|$. Then there is a sequence $\{n_j\}_{j=0}^\infty$ of positive integers for which

$$\lim_{j \rightarrow \infty} \delta_{n_j} = 0 .$$

The next lemma is found in [5] page 45.

LEMMA 4.4. *Let ω be any modulus of continuity. Then there is a concave modulus of continuity $\bar{\omega}$ with the same domain of definition as ω for which*

$$(8) \quad \frac{1}{2}\bar{\omega}(h) \leq \omega(h) \leq \bar{\omega}(h) .$$

The next lemma is well-known. We first define for $r = 1, 2, \dots$

$$(9) \quad x_+^r = \begin{cases} 0 & \text{for } x \leq 0 \\ x^r & \text{for } x > 0 . \end{cases}$$

LEMMA 4.5. *There is a constant $C_r > 0$ for which*

$$(10) \quad E_n(x_+^r) \geq C_r/(n+1)^r .$$

Proof. This is an easy consequence of a theorem of S.N. Bernstein [1].

LEMMA 4.6. *If there are m non-overlapping intervals I_1, \dots, I_m contained in $[a, b]$ each with length $l, i = 1, \dots, m$ respectively, then for each positive integer l there must be at least $[m(l-1)/l]$ intervals I_i for which $l_i \leq (l(b-a)/m)$.*

Proof. The proof of this is elementary and is omitted.

LEMMA 4.7. *Let $m \geq 2$ be an integer and let $z_0 < z_1 < \dots < z_m$ be given. Define $h[z_0, \dots, z_m] = \sum_{j=0}^m \prod_{\substack{k=0 \\ k \neq j}}^m |z_j - z_k|^{-1}$. Then*

$$(11) \quad (z_m - z_0)h[z_0, \dots, z_m] \geq (m+1)(z_m - z_0)^{-m+1}$$

$$(12) \quad (z_m - z_0)(z_m - z_1)h[z_0, \dots, z_m] \geq (z_m - z_0)^{-m+2}$$

$$(13) \quad (z_m - z_0)(z_{m-1} - z_0)h[z_0, \dots, z_m] \geq (z_m - z_0)^{-m+2}.$$

Proof. The proof of (11) is easy. The proofs of (12) and (13) are obtained by considering the terms $j = 1$ and $j = 0$ in the sum respectively.

LEMMA 4.8. *If $f[x_0, \dots, x_{r+1}] \geq 0$ for all $x_0 < \dots < x_{r+1}$ in $[-1 - \epsilon, 1 + \epsilon]$ for some $\epsilon > 0$ then $f[t_0, \dots, t_r]$ is bounded on $[-1, +1]$.*

Proof. Use the above mentioned result in [2] that

$$f \in C^{r-1}(-1 - \epsilon, 1 + \epsilon)$$

and therefore that $f^{(r-1)}$ is convex on $(-1 - \epsilon, 1 + \epsilon)$.

We now proceed with the proof of Theorem 3.1 and its corollaries. Let f have bounded r th order divided differences and nonnegative $(r + 1)$ -st order divided differences on $[-1, +1]$. Assume that for n sufficiently large we have $p_n^{(r+1)}(x) \geq 0$ on $[-1, +1]$. We will show that this gives a constant $M > 0$ for which

$$E_n(f) \leq M/(n + 1)^{r+1} \quad \text{for } n = 0, 1, 2, \dots.$$

This will give Theorem 3.1. Corollary 3.1(a) will then follow from Theorem 3.1 and Lemma 4.2. Corollary 3.1(b) follows from Theorem 3.1 and Lemma 4.8.

Proof of Theorem 3.1. Let $n \geq r$ and let $x_0 < x_1 < \dots < x_{n+1}$ be a Chebyshev alternation for f . Assume that there is a positive integer N so that for all $n \geq N$ we have $p_n^{(r+1)}(x) \geq 0$ on $[-1, +1]$, and let $n \geq N$.

