Pacific Journal of Mathematics

THE CENTRALISER OF $E \otimes_\lambda F$

Anthony William Wickstead

Vol. 65, No. 2 October 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 2, 1976

THE CENTRALISER OF $E \otimes_\lambda F$

A. W. WICKSTEAD

If E is a real Banach space then $\mathscr{B}(E)$ is the space of all bounded linear operators on E, and $\mathscr{Z}(E)$ the subspace of M-bounded operators, i.e. the centraliser of E. Two Banach spaces E and F are considered as well as the tensor product $E \otimes_\lambda F$. There is a natural mapping of the algebraic tensor product $\mathscr{Z}(E) \odot \mathscr{Z}(F)$ into $\mathscr{Z}(E \otimes_\lambda F)$. It is shown that $\mathscr{Z}(E \otimes_\lambda F)$ is precisely the strong operator closure, in $\mathscr{B}(E \otimes_\lambda F)$, of its image.

1. Definitions and statement of results. A linear operator T on a real Banach space E is *M-bounded* if there is $\lambda > 0$ such that if $e \in E$ and D is a closed ball in E containing λe and $-\lambda e$, then $Te \in D$. The *centraliser* of E, $\mathscr{Z}(E)$, is the commutative Banach algebra of all M-bounded linear operators on E. Let K denote the unit ball of E^*, the Banach dual of E, equipped with the weak* topology. We denote the set of extreme points of a convex set C by $\mathscr{E}(C)$. In [2], Theorem 4.8 it is shown that a bounded linear operator T on E is M-bounded if and only if each point of $\mathscr{E}(K)$ is an eigenvalue for T^*, the adjoint of T. Thus there is a real valued function $\tilde{T}$ on $\mathscr{E}(K)$ such that $T^*p = \tilde{T}(p)p (p \in \mathscr{E}(K))$.

An *L-ideal* in a real Banach space is a subspace I with a complementary direct summand J such that $\|i\| + \|j\| = \|i + j\| (i \in I, j \in J)$. The sets $I \cap \mathscr{E}(K)$ for I a weak*-closed L-ideal in E^* form the closed sets of the structure topology on $\mathscr{E}(K)$. The map $T \mapsto \tilde{T}$ is an isometric algebra isomorphism of $\mathscr{Z}(E)$ onto the bounded structurally continuous real valued functions on $\mathscr{E}(K)$ with the supremum norm and pointwise multiplication ([2], Theorem 4.9).

We shall consider two Banach spaces E and F, K will retain its meaning and M will denote the corresponding subset of F^*. We use $E \odot F$ to denote the algebraic tensor product of E and F. We shall consider the norm

$$\left\| \sum_{i=1}^{n} e_i \otimes f_i \right\|_\lambda = \sup \left\{ \left| \sum_{i=1}^{n} k(e_i) m(f_i) \right| : k \in K,\ m \in M \right\} .$$

$E \odot_\lambda F$ will denote $E \odot F$ with this norm, and $E \otimes_\lambda F$ its completion.

We may identify $E \otimes_\lambda F$ concretely in a number of ways. The formula $(k, m) \mapsto \sum_{i=1}^{n} k(e_i) m(f_i)$ defines a real valued function on $K \times M$. Such functions are continuous and affine in each variable. $\|\sum_{i=1}^{n} e_i \otimes f_i\|_\lambda$ is the same as the supermum norm for such a function, so we may identify $E \otimes_\lambda F$ with a subspace H, the closure of these functions, in $C(K \times M)$, the continuous real valued functions on $K \times M$. We shall have need to call upon:

LEMMA. *Every extreme point of the unit ball of H^* is of the form $h \mapsto h(p, q)(p \in \mathscr{E}(K), q \in \mathscr{E}(M))$.*

Let $R: C(K \times M)^* \to H^*$ be the restriction map, and let B be the unit ball of $C(K \times M)^*$. If f is an extreme point of the unit ball of H^*, then $R^{-1}f \cap B$ is a weak* closed face of B which is nonempty by the Hahn-Banach theorem. By the Krein-Milman theorem, $R^{-1}f \cap B$ has an extreme point, which must be extreme in the unit ball of $C(K \times M)^*$, so is of the form $h \mapsto \pm h(p, q)$ for $p \in K, q \in M$. By replacing p *by* $-p$, if necessary, we may ensure a positive sign. If p (say) is not extreme, then $p = 1/2(p_1 + p_2)$, $p_1, p_2 \in K$, $p_1 \neq p_2$. $h(p, q) = 1/2h(p_1, q) + 1/2h(p_2, q)(h \in H)$ as these functions are affine in each variable. As the functions of H separate the points of $K \times M$, this contradicts the extremality.

