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Ogawa has defined sets of Weierstrass points of a holomorphic vector bundle on a compact complex manifold. We generate nontrivial examples of such sets of Weierstrass points by considering the canonical bundle on a product of Riemann surfaces.

In the first section, we review Ogawa's definition and some classical facts about Weierstrass points on Riemann surfaces. In §2, we prove our theorems and consider an example to illustrate the proofs. Finally, we remark that a connection between Weierstrass points on a Riemann surface and fixed points of a periodic automorphism does not seem to extend to higher dimensions.

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**1.** Let $M$ denote a connected, compact complex manifold of (complex) dimension $n$. Let $E$ denote a holomorphic vector bundle on $M$ of rank $q$. Let $J^k(E)$, $k = 0, 1, \cdots$, denote the holomorphic vector bundle of $k$-jets of $E$(cf. **[7]**). Put $R_k = \text{rank } J^k(E) = q\cdot(n+k)!/n!k!$. Suppose that $\Gamma(E)$, the vector space of global holomorphic sections of $E$, is of dimension $d > 0$. Consider the trivial bundle $M \times \Gamma(E)$ and the map

$$j_k : M \times \Gamma(E) \to J^k(E)$$

which at a point $P \in M$ takes a section to its $k$-jet at $P$. Put $\mu = \min(d, R_k)$.

DEFINITION. (cf. **[6, 3]**). For $1 \leqq r \leqq \mu$, let $W_k^r(E)$ denote the reduced closed analytic subspace of $M$ defined by the vanishing of the exterior power $\Lambda^{\mu-r+1} j_k$.

The points of $W_k^r(E)$ are those $P \in M$ such that the rank of $j_{k,P}$ is at most $\mu - r$.

PROPOSITION 1. *Either $W_k^r(E)$ is empty or each component has codimension at most $r(|d - R_k| + r)$ in $M$.*

*Proof.* [**2**, Proposition 4].

Next, we need to review some facts from the classical theory of

Weierstrass points. We refer the reader to **[1]** for details. Let $C$ denote a compact Riemann surface of genus $g > 1$ and let $P$ be a point on $C$. Let $t$ denote a local coordinate at $P$ on $C$ (so $t(P) = 0$). Suppose the sequence of gaps at $P$ is $1, s_2, s_3, \cdots, s_g$. Then we may choose a basis $\omega_1, \cdots, \omega_g$ of holomorphic differentials on $C$ such that, writing $\omega_j = f_j(t)dt$ locally at $P$, we have that $f_1(0) = 1$ and the order of $f_j$ at $P$ is $s_j - 1$ for $j = 2, \cdots, g$. We will call such a basis of holomorphic differentials *special with respect to P.*

Let $K$ denote the canonical bundle on $C$. Then the matrix of the map $j_k$: $C \times \Gamma(K) \to J^k(K)$ locally at $P$ with respect to the above basis $\{\omega_j\}$ is

$$[f_j^{(i)}(t)] \quad \begin{matrix} i = 0, \cdots, k \\ j = 1, \cdots, g \end{matrix}$$

where $f_j^{(i)}(t)$ denotes the $i^{\text{th}}$ derivative of $f$ with respect to $t$. Note that, by our choice of basis, when we evaluate this matrix at $P$ we get a lower triangular matrix. The next proposition follows easily from the form of this matrix and the choice of our basis $\{\omega_j\}$.

PROPOSITION 2. *Suppose $k \leqq g - 1$. Then $P \in W_k^1(K)$ if and only if $s_j > j$ for some $j = 2, 3, \cdots, k + 1$.*

**2.** Let $C_i$ be a compact Riemann surface of genus $g_i > 1$, $i = 1, \cdots, n$. Denote by $K_i$ the canonical bundle on $C_i$. Put $X = C_1 \times C_2 \times \cdots \times C_n$ and let $K$ denote the canonical bundle on $X$. Then $X$ is a connected, compact complex manifold of dimension $n$ and $\dim_{\mathbf{C}} \Gamma(K) = \Pi_{i=1}^n g_i$. Put $R_k = \operatorname{rank} J^k(K)$.

