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**AN ESTIMATE OF THE NIELSEN NUMBER AND AN
EXAMPLE CONCERNING THE LEFSCHETZ FIXED POINT
THEOREM**

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Given a map $f: X \rightarrow X$ of a compact ANR and any finite connected regular covering $p: \tilde{X} \rightarrow X$ to which f admits lifts, then one can compute a certain homotopy invariant $N_H(f)$ if the Lefschetz numbers of the lifts and the relation of the lifts to the covering transformations are known. $H = p_* \pi_1(\tilde{X})$. Every map homotopic to f has at least $N_H(f)$ fixed points. If X is a finite polyhedron, then $N_H(f) \leq N(f)$, the Nielsen number. The smaller invariant is easier to compute by virtue of its smallness, but it is adequate to discern for example homeomorphisms, h , of manifolds in all dimensions with $L(h) = 0$ and $N(h) \geq 2$.

1. Introduction. It is known that if X is simply-connected and either a compact topological manifold [2] or a finite polyhedron satisfying the Shi condition [1, p. 139], then the converse of the Lefschetz Fixed Point Theorem is valid, i.e. if the Lefschetz number $L(f)$ of a map $f: X \rightarrow X$ is zero, then there is a map $g: X \rightarrow X$ homotopic to f which has no fixed points. This converse remains valid if the condition of simple-connectivity is relaxed to that of Jiang [1, p. 141].

Our objective here is to give examples of manifolds M^n in all dimensions which admit self-maps f (homeomorphisms, in fact) with $L(f) = 0$ such that every map homotopic to f has two or more fixed points.

We will use an approach due to G. Hirsch [3] which detects essential Nielsen classes using two-fold covers. In the following section we outline a generalization of this procedure.

2. The generalized Hirsch method. Let X be a compact ANR and $p: \tilde{X} \rightarrow X$ a finite connected regular covering of X . Let $H = p_* \pi_1(\tilde{X})$. For maps $f: X \rightarrow X$ which admit lifts \tilde{f} ,

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{f} & X
 \end{array}$$

we will define a number $N_H(f)$ which is no larger than the Nielsen

number $N(f)$ and which is easier to compute because it may be smaller and because it is defined with reference to $\tilde{f}_*: H_*(\tilde{X}) \rightarrow H_*(\tilde{X})$ rather than to the local fixed point index.

Let f be as mentioned, and notice that since p is regular, we have assumed there is a collection

$$\mathcal{C} = \{\tilde{f} \mid p\tilde{f} = fp\}$$

of lifts having as many members as the multiplicity of p . Let $\text{Fix}(g)$ denote the set of points fixed by a map g .

If $\tilde{f} \in \mathcal{C}$, then

$$p \text{Fix}(\tilde{f}) \subset \text{Fix}(f).$$

If $\tilde{f}, \tilde{f}' \in \mathcal{C}$ and

$$p \text{Fix}(\tilde{f}) \cap p \text{Fix}(\tilde{f}') \neq \phi,$$

then there is a covering transformation $\gamma: \tilde{X} \rightarrow \tilde{X}$ such that

$$(1) \quad \tilde{f}'\gamma = \gamma\tilde{f}.$$

Whenever the conjugacy relation (1) prevails, we find that

$$p \text{Fix}(\tilde{f}) = p \text{Fix}(\tilde{f}').$$

It is convenient to summarize this situation in the following way. The group G of covering transformations acts on \mathcal{C} by conjugation, partitioning \mathcal{C} into a collection \mathcal{C}/G of (let us say k) conjugacy classes.

To each class $[\tilde{f}] \in \mathcal{C}/G$ we may associate the subset

$$p \text{Fix}(\tilde{f}) \subset \text{Fix}(f)$$

independently of the representative \tilde{f} . These various subsets of $\text{Fix}(f)$ are mutually disjoint; and moreover,

$$(2) \quad \text{Fix}(f) = \bigcup_{[\tilde{f}] \in \mathcal{C}/G} p \text{Fix}(\tilde{f}).$$

Likewise, to each class $[\tilde{f}] \in \mathcal{C}/G$ we may associate the number

$$L([\tilde{f}]) = L(\tilde{f})$$

independently of the representative \tilde{f} . These numbers constitute an unordered k -tuple $\mathcal{L}_H(f)$.