COROLLARY.

$$\left\| \sum_{i=1}^{n} e_i \otimes f_i \right\|_\lambda = \sup \left\{ \left| \sum_{i=1}^{n} p(e_i)q(f_i) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\}.$$

We consider the centraliser of $E \bigotimes_\lambda F$. We have quite easily:

PROPOSITION. *If $S_i \in \mathscr{Z}(E)$, $T_i \in \mathscr{Z}(F)(1 \leqq i \leqq n)$ there is $U \in \mathscr{Z}(E \bigotimes_\lambda F)$ such that if $e_j \in E, f_j \in F(1 \leqq j \leqq m)$ then $U(\sum_{j=1}^m e_j \otimes f_j) = \sum_{j=1}^m \sum_{i=1}^n (S_i e_j) \otimes (T_i e_j)$.*

To show that U exists (as a bounded linear operator) we need only show that the linear operator defined on $E \odot_\lambda F$ by this formula is bounded. This is so because,

$$\begin{aligned}
&\left\| \sum_{i,j} (S_i e_j) \otimes (T_i f_j) \right\|_\lambda \\
&\quad = \sup \left\{ \left| \sum_{i,j} p(S_i e_j) q(T_i e_j) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\} \\
&\quad = \sup \left\{ \left| \sum_{i,j} \tilde{S}_i(p) \tilde{T}_i(p) p(e_j) q(f_j) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\} \\
&\quad \leqq \sup \left\{ \sum_i |\tilde{S}_i(p)| \, |\tilde{T}_i(p)| \left| \sum_j p(e_j) q(f_j) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\} \\
&\quad \leqq \sum_i \|S_i\| \, \|T_i\| \sup \left\{ \left| \sum_j p(e_j) q(f_j) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\} \\
&\quad = \sum_i \|S_i\| \, \|T_i\| \, \|\sum_j e_j \otimes f_j\|_\lambda .
\end{aligned}$$

It remains to show that each extreme point of the unit ball of $(E\boldsymbol{\otimes}_\lambda F)^*$ is an eigenvalue for U^*. If we denote by $p\otimes q$ the functional $\sum_j e_j\otimes f_j \mapsto \sum_j p(e_j)q(f_j)$ then we have

$$\begin{aligned}U^*(p\otimes q)\Big(\sum_j e_j\otimes f_j\Big) &= (p\otimes q)U\Big(\sum_j e_j\otimes f_j\Big)\\ &= (p\otimes q)\sum_{i,j}(S_i e_j)\otimes(T_i f_j)\\ &= \sum_{i,j} p(S_i e_j)q(T_i f_j)\\ &= \sum_{i,j}\tilde{S}_i(p)\tilde{T}_i(p)p(e_j)q(f_j)\\ &= \Big[\sum_i \tilde{S}_i(p)\tilde{T}_i(p)\Big]\Big[(p\otimes q)\Big(\sum_j e_j\otimes f_j\Big)\Big].\end{aligned}$$

It is immediate that $U^*(p\otimes q) = [\sum_i \tilde{S}_i(p)\tilde{T}_i(p)](p\otimes q)$.

We thus have an embedding of $\mathscr{Z}(E)\odot\mathscr{Z}(F)$ in $\mathscr{Z}(E\boldsymbol{\otimes}_\lambda F)$ in an obvious way. The remainder of this paper is devoted to a proof of the following result.

THEOREM. *$\mathscr{Z}(E\boldsymbol{\otimes}_\lambda F)$ is the closure, for the strong operator topology, of the canonical copy of $\mathscr{Z}(E)\odot\mathscr{Z}(F)$ in $\mathscr{B}(E\boldsymbol{\otimes}_\lambda F)$.*

2. The proof.

For this proof we shall identify the element $\sum_{i=1}^n e_i\otimes f_i\in E\odot F$ with the function $k\mapsto\sum_{i=1}^n k(e_i)f_i$ from K into F. This is continuous affine function vanishing at 0. The set of all F-valued continuous affine functions of K which vanish at 0 we shall denote by $A_0(K, F)$, and norm it by $\|a\| = \sup\{\|a(k)\|: k\in K\}$, which corresponds to the norm on $E\odot_\lambda F$. We may thus identify $E\boldsymbol{\otimes}_\lambda F$ whith the closure, H, in $A_0(K, F)$ of the functions with finite dimensional range.