THEOREM 1. *Suppose $k \leqq \min_{1 \leqq i \leqq n} \{g_i - 1\}$. Let $\mathbf{P} = (P_1, \cdots, P_n) \in X$. Then $\mathbf{P} \in W_k^1(K)$ if and only if $P_i \in W_k^1(K_i)$ for some $i = 1, \cdots, n$.*

*Proof.* The notation in the general case is very complicated. We will prove the theorem for the case $n = 2$. It is not hard to see that the general case may be demonstrated by a completely similar argument with no new ideas necessary.

So, let $C$ and $D$ be compact Riemann surfaces of genera $g > 1$ and $h > 1$ respectively, and suppose, without loss of generality, that $g$ is greater than or equal to $h$. Let $K_C$ (resp. $K_D$) denote the canonical bundle on $C$ (resp. $D$). Put $X = C \times D$ and let $(P, Q) \in X$. Let $t$ (resp. $u$) denote a local coordinate at $P$ on $C$ (resp. at $Q$ on $D$). Let $\alpha_i = \varphi_i(t)dt$, $i = 1, \cdots, g$, denote the basis of holomorphic differentials on

$C$ special with respect to $P$ and let $\beta_j = \psi_j(u)du$, $j = 1, \cdots, h$, denote the basis of holomorphic differentials on $D$ special with respect to $Q$.

Let $\pi_1\colon X \to C$ and $\pi_2\colon X \to D$ denote the respective projection maps. Put

$$\omega_{ij} = \pi_1^*\alpha_i \wedge \pi_2^*\beta_j \qquad \begin{array}{l} i = 1, \cdots, g \\ j = 1, \cdots, h. \end{array}$$

Then the $\omega_{ij}$ are a basis of holomorphic 2-forms on $X$ and locally at $(P, Q)$ we may write $\omega_{ij} = \varphi_i(t)\psi_j(u)dt \wedge du$.

Let $K$ denote the canonical bundle on $X$ and suppose $0 \leqq k \leqq h-1$. (Note then that $R_k < h^2 \leqq gh$.) Consider the map $j_k\colon X \times \Gamma(K) \to J^k(K)$. Denote by $D^{l,m}$ the differential operator $\partial^{l+m}/\partial t^l \partial u^m$. The entries of the matrix of $j_k$ locally at $(P, Q)$ are then $D^{l,m}(\varphi_i(t)\psi_j(u))$, where $1 \leqq i \leqq g$, $1 \leqq j \leqq h$, and where $l$, $m$ are nonnegative integers such that $l + m \leqq k$. It is not hard to see that after suitably ordering the basis elements of $\Gamma(K)$ and $J^k_{(P,Q)}(K)$ the matrix of $j_k$ when evaluated at $(P, Q)$ is a lower triangular matrix with diagonal entries

$$D^{l,m}(\varphi_{l+1}(t)\psi_{m+1}(u))\big|_{(0,0)} = \varphi_{l+1}^{(l)}(0) \cdot \psi_{m+1}^{(m)}(0).$$

More precisely, we order the operators $D^{l,m}$ "lexicographically in each degree"; i.e. $D^{l,m}$ comes before $D^{l',m'}$ if $l + m < l' + m'$ or if $l + m = l' + m'$ and $l > l'$. Similarly, $\omega_{ij}$ comes before $\omega_{i'j'}$ if $i + j < i' + j'$ or if $i + j = i' + j'$ and $i > i'$.

Now, the rank of this matrix at $(P, Q)$ is less than $R_k$ (the maximum possible rank) if and only if $\varphi_{l+1}^{(l)}(0) = 0$ for some $l = 0, 1, \cdots, k$ or $\psi_{m+1}^{(m)}(0) = 0$ for some $m = 0, 1, \cdots, k$. But, by Proposition 2, $\varphi_{l+m}^{(l)}(0) = 0$ for some $l, 0 \leqq l \leqq k$, if and only if $P \in W_k^1(K_C)$ and $\psi_{m+1}^{(m)}(0) = 0$ for some $m, 0 \leqq m \leqq k$, if and only if $Q \in W_k^1(K_D)$. Hence $(P, Q) \in W_k^1(K)$ if and only if $P \in W_k^1(K_C)$ or $Q \in W_k^1(K_D)$.