THEOREM 1. $\mathcal{L}_H(f)$ is a homotopy invariant.

Proof. Let $F: X \times I \rightarrow X$ be a homotopy with

$$F(x, 0) = f_0(x), \quad F(x, 1) = f_1(x).$$

For $i = 0, 1$, let

$$\mathcal{C}_i = \{\tilde{f}_i \mid p\tilde{f}_i = f_i p\}$$

be the collection of lifts of f_i .

For each lift $\tilde{f}_0 \in \mathcal{C}_0$ of f_0 there is a unique homotopy \tilde{F} completing the diagram

$$\begin{array}{ccc} \tilde{X} \times I & \xrightarrow{\tilde{F}} & \tilde{X} \\ \downarrow p \times 1 & & \downarrow p \\ X \times I & \xrightarrow{F} & X \end{array}$$

and satisfying the initial condition

$$\tilde{F}(x, 0) = \tilde{f}_0(x).$$

By associating with \tilde{f}_0 the other end of this homotopy

$$\tilde{f}_1(x) = \tilde{F}(x, 1),$$

we may define a one-to-one correspondence

$$\eta: \mathcal{C}_0 \rightarrow \mathcal{C}_1.$$

Corresponding lifts have the same Lefschetz number, and a conjugate pair of lifts in \mathcal{C}_0 correspond with a pair in \mathcal{C}_1 which are also conjugate. This completes the proof of Theorem 1.

DEFINITION. Let $N_H(f)$ be the number of classes $[\tilde{f}] \in \mathcal{C}/G$ for which $L(\tilde{f}) \neq 0$.

THEOREM 2. Every map homotopic to f has at least $N_H(f)$ fixed points.

Proof. By Theorem 1, we need only consider f itself. Certainly $p \text{Fix}(\tilde{f}) \neq \phi$ whenever $L(\tilde{f}) \neq 0$, and these subsets of $\text{Fix}(f)$ are mutually disjoint if they derive from different conjugacy classes.

COROLLARY 1 (Hirsch). *Let $f: X \rightarrow X$ be a map of a compact ANR and $p: \tilde{X} \rightarrow X$ a connected two-fold covering of X . Suppose*

- (1) f lifts to $\tilde{f}, \tilde{f}': \tilde{X} \rightarrow \tilde{X}$,
- (2) if $p(a) = p(b)$ and $a \neq b$, then $\tilde{f}(a) \neq \tilde{f}(b)$,
- (3) $L(\tilde{f}) \neq 0 \neq L(\tilde{f}')$.

Then every map homotopic to f has two or more fixed points.

Proof. If γ is the nontrivial covering transformation, then by (2), $\tilde{f}\gamma = \tilde{f}'\gamma$; $[\tilde{f}] \neq [\tilde{f}']$.

3. Relation of the Hirsch method to the Nielsen number. Suppose X, \tilde{X}, p, f are as above. Let us retain the notation used before, and let $N(f)$ denote the Nielsen number of f . Later in this section, we will prove the following.

THEOREM 3. *If X is a finite polyhedron, then*

$$N(f) \geq N_H(f).$$

Recall [1] that $N(f)$ is the number of fixed point (equivalence) classes $F \subset \text{Fix}(f)$ for which the local fixed point index $i(F) \neq 0$. Here the equivalence relation is this: $x_0, x_1 \in \text{Fix}(f)$ are equivalent if there is a path $\omega: I \rightarrow X$ with $\omega(0) = x_0, \omega(1) = x_1$, and

$$[\omega(f\omega)^{-1}] = 1 \in \pi_1(X).$$

The other invariant, $N_H(f)$, is also related to an equivalence relation on $\text{Fix}(f)$ as defined by the partition (2).