If $\sum_{i=1}^n S_i\otimes T_i\in\mathscr{Z}(E)\odot\mathscr{Z}(F)$ then $\pi: p\mapsto\sum_{i=1}^n\tilde{S}_i(p)T_i$ is a function from $\mathscr{E}(K)$ into $\mathscr{Z}(F)$ which is bounded and continuous for the structure topology on $\mathscr{E}(K)$ and the strong operator topology on $\mathscr{Z}(F)$. If U is the image of $\sum_{i=1}^n S_i\otimes T_i$ in $\mathscr{Z}(H)$ (using the proposition and the identification of H with $E\otimes_\lambda F$) then we have

$$(Uh)(p) = \pi(p)h(p) \quad (h\in H,\ p\in\mathscr{E}(K)).$$

This is because, if $\varepsilon > 0$, we may find $\sum_{j=1}^m e_j\otimes f_j\in E\odot F$ with $\|h - \sum_{j=1}^m e_j\otimes f_j\|_\lambda < \varepsilon$ and then

$$\begin{aligned}\|(Uh)(p) - \pi(p)h(p)\| \le{}& \Big\|(Uh)(p) - U\Big(\sum_{j=1}^m e_j\otimes f_j\Big)(p)\Big\|\\ &+\Big\|U\Big(\sum_{j=1}^m e_j\otimes f_j\Big)(p) - \pi(p)h(p)\Big\|.\end{aligned}$$

But

$$\begin{aligned}U\Big(\sum_{j=1}^{m} e_j \otimes f_j\Big)(p) &= \sum_{i,j} (S_i e_j) \otimes (T_i e_j)(p)\\ &= \sum_{i,j} p(S_i e_j)(T_i e_j)\\ &= \sum_{i,j} \tilde{S}_i(p)p(e_j)(T_i e_j)\\ &= \Big(\sum_i \tilde{S}_i(p)T_i\Big)\Big(\sum_j p(e_j)f_j\Big)\\ &= \pi(p)\Big(\Big(\sum_{j=1}^{m} e_j \otimes f_j\Big)(p)\Big)\,.\end{aligned}$$

Thus $\|(Uh)(p) - \pi(p)h(p)\| \leqq \|U\|\varepsilon + \|\pi(p)\|\,\|\sum_{j=1}^{m} e_j \otimes f_j - h)(p)\| \leqq (\|U\| + \|\pi(p)\|)\varepsilon$, which can be made as small as desired, so that $(Uh)(p) = \pi(p)h(p)$.

Let $V(K)$ denote the set of extreme points, p, of K for which there is $x \in E$ with $p(x) = \|x\|$, then $V(K)$ is weak* dense in $\mathscr{E}(K)$. To show this it will suffice to prove that $K = \overline{\text{co}}(V(K))$, the weak* closed convex hull of $V(K)$, for then $\mathscr{E}(K) \subset \overline{V(K)}$ by Milman's theorem. If $\overline{\text{co}}(V(K)) \neq K$ we may, by Hahn-Banach separation, find $x \in E$ with $k(x) \leqq \alpha < k_0(x)$ for some real α, all $k \in \overline{\text{co}}(V(K))$ and some $k_0 \in K$. Then $\{k \in K: k(x) = \|x\|\}$ is a nonempty weak* closed face of K. This possesses an extreme point, which cannot lie in $\overline{\text{co}}(V(K))$, yet which is in $V(K)$ by its construction, a contradiction.

If $p \in V(K)$, $q \in V(M)$ then $p \otimes q$ is extreme in the unit ball of $(E \bigotimes_\lambda F)^*$. Fix $e \in E$, $f \in F$ with $\|e\| = e(p) = 1$, $\|f\| = f(p) = 1$. Define injections $P: E \rightarrow E \bigotimes_\lambda F$, $Q: F \rightarrow E \bigotimes_\lambda F$ by $P(x) = x \otimes f$, $Q(y) = e \otimes y$. P, Q are isometric injections so the image of the unit ball of $(E \bigotimes_\lambda F)^*$ under P^* (respectively Q^*) is K (respectively M). P^*, Q^* are continuous and affine, so $P^{*-1}(p)$ and $Q^{*-1}(q)$ intersect the unit ball of $(E \bigotimes_\lambda F)^*$ in weak* closed faces, as must $P^{*-1}(p) \cap Q^{*-1}(q)$. This intersection is nonempty, for $P^*(p \otimes q) = p$, $Q^*(p \otimes q) = q$. This is because for $x \in E$, $(P^*(p \otimes q))(x) = (p \otimes q)(Px) = (p \otimes q)(x \otimes f) = p(x)q(f) = p(x)$, with a similar proof for Q^*. This face must have an extreme point which is extreme in the unit ball of $(E \bigotimes_\lambda F)^*$, so is $p' \otimes q'$ for $p' \in \mathscr{E}(K)$, $q' \in \mathscr{E}(M)$. But now $p = P^*(p \otimes q) = P^*(p' \otimes q') = p'$ and also $q = q'$, so that $p \otimes q$ is itself extreme.