THEOREM 2. *With the notation of Theorem* 1, *suppose* $k > \min_{1 \leqq i \leqq n}\{g_i - 1\}$ *and* $R_k \leqq \prod_{i=1}^n g_i$. *Then* $W_k^1(K) = X$.

*Proof.* Again, we will prove this only for $n = 2$. With notation as in the proof of Theorem 1, we have that the matrix of $j_k$ when evaluated at $(P, Q)$ will be a lower triangular matrix with a term of the form

$$D^{0,k}(\varphi_l(t)\psi_h(u))\big|_{(0,0)} = \varphi_l(0)\psi_h^{(k)}(0)$$

with $l > 1$, as the last entry on the diagonal. But $\varphi_l(0) = 0$ for $l > 1$, so the mapping $j_k$ fails to have maximal rank at every point of $X$. Hence $W_k^1(K) = X$.

To illustrate the proofs of the above theorems, we consider the following example. With notation as in Theorem 1, we suppose $g = h = 4$. We order the basis $\{\omega_{ij}\}$ of holomorphic 2-forms on $X$ as follows: $\omega_{11}$, $\omega_{21}$, $\omega_{12}$, $\omega_{31}$, $\omega_{22}$, $\omega_{13}$, $\omega_{41}$, $\omega_{32}$, $\omega_{23}$, $\omega_{14}$, $\omega_{42}$, $\omega_{33}$, $\omega_{24}$, $\omega_{43}$, $\omega_{34}$, $\omega_{44}$. The matrix of $j_4$ evaluated at $(P, Q)$ is a $15 \times 16$ lower triangular matrix with diagonal entries: 1, $\varphi_2'(0)$, $\psi_2'(0)$, $\varphi_3''(0)$, $\varphi_2'(0)\psi_2'(0)$, $\psi_3''(0)$, $\varphi_4'''(0)$, $\varphi_3''(0)\psi_2'(0)$, $\varphi_2'(0)\psi_3''(0)$, $\psi_4'''(0)$,0, 0, 0, 0, 0.

It is then clear that the conclusions of Theorems 1 and 2 hold.

**3.** Let $C$ be a compact Riemann suface of genus $g > 1$. Let $\sigma$ be a periodic automorphism (conformal homeomorphism) of $C$ of order $n$. Put $C^* = C/\langle\sigma\rangle$ and let $g^*$ denote the genus of $C^*$. In **[8]**, Schoeneberg proves the following theorem (also cf. **[4]**):

THEOREM. *Let P denote a fixed point of $\sigma$. Then P is a Weierstrass point of C if $g^* \neq [g/n]$, where $[x]$ denotes the greatest integer in x.*

We remark here that this result does not seem to generalize to higher dimensions. Indeed, consider $C \times C$, an algebraic manifold of geometric genus $g^2$. Let $C(2)$ denote the second symmetric product of $C$ with itself; i.e. $C(2) = C \times C/S_2$. Then $C(2)$ is an algebraic manifold of geometric genus $g(g-1)/2$ **[5]**. Note that $g(g-1)/2 < [g^2/2]$. Now, the set of fixed points of $C \times C$ under the action of $S_2$ is the diagonal, while, by Theorem 1, the nontrivial set of Weierstrass points of the canonical bundle on $C \times C$, the set $W^1_{g-1}(K_{C\times C})$, consists of all points $(P, Q)$ such that either $P$ or $Q$ is a Weierstrass point of $C$. Thus not all fixed points are Weierstrass points in this case. We do not see any good way of generalizing Schoeneberg's Theorem to higher dimensions.

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