Now

$$f(x_i) = p_n(x_i) + \varepsilon(-1)^i E_n(f)$$

for $i = 0, 1, \dots, n + 1$ where $\varepsilon = \pm 1$ is fixed relative to i . Let g be any function which satisfies

$$g(x_i) = (-1)^i \quad \text{for } i = 0, 1, \dots, n + 1.$$

Then

$$(14) \quad f(x_i) = p_n(x_i) + \varepsilon E_n(f)g(x_i)$$

for $i = 0, 1, 2, \dots, n + 1$.

From [3] p. 247 we have the identity

$$(15) \quad F[x_0, \dots, x_m] = \sum_{j=0}^m F(x_j) \prod_{\substack{k=0 \\ k \neq j}}^m (x_j - x_k)^{-1}.$$

If $i + r + 1 \leq n + 1$ we have

$$(16) \quad g[x_i, \dots, x_{i+r+1}] = \sum_{j=0}^{r+1} (-1)^{i+j} \prod_{\substack{k=0 \\ k \neq j}}^{r+1} (x_{i+j} - x_{i+k})^{-1}.$$

We note that all terms in the sum on the right of (16) have the same sign. If ε is as in (14) and if

$$(17) \quad (-1)^i \varepsilon \prod_{k=1}^{r+1} (x_i - x_{i+k})^{-1} > 0$$

we have from (16)

$$(18) \quad \varepsilon g[x_i, \dots, x_{i+r+1}] = h[x_i, \dots, x_{i+r+1}]$$

where h is as in Lemma 4.7.

From (11) and (17) we have

$$(19) \quad \varepsilon(x_{i+r+1} - x_i)g[x_i, \dots, x_{i+r+1}] \geq (r+2)(x_{i+r+1} - x_i)^{-r}.$$

Now using (14), (17), and (19) and the assumption that $p[x_i, \dots, x_{i+r+1}] \geq 0$ we have

$$(20) \quad (x_{i+r+1} - x_i)f[x_i, \dots, x_{i+r+1}] \geq E_n(f)(x_{i+r+1} - x_i)^{-r}(r+2).$$

There are at least $t_n = [(n - r + 1)/2]$ points x_i in $[-1, +1]$ for which (17) holds. We now consider non-overlapping sets $\{x_i, \dots, x_{i+r+1}\}$ where (17) holds for x_i . There are at least

$$m = \left\lceil \frac{t_n}{r+2} \right\rceil$$

such sets, and by Lemma 4.6 there are at least $[m/2]$ such sets with $x_{i+r+1} - x_i \leq 4/m$. It is clear that there is a constant $K > 0$ for which

$$(21) \quad \frac{4}{m} \leq \frac{K}{n} \quad \text{for } m \geq 1.$$

Thus $x_{i+r+1} - x_i \leq K/n$ for n sufficiently large.

Now we sum (20) over all such sets and use this to get

$$(22) \quad K_1 \left[\frac{m}{2} \right] \left(\frac{n}{K} \right)^r E_n(f) \leq \sum_i (x_{i+r+1} - x_i) f[x_i, \dots, x_{i+r+1}].$$

Clearly there is $K_2 > 0$ for which

$$\begin{aligned}
 (23) \quad E_n(f) &\leq \frac{K_2}{n^{r+1}} \sum_i (x_{i+r+1} - x_i) f[x_i, \dots, x_{i+r+1}] \\
 &= \frac{K_2}{n^{r+1}} \sum_i (f[x_{i+1}, \dots, x_{i+r+1}] - f[x_i, \dots, x_{i+r}]) \\
 &\leq \frac{2K_2 M^*}{n^{r+1}}
 \end{aligned}$$

where $M^* = \max \{|f[t_0, \dots, t_r]| : -1 \leq t_0 < \dots < t_r \leq 1\}$. This proves Theorem 3.1.