It follows that if $U \in \mathscr{Z}(H)$ then all points $p \otimes q$ for $p \in \mathscr{E}(K)$, $q \in \mathscr{E}(M)$ are eigenvectors for U^*. For let $p_\gamma \rightarrow p$, $q_\delta \rightarrow q$ be nets with $p_\gamma \in V(K)$, $q_\delta \in V(M)$. The continuity of the map $(k, m) \mapsto k \otimes m$ from $K \times M$ into $(E \bigotimes_\lambda F)^*$ implies that $p_\gamma \otimes q_\delta \rightarrow p \otimes q$. But $U^*(p_\gamma \otimes q_\delta) = \tilde{U}(p_\gamma \otimes q_\delta)(p_\gamma \otimes q_\delta)$. The reals $\tilde{U}(p_\gamma \otimes q_\delta)$ are bounded (by $\|U\|$) so we may suppose (by choosing a subnet if necessary) that $\tilde{U}(p_\gamma \otimes q_\delta) \rightarrow \lambda$. Now $U^*(p \otimes q) = \lim U^*(p_\gamma \otimes q_\delta) = \lim \tilde{U}(p_\gamma \otimes q_\delta) \lim (p_\gamma \otimes q_\delta) = \lambda(p \otimes q)$.

Suppose $U \in \mathscr{Z}(H)$, $p \in \mathscr{E}(K)$ and $h, h' \in H$ with $h(p) = h'(p)$. If

$q \in \mathscr{E}(M)$ then

$$\begin{aligned} q((Uh)(p) &= (p \otimes q)(Uh) = \tilde{U}(p \otimes q)((p \otimes q)(h)) \\ &= \tilde{U}(p \otimes q)(q(h(p))) \\ &= \tilde{U}(p \otimes q)(q(h'(p))) = q((Uh')(p)) . \end{aligned}$$

Thus $(Uh)(p) = (Uh')(p)$. We may thus define a linear operator $\pi(p)$ on F by $\pi(p)y = (Uh)(p)$ whenever $h(p) = y$. $\pi(p)$ is clearly linear, is well defined, and has domain the whole of F since we may take $h = e \otimes y$ where $e(p) = 1$.

$\pi(p)$ has norm at most $||U||$, for we may find $e_n \in E$ with $e_n(p) = 1$, $||e_n|| \leqq (n+1)/n$, and then

$$\begin{aligned} ||\pi(p)y|| = ||U(e_n \otimes y)(p)|| &\leqq ||U(e_n \otimes y)|| \\ &\leqq ||U|| \, ||e_n \otimes y|| = ||U|| \, ||y||(n+1)/n . \end{aligned}$$

Thus $||\pi(p)y|| \leqq ||U|| \, ||y||$. In fact $\pi(p) \in \mathscr{E}(F)$ because if $y \in F$, $q \in \mathscr{E}(M)$ and $e \in E$ with $p(e) = 1$ then

$$\begin{aligned} q(\pi(p)y) &= q(U(e \otimes y)(p)) = (p \otimes q)(U(e \otimes y)) \\ &= \tilde{U}(p \otimes q)(p \otimes q)(e \otimes y) = \tilde{U}(p \otimes q)q(y) . \end{aligned}$$

We thus have a function $\pi: \mathscr{E}(K) \to \mathscr{L}(F)$ with $(Uh)(p) = \pi(p)h(p)(p \in \mathscr{E}(K))$. Also π is norm bounded, and we let $||\pi||$ denote $\sup\{||\pi(p)||: p \in \mathscr{E}(K)\}$.

π is continuous for the structure topology on $\mathscr{E}(K)$ and the weak operator topology on $\mathscr{L}(F)$. Suppose $y \in F$, $g \in F^*$ and $x \in E$ then $k \mapsto g(U(x \otimes y)(k))$ is a continuous affine function on K vanishing at 0, so may be identified with an element of E. If $p \in \mathscr{E}(K)$ then

$$\begin{aligned} g(U(x \otimes y)(p)) &= g(\pi(p)(x \otimes y)(p)) \\ &= g(\pi(p)x(p)y) = x(p)(g(\pi(p)y)) . \end{aligned}$$

Thus $x \mapsto g(U(x \otimes y))$ is an element of $\mathscr{L}(E)$, so the function $p \mapsto g(\pi(p)y)$ is structurally continuous.