LEMMA 1. *For $x_0, x_1 \in \text{Fix}(f)$, the following are equivalent:*

- (1) $x_0, x_1 \in p \text{Fix}(\tilde{f})$ for some $\tilde{f} \in \mathcal{C}$
- (2) there is a path $\omega: I \rightarrow X$ with $\omega(0) = x_0, \omega(1) = x_1$, and

$$[\omega(f\omega)^{-1}] \in H = p_{\#}\pi_1(\tilde{X}).$$

Proof. Supposing (1), choose $\tilde{x}_i \in p^{-1}(x_i) \cap \text{Fix}(\tilde{f})$, for $i = 0$ and 1 , and next choose a path $\tilde{\omega}: I \rightarrow \tilde{X}$ with $\tilde{\omega}(0) = \tilde{x}_0, \tilde{\omega}(1) = \tilde{x}_1$. Then $\omega = p\tilde{\omega}$ satisfies (2).

Supposing (2), choose any $\tilde{x}_0 \in p^{-1}(x_0)$ and then choose $\tilde{f} \in \mathcal{C}$ with $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$. The paths ω and $f\omega$ lift to paths starting at \tilde{x}_0 and ending at a common point $\tilde{x}_1 = \tilde{f}(\tilde{x}_1) \in p^{-1}(x_1)$. So $x_0, x_1 \in p \text{ Fix}(\tilde{f})$.

A result of this first Lemma is that each Nielsen class F has the property

$$F \subset p \text{ Fix}(\tilde{f})$$

for some $[\tilde{f}] \in \mathcal{C}/G$. And so each of the sets $p \text{ Fix}(\tilde{f})$ is the union of several Nielsen classes.

DEFINITION. For $[\tilde{f}] \in \mathcal{C}/G$, let $m([\tilde{f}])$ be the number of covering transformations $\gamma \in G$ for which $\tilde{f} = \gamma\tilde{f}\gamma^{-1}$. That is,

$$m([\tilde{f}]) = \text{order}(\text{stabilizer of } \tilde{f} \text{ in } G).$$

LEMMA 2. If $\tilde{f} \in \mathcal{C}$ and $x \in p \text{ Fix}(\tilde{f})$, then

$$m([\tilde{f}]) = \text{cardinality}(p^{-1}(x) \cap \text{Fix}(\tilde{f})).$$

Proof. Where $\tilde{x} \in p^{-1}(x) \cap \text{Fix}(\tilde{f})$, it is easily shown that $\gamma\tilde{x} \in \text{Fix}(\tilde{f})$ if and only if $\tilde{f}\gamma = \gamma\tilde{f}$.

LEMMA 3. If $\text{Fix}(f)$ is finite and $\tilde{f} \in \mathcal{C}$, then

$$L(\tilde{f}) = m([\tilde{f}]) \cdot i(p \text{ Fix}(\tilde{f}))$$

where $i(p \text{ Fix}(\tilde{f}))$ is the local fixed point index of f in a closed neighborhood of $p \text{ Fix}(\tilde{f})$ disjoint with the remainder of $\text{Fix}(f)$.

Proof. Since p is locally a homeomorphism, the index of a fixed point $x \in p \text{ Fix}(\tilde{f})$ is the same as that of each of the $m([\tilde{f}])$ points in $p^{-1}(x) \cap \text{Fix}(\tilde{f})$.

Proof of Theorem 3. Since both $N(f)$ and $N_H(f)$ are homotopy invariant, we may assume that $\text{Fix}(f)$ is finite (Hopf construction for finite polyhedra, [1]). By Lemma 3, $L(\tilde{f})$ is a nonzero multiple of the sum of the indices $i(F)$, $F \subset p \text{ Fix}(\tilde{f})$. Thus if $L(\tilde{f}) \neq 0$, then $i(F) \neq 0$ for at least one of the fixed point classes $F \subset p \text{ Fix}(\tilde{f})$.