For the proof of Theorem 3.2 we use (12) and (13) and the fact that $f^{(r-1)}$ is of bounded variation. The proof proceeds as above except that $f[x_i, \dots, x_{i+r+1}]$ is written in terms of $(r-1)$ st order divided differences and therefore in terms of $f^{(r-1)}$. We omit the details here.

Corollary 3.2 is a simple consequence of Lemma 4.2 and Theorem 3.2.

For the proof of Theorems 3.3 and 3.4 we may as well assume that ω is concave in view of (8). The proofs will be done simultaneously. We will work on $[-2, 2]$ here instead of on $[-1, 1]$.

Proofs of Theorem 3.3 and Theorem 3.4. Let $\varepsilon > 0$ be given and let ω be any concave modulus of continuity. Define

$$g(x) = \begin{cases} \varepsilon(x^2 + 5x + 1) & \text{on } [-2, -1] \\ (x-1)^2 + |x| + (5+3\varepsilon)x & \text{on } [-1, +1] \\ 3(2+\varepsilon)x^2 + \omega(1) - \omega(2-x) & \text{on } [1, 2]. \end{cases}$$

g is easily seen to be continuous, increasing, and convex on $[-2, 2]$. Moreover, $g'(0)$ does not exist.

Let g_r be an r th order integral of g . Then $g_r \in C^r[-2, 2]$ and

$$g_r[t_0, \dots, t_{r+1}] \geq \frac{\varepsilon}{(r+1)!}$$

for

$$-2 \leq t_0 < \dots < t_{r+1} \leq 2$$

and

$$g_r[t_0, \dots, t_{r+2}] \geq \frac{2\varepsilon}{(r+2)!}$$

for

$$-2 \leq t_0 < \dots < t_{r+1} < t_{r+2} \leq 2.$$

We will show that there are infinitely many n for which we do not have $p_n^{(r+1)}(x) \geq 0$ on $[-2, +2]$ and infinitely many n for which we do not have $p_n^{(r+2)}(x) \geq 0$ on $[-2, +2]$, where p_n is the polynomial from H_n of best approximation to g_r . This will be sufficient for the proofs of both theorems in view of the fact that for $0 \leq h \leq 1$

$$(24) \quad \omega(h) \leq \omega(g, h) \leq K\omega(h),$$

which is easy to show. The proof of (24) is essentially the same as the proof of (16) in [12]. It is easy to see that on $[-1, +1]$ we have $g_r(x) = Cx_+^{r+1} + Dq_r(x)$ where $q_r \in H_{r+2}$, and where C depends only on r . In view of this and Lemma 4.5 we have

$$(25) \quad E_n(g_r) \geq \frac{K_r}{(n+1)^{r+1}} \quad \text{for } n = 0, 1, \dots,$$

where K_r depends only on r .

If $-2 \leq t_0 < \dots < t_{r+1} \leq -1$ then

$$(26) \quad g_r[t_0, \dots, t_{r+1}] \leq \frac{3\varepsilon}{(r+1)!}$$

and if $-2 \leq t_0 < \dots < t_{r+2} \leq -1$ then

$$(27) \quad g_r[t_0, \dots, t_{r+2}] = \frac{2\varepsilon}{(r+2)!}.$$

Now assume that $p_n^{(r+1)}(x) \geq 0$ on $[-2, +2]$ for n sufficiently large. Then as in the proof of Theorem 3.1 we choose a Chebyshev alternation for such n

$$-2 \leq x_0 < x_1 < \dots < x_{n+1} \leq 2$$

and for g_r and obtain

$$(28) \quad g_r[x_i, \dots, x_{i+r+1}] \geq \sigma E_n(g_r)y[x_i, \dots, x_{i+r+1}]$$

where $\sigma = \pm 1$ is independent of i , and y is any function for which $y(x_i) = (-1)^i i = 0, 1, \dots, n+1$.