By [2], Proposition 3.10 π has an extension, $\bar{\pi}$, to $\overline{\mathscr{E}(K)}\backslash\{0\}$ which is continuous for the weak* topology on $\overline{\mathscr{E}(K)}\backslash\{0\}$ and the weak operator topology on $\mathscr{L}(F)$ (the result there is stated for real valued functions but the proof remains valid in this context). We note for later reference that $\pi\mathscr{E}(K)) = \bar{\pi}(\overline{\mathscr{E}(K)}\backslash\{0\})$. We propose now to show $\bar{\pi}$ is still continuous when $\mathscr{L}(F)$ is given its strong operator topology.

Provisionally we define $\tilde{\pi}(k)$, for $k \in \overline{\mathscr{E}(K)}\backslash\{0\}$, to be that linear operator on F such that

$$\tilde{\pi}(k)y = U(x \otimes y)(k)/k(x)$$

with $x \in E$, $k(x) > 0$. This definition coincides with that of π if $k \in \mathscr{E}(K)$, and is well defined because if $k_\gamma \in \mathscr{E}(K)$ and $k_\gamma \to k$ for the weak* topology then

$$\tilde{\pi}(k)y = U(x \otimes y)(k)/k(x) = \lim U(x \otimes y)(k_\gamma)/k_\gamma(x)$$
$$= \lim \pi(k_\gamma)y \ .$$

Clearly $\tilde{\pi}(k)$ acts linearly on F, and it is bounded because

$$\begin{aligned} ||(\tilde{\pi}(k)y)|| &= ||U(x \otimes y)(k)||/|k(x)| \\ &= \lim ||U(x \otimes y)(k_\gamma)||/|k_\gamma(x)| \\ &= \lim ||\pi(k_\gamma)y|| \leqq ||\pi|| \, ||y|| \ . \end{aligned}$$

Also $||\tilde{\pi}|| = \sup \{||\pi(k)||: k \in \overline{\mathscr{E}(K)}\backslash\{0\}\} = ||\pi||$. $\tilde{\pi}$ is locally a quotient of a function that is clearly strong operator continuous and a non-vanishing scalar function, so is strong operator continuous. In fact $\tilde{\pi}$ is the same as $\bar{\pi}$ as both are extensions of π to $\overline{\mathscr{E}(K)}\backslash\{0\}$ which are continuous for the weak* topology on $\overline{\mathscr{E}(K)}\backslash\{0\}$ and the weak operator topology on $\mathscr{L}(F)$.

We do not know if π itself is continuous when $\mathscr{L}(F)$ is given the strong operator topology. All that we shall require is that if $D \subset \mathscr{E}(K)$ and 0 does not lie in the weak* closure of D, then $\pi|_D$ is continuous for the structure topology on D and the strong operator topology on $\mathscr{L}(F)$. For suppose $d_\gamma, d \in D$ and $d_\gamma \to d$ for the structure topology, then $\pi(d_{\gamma'}) \to \pi(d)$ for the weak operator topology whenever $(d_{\gamma'})$ is a subnet of (d_γ). Let $(d_{\gamma''})$ be a weak* convergent subnet of $(d_{\gamma'})$ with limit $d' \neq 0$, which exists as K is weak* compact. Then $\pi(d_{\gamma''}) \to \pi(d)$ for the weak operator topology whilst $\pi(d_{\gamma''}) = \bar{\pi}(d_{\gamma''}) \to \bar{\pi}(d')$ for the strong operator topology, and hence also for the weak operator topology. Thus $\pi(d) = \bar{\pi}(d')$ and $\pi(d_{\gamma''}) \to \pi(d)$ for the strong operator topology. I.e. every subnet of $(\pi(d_\gamma))$ has a subnet converging to $\pi(d)$, so in fact $\pi(d_\gamma) \to \pi(d)$ for the strong operator topology.