REMARK. If $p: \tilde{X} \rightarrow X$ is the universal covering space ($\pi_1(X)$ finite and $H = \{1\}$), then the nonempty sets $p \text{ Fix}(\tilde{f})$ are precisely the Nielsen fixed point classes, and $N(f) = N_H(f)$. This is the case covered by Jiang [4].

4. An example in dimension two. We will exhibit a homeomorphism $f: X \rightarrow X$ of the orientable surface of genus 2 for which $L(f) = 0$ and to which Corollary 1 applies. We will employ certain elementary surface homeomorphisms that have been described by W. B. R. Lickorish [5]:

Let the 2-manifold M contain an annulus, A , one of the boundary components of which is a simple closed curve c . There is a homeomorphism of A to itself, fixed on the boundary of A , which sends radial arcs onto arcs which spiral once [or several times] around A (see Figure 1). This can be extended to a homeomorphism of M to itself, by the identity on $M - A$. Intuitively this homeomorphism can be thought of as the process of cutting M along c , twisting one of the now free ends, and then gluing together again.

Our double covering of X is by \tilde{X} , the orientable surface of genus 3. As indicated in Figure 2, such a covering is obtained by wrapping the center hole of \tilde{X} twice around the left hole of X . Alternatively, this projection may be regarded as the process of cutting \tilde{X} along the two unlabeled simple closed curves and then mapping each half onto X by first identifying its two boundary components and then mapping the resulting space homeomorphically onto X so that the identified curve maps onto the unlabeled simple closed curve in X . The other curves in Figure 2 are the free Abelian generators of $H_1(X)$ and $H_1(\tilde{X})$.

The homeomorphism $f: X \rightarrow X$ is a composition of Lickorish twists. On the left hole of X first perform a single twist at β_1 , twisting in the direction that sweeps α_1 backward along β_1 , and then perform a double twist at α_1 in the direction that sweeps β_1 forward along α_1 . The effect of this composition on the two generators α_1, β_1 is described by the matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

Do nothing to the other hole of X so that α_2, β_2 are transformed by the identity matrix. Since f is an orientation preserving homeomorphism,

$$f_* = id: H_2(X) \rightarrow H_2(X).$$

And thus $L(f) = 1 - 2 + 1 = 0$.

The lifts \tilde{f}, \tilde{f}' of f are each a composition of twists (and covering transformations). We may describe the more obvious one, \tilde{f} , as follows. On the center hole, \tilde{f} is two twists at $\tilde{\beta}_2$ in the direction that sweeps $\tilde{\alpha}_2$ backward along $\tilde{\beta}_2$ followed by a single twist at $\tilde{\alpha}_2$ in the direction that sweeps $\tilde{\beta}_2$ forward along $\tilde{\alpha}_2$. The associated matrix is