Now by Lemma 4.3 there is a sequence $\{n_j\}_{j=0}^\infty$ for which $\lim_{j \rightarrow \infty} \delta_{n_j} = 0$. Thus for j sufficiently large $1/4$ of the $n_j + 2$ Chebyshev alternation points for g_r lie in $[-2, -1]$. Thus there is a constant K depending only on r such that for j sufficiently large there are $r+2$ alternation points x_i, \dots, x_{i+r+1} in $[-2, -1]$ with

$$(29) \quad x_{i+r+1} - x_i \leq \frac{K}{n_j + 1}$$

and for which

$$(30) \quad \sigma y[x_i, \dots, x_{i+r+1}] \geq 0.$$

An application of (11) now gives

$$(31) \quad \sigma y[x_i, \dots, x_{i+r+1}] \geq \frac{(r+2)}{K^{r+1}}(n_j + 1)^{r+1}.$$

Thus from (26), (28), and (31) we get for j sufficiently large

$$(32) \quad E_{n_j}(g_r) \leq \frac{K^{r+1}}{(r+2)!} \cdot 3\varepsilon \left(\frac{1}{(n_j + 1)^{r+1}} \right).$$

This together with (25) gives

$$K_r \leq \frac{3K^{r+1}}{(r+2)!} \varepsilon.$$

But for ε sufficiently small this can easily be violated. Thus we have a contradiction.

To show that we cannot have $p_n^{\langle r+2 \rangle}(x) \geq 0$ for n sufficiently large we proceed in similar fashion. We use (27) and obtain a sequence $\{n_j\}_{j=0}^\infty$ for which

$$(33) \quad E_{n_j}(g_r) \leq \frac{2C_r^{r+2}}{(r+3)!} \cdot \frac{\varepsilon}{(n_j + 1)^{r+2}}.$$

This together with (25) gives an obvious contradiction. We omit the proof of (33) since it is the same as the proof of (32).

We remark that the existence of a $g \in C[-2, 2]$ such that (24) holds implies the existence of $A > 1$, $B > 0$ such that

$$\omega(h) \leq \omega(Ag, h) \leq B\omega(h),$$

for $0 \leq h \leq 4$. Thus both theorems are proven.

REFERENCES

1. S. N. Bernstein, *Collected Works*, AEC Tr. No. 3460, (1952), 175.
2. R. P. Boas and D. V. Widder, *Functions with positive differences*, Duke Math. J. **7** (1940), 496-503.
3. E. Isaacson and H. B. Keller, *Analysis of Numerical Methods*, John Wiley and Sons, New York, 1966.
4. M. I. Kadec, *On the distribution of points of maximum deviation in the approximation of continuous functions by polynomials*, Amer. Math. Soc. Translations (2), **26** (1963), 231-234.

5. G. G. Lorentz, *Approximation of Functions*, Holt, Rinehart and Winston, New York, 1966.
6. ———, Monotone approximation, *Inequalities III*, (O. Shisha, Ed.), pp. 201-215, Academic Press Inc., 1969.
7. G. G. Lorentz and K.L. Zeller, *Monotone approximation by algebraic polynomials*, Trans. Amer. Math. Soc., **149** (1970), 1-18.
8. G. G. Lorentz and K. L. Zeller, *Degree of approximation by monotone polynomials II*, J. Approximation Theory, **2** (1969), 265-269.
9. J. A. Roulier, *Monotone and weighted approximation*, Dissertation, Syracuse University, 1968.
10. ———, *Polynomials of best approximation which are monotone*, J. Approximation Theory, **9** (1973), 212-217.
11. ———, *Best approximation to functions with restricted derivatives*, J. Approximation Theory (to appear).
12. ———, *Negative theorems on monotone approximation*, Proc. Amer. Math. Soc., **55** (1976), 37-43.
13. S. B. Steckin, *On the order of the best approximation of continuous functions*, Izv., **15** (1951), 219-242.

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