We now seek, given $h_i \in H (i = 1, 2, \cdots, n)$ and $\varepsilon > 0$, to find $\pi': \mathscr{E}(K) \to \mathscr{L}(F)$ which is of finite dimensional range and continuous for the structure topology, such that

$$||\pi'(p)h_i(p) - \pi(p)h_i(p)|| \leqq \varepsilon \quad (p \in \mathscr{E}(K), 1 \leqq i \leqq n) \ .$$

π' is the image of an element of $\mathscr{L}(E) \odot \mathscr{L}(F)$ so defines an element U' of the copy of $\mathscr{L}(E) \odot \mathscr{L}(F)$ in $\mathscr{E}(E \otimes_\lambda F)$. We then have

$$||(U'h_i)(p) - (Uh_i)(p)|| \leqq \varepsilon \quad (p \in \mathscr{E}(K), 1 \leqq i \leqq n) \ .$$

The function $k \mapsto ||(U'h_i)(k) - (Uh_i)(k)||$ on K is continuous and convex, so by [1], Lemma II.7.1, $||(U'h_i) - (Uh_i)|| \leqq \varepsilon (1 \leqq i \leqq n)$. This will show that U is in the strong operator closure of the copy of $\mathscr{L}(E) \odot$

$\mathscr{K}(F)$ in $\mathscr{K}(E \otimes_\lambda F)$.

We first prove that [3], Proposition 4.8 remains valid in this context. I.e. if $x \in E$ then $P = \{p \in \mathscr{E}(K): |p(x)| \geqq \alpha\}$ is structurally compact provided $\alpha > 0$. If $(C_s)_{s \in S}$ is a family of nonempty structurally closed subsets of P with the finite intersection property, let $C_s = P \cap F_s$ with each F_s a weak* closed L-ideal in E^*. Set $Q = \{k \in K: |k(x)| \geqq \alpha\}$ then each $F_s \cap Q$ is nonempty and this family has the finite intersection property. As Q is weak* compact and these sets are weak* closed, $\bigcap(F_s \cap Q) = (\bigcap F_s) \cap Q \neq \emptyset$. $\bigcap F_s$ is a weak* closed L-ideal and for some $k \in K \cap (\bigcap F_s) |k(x)| \geqq \alpha$. But x attains its supremum at an extreme point, p, of $K \cap (\bigcap F_s)$ which is an extreme point of K by [2], Proposition 1.15. As $K \cap (\bigcap F_s)$ is symmetric, $p(x) \geqq \alpha$ so that $p \in E(K) \cap (\bigcap F_s) = \bigcap (p \cap F_s) = \bigcap C_s$. We note also that such a set P does not contain 0 in its weak* closure, so $\pi|_P$ is continuous for the strong operator topology.

Given $h_i \in H$, $\delta > 0$, we may find a weak* closed subset Q_i of $\overline{\mathscr{E}(K)}$, not containing 0 and with $Q_i \cap \mathscr{E}(K)$ structurally compact, such that $\|h_i(k)\| < \delta$ if $k \in \mathscr{E}(K) \backslash Q_i$. For we can find $\sum_{j=1}^m e_j \otimes f_j \in E \odot F$ with $\|\sum_{j=1}^m k(e_j)f_j - h_i(k)\| < \delta/2 (k \in K)$. Now let $P_j = \{k \in \mathscr{E}(K): |k(e_j)| \, \|f_j\| \geqq \delta/2m\}$, which is weak* closed, does not contain 0, and is such that $P_j \cap \mathscr{E}(K)$ is structurally compact. Define $Q_i = \bigcup_{i=1}^m P_j$, then Q_i will have all the desired properties except possibly that on the norm. If $k \in \overline{\mathscr{E}(K)} \backslash Q_i$ then

$$\begin{aligned} \|h_i(k)\| &\leqq \left\| \sum_{j=1}^m k(e_j) f_j \right\| + \left\| \sum_{j=1}^m k(e_j) f_j - h_i(k) \right\| \\ &< \sum_{j=1}^m |k(e_j)| \, \|f_j\| + \delta/2 \\ &\leqq m(\delta/2m) + \delta/2 = \delta \,. \end{aligned}$$

We may thus find a weak* open neighbourhood of 0 in $\overline{\mathscr{E}(K)}$, O_0, with structurally compact complement in $\mathscr{E}(K)$, such that $O_0 \subset \{k \in \overline{\mathscr{E}(K)}: \|h_i(k)\| < \varepsilon/(2\|\pi\| + 1)(1 \leqq i \leqq n)\}$. Indeed if we take $\delta = \varepsilon/(2\|\pi\| + 1)$ and choose Q_i as above we take O_0 to be $\overline{\mathscr{E}(K)} \backslash \bigcup_{i=1}^n Q_i$, which has the desired properties. If $k \in \overline{\mathscr{E}(K)}$ we let $U_k = \{T \in \mathscr{K}(F): \|T(h_i(k))\| < \varepsilon/3 (1 \leqq i \leqq n)\}$, an open symmetric neighbourhood of the origin in $\mathscr{K}(F)$ for the strong operator topology. Thus $\bar{\pi}^{-1}(\bar{\pi}(k) + U_k)$ is an open subset of $\overline{\mathscr{E}(K)} \backslash \{0\}$ (by the continuity of $\bar{\pi}$ for the strong operator topology) and hence of $\overline{\mathscr{E}(K)}$. The set $\overline{\mathscr{E}(K)} \cap \bigcap_{i=1}^n h_i^{-1}(h_i(k) + B)$ (where B is the open ball in F of centre the origin and radius $\varepsilon/(3(\|\pi\| + 1))$) is also weak* open, hence so is