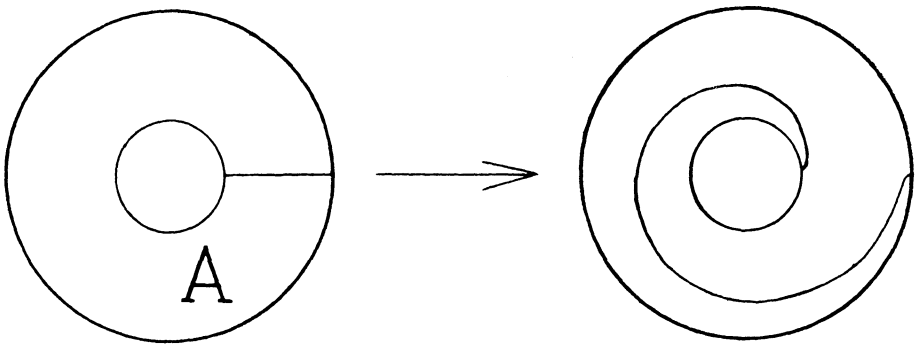


FIGURE 1

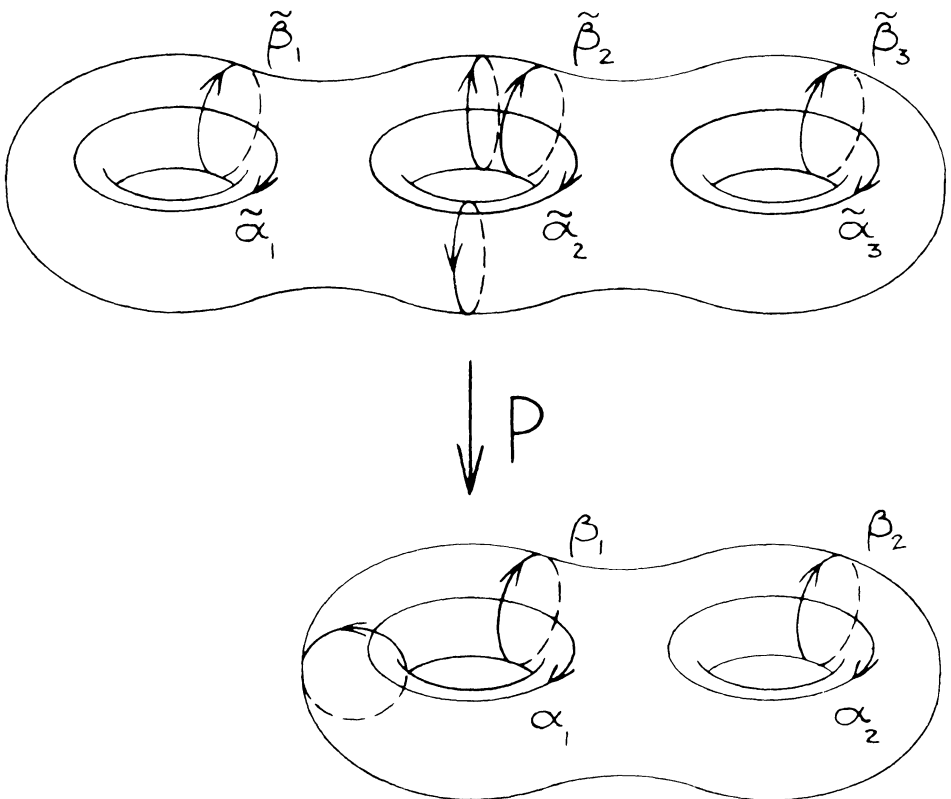


FIGURE 2

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}.$$

The other holes are undisturbed. We find $L(\tilde{f}) = 1 - 4 + 1 = -2$.

Since $\tilde{f}'_* = (\gamma\tilde{f})_* = \gamma_*\tilde{f}'_*$, it is easily shown that $L(\tilde{f}') = +2$. Thus all of the conditions of Corollary 1 are satisfied.

5. Other examples. Let $f, X, p, \tilde{X}, \tilde{f}, \tilde{f}'$ be as in the previous section. For $n \geq 3$, let

$$M^n = X \times S^{n-2}.$$

Let $g: S^{n-2} \rightarrow S^{n-2}$ be a homeomorphism for which $L(g) \neq 0$. And let us consider the homeomorphism

$$h = f \times g: M^n \rightarrow M^n.$$

There is the double covering

$$p \times 1: \tilde{X} \times S^{n-2} \rightarrow X \times S^{n-2},$$

and h lifts to the homeomorphisms

$$\tilde{f} \times g, \tilde{f}' \times g: \tilde{X} \times S^{n-2} \rightarrow \tilde{X} \times S^{n-2}.$$

Also

$$L(h) = L(f) \cdot L(g) = 0$$

and

$$L(\tilde{f} \times g) = L(\tilde{f}) \cdot L(g) \neq 0 \neq L(\tilde{f}') \cdot L(g) = L(\tilde{f}' \times g).$$

So we may again conclude that although $L(h) = 0$, $N(h) \geq 2$.

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