$$O_k = (\bar{\pi}^{-1}(\bar{\pi}(k) + U_k)) \cap \bigcap_{i=1}^n h_i^{-1}(h_i(k) + B)$$

for each $k \in \overline{\mathscr{E}(K)}\backslash\{0\}$, and we have $k \in O_k$. Now let $\{0, k_1, k_2, \cdots, k_r\}$ be a finite set of distinct points of $\overline{\mathscr{E}(K)}$ with $\overline{\mathscr{E}(K)} = O_0 \cup \bigcup_{j=1}^{r} O_{k_j}$.

Let $W = \bigcap_{j=1}^{r} U_{k_j}$, an open convex symmetric neighbourhood of the origin in $\mathscr{L}(F)$ for the strong operator topology. Because $\mathscr{E}(K)\backslash O_0$ is structurally compact and π is continuous on this for the strong operator topology on $\mathscr{L}(F)$, $\pi(\mathscr{E}(K)\backslash O_0)$ is strong operator compact. Thus there exist $\{T_1, T_2, \cdots, T_s\} \subset \mathscr{L}(F)$ such that $\bigcup_{i=1}^{s}(T_i + W/2) \supset \pi(\mathscr{E}(K)\backslash O_0)$. Define G to be the linear span of $\{T_i: 1 \leqq i \leqq s\}$ in $\mathscr{L}(F)$, and let Φ be defined on $\pi(\mathscr{E}(K)\backslash O_0)$ with values in 2^G by

$$\Phi(S) = \{g \in G: \|g\| < \|\pi\| + 1,\ g - S \in W/2\}^- .$$

For some i, $T_i - S \in W/2$ and $T_i \in \pi(\mathscr{E}(K)\backslash O_0)$ so $\|T_i\| \leqq \|\pi\|$, so that $\Phi(S)$ is certainly nonempty. It is clear that $\Phi(S)$ is closed and convex.

We show that Φ is lower semi-continuous, for the unique vector topology on G, and the weak and strong operator topologies on $\pi(\mathscr{E}(K)\backslash O_0)$ which coincide by the compactness of $\pi(\mathscr{E}(K)\backslash O_0)$ for the latter topology. If $D \subset G$ is open we must show that $\{S \in \pi(\mathscr{E}(K)\backslash O_0): \Phi(S) \cap D \neq \varnothing\}$ is open. Suppose $S_0 \in \pi(\mathscr{E}(K)\backslash O_0)$ with $\Phi(S_0) \cap D \neq \varnothing$. By the definition of Φ, we can find $x_0 \in D$ with $\|x_0\| < \|\pi\| + 1$, $x_0 - S_0 \in W/2$. As W is open, there is a symmetric strong operator neighbourhood of the origin in $\mathscr{L}(F)$, V, such that $x_0 - S_0 + V \subset W/2$. Now if $S \in (S_0 + V) \cap \pi(\mathscr{E}(K)\backslash O_0)$ we claim $\Phi(S) \cap D \neq \varnothing$, for $x_0 - S = (x_0 - S_0) + (S_0 - S) \in (x_0 - S_0) + V \subset W/2$. It is now clear that $x_0 \in \Phi(S) \cap D$, completing the proof that Φ is lower semi-continuous.

As G is finite dimensional we can apply a selection theorem (e.g. [4], Theorem 3.2′) to assert the existence of a continuous selection for Φ, ϕ. We note that $\phi(\pi(\mathscr{E}(K)\backslash O_0))$ is contained in the closed ball in G of centre the origin and radius $\|\pi\| + 1$. We extend ϕ to ψ defined on the whole of $\pi(\mathscr{E}(K))$ with values in the same ball and with ψ continuous for the weak operator topology on $\pi(\mathscr{E}(K))$. Let $\beta(\pi(\mathscr{E}(K)))$ be the Stone-Čech compactification of $\pi(\mathscr{E}(K))$ (for the weak operator topology), and ρ the natural injection of $\pi(\mathscr{E}(K))$ into $\beta(\pi(\mathscr{E}(K)))$. Since the weak operator topology is uniformisable ρ is a homeomorphism, so that $\phi\circ\rho^{-1}$ is a continuous function from the closed set $\rho(\pi(\mathscr{E}(K)\backslash O_0))$ into G. Let σ be a continuous extension of $\phi\circ\rho^{-1}$ to the whole of $\beta(\pi(\mathscr{E}(K)))$ with values in the required ball in G, which exists by Tietze's extension theorem. Now $\psi = \sigma\circ\rho$ is the desired function. Define $\pi' = \psi\circ\pi$, a function from $\mathscr{E}(K)$ into G that is bounded and continuous for the structure topology on $\mathscr{E}(K)$, since π is continuous for the structure topology on $\mathscr{E}(K)$ and the weak operator topology on $\mathscr{L}(F)$ whilst ψ is continuous for the weak operator topology on $\pi(\mathscr{E}(K))$. We claim π' has the required property.

If $p \in \mathscr{E}(K)\backslash O_0$ then $p \in O_{k_j}$ for some j. Then $\|h_i(p) - h_i(k_j)\| < \varepsilon/3(\|\pi\| + 1)$ and we also have $\pi'(p) - \pi(p) \in \overline{W/2} \subset W$. Thus for $1 \leqq i \leqq n$,

$$\begin{aligned}
&\|\pi(p)h_i(p) - \pi'(p)h_i(p)\| \\
&\quad \leqq \|\pi(p)h_i(p) - \pi(p)h_i(k_j)\| + \|\pi(p)h_i(k_j) - \pi'(p)h_i(k_j)\| \\
&\qquad + \|\pi'(p)h_i(k_j) - \pi'(p)h_i(p)\| \\
&\quad \leqq \|\pi(p)\|\,\|h_i(p) - h_i(k_j)\| + (\varepsilon/3) + \|\pi'(p)\|\,\|h_i(k_j) - h_i(p)\| \\
&\qquad\quad (\text{since } \pi(p) - \pi'(p) \in W \subset U_{k_j}) \\
&\quad \leqq \|\pi\|(\varepsilon/3(\|\pi\| + 1)) + (\varepsilon/3) + (\|\pi\| + 1)(\varepsilon/3(\|\pi\| + 1)) \\
&\quad < \varepsilon\,.
\end{aligned}$$

On the other hand if $p \in O_0 \cap \mathscr{E}(K)$ then

$$\begin{aligned}
&\|\pi(p)h_i(p) - \pi'(p)h_i(p)\| \\
&\quad \leqq (\|\pi'(p)\| + \|\pi(p)\|)\|h_i(p)\| \\
&\quad \leqq (2\|\pi\| + 1)(\varepsilon/(2\|\pi\| + 1)) = \varepsilon\,.
\end{aligned}$$

Thus π' has the desired properties.

So far we have shown that $\mathscr{Z}(E \otimes_\lambda F)$ is contained in the strong operator closure in $\mathscr{B}(E \otimes_\lambda F)$ of the copy of $\mathscr{Z}(E) \odot \mathscr{Z}(F)$ there. It remains only to show that for any Banach space, X, $\mathscr{Z}(X)$ is strong operator closed in $\mathscr{B}(X)$. Indeed if $T_\lambda \to T$ for the strong operator topology with $T_\gamma \in \mathscr{Z}(X)$, p is an extreme point of the unit ball of X^* and $x \in X$, then

$$(T^*p)(x) = \lim\,(T_\gamma^*p)(x) = \lim\,\tilde{T}_\gamma(p)p(x)\,.$$

Thus $\lim \tilde{T}_\gamma(p)$ exists and $T^*p = (\lim \tilde{T}_\gamma(p))p$, so $T \in \mathscr{Z}(X)$.

References

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse der Matematik 57, Springer Verlag, Germany, 1971.

2. E. M. Alfsen and E. G. Effros, *Structure in real Banach spaces II*, Ann. of Math., **96** (1972), 129-173.

3. E. G. Effros, *On a class of real Banach spaces*, Israel J. Math., **9** (1971), 430-458.

4. E. Michael, *Continuous selections I*, Ann. of Math., **63** (1956), 361-382.

Received November 18, 1974 and in revised form March 17, 1976.

THE QUEEN'S UNIVERSITY OF BELFAST, NORTHERN IRELAND

PACIFIC JOURNAL OF MATHEMATICS

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 65, No. 2 October, 1976