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DIVISION OF DISTRIBUTIONS

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This paper deals with division in an associative commutative algebra containing the distributions in \mathbb{R}^n .

1. Introduction. In [5] and [6], a family $(A_{p, \lambda} \mid p \in \bar{N}^n, \lambda \in \Lambda)$ of associative, commutative algebras with unit element were constructed, with the following main properties:

- (1) $\mathcal{D}'(\mathbb{R}^n) \subset A_{p, \lambda}, \forall p \in \bar{N}^n, \lambda \in \Lambda$,
(here, $N = \{0, 1, 2, \dots\}$, $\bar{N} = N \cup \{\infty\}$ and $n \in N, n \geq 1$);
- (2) The multiplication in each of the algebras $A_{p, \lambda}, p \in \bar{N}^n, \lambda \in \Lambda$, induces on $\mathcal{C}^\infty(\mathbb{R}^n)$ the usual multiplication of functions and the function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$, with $\psi(x) = 1, \forall x \in \mathbb{R}^n$, is the unit element in the algebras;
- (3) for each $\lambda \in \Lambda$, there exist linear mappings $D^p: A_{q+p, \lambda} \rightarrow A_{q, \lambda}$, with $p \in N^n, q \in \bar{N}^n$, such that

(3.1) D^p satisfies on $A_{q+p, \lambda}$ the Leibnitz rule of product derivative.

(3.2) D^p is the usual distribution derivative on $\mathcal{C}^\infty(\mathbb{R}^n) \oplus \mathcal{D}'_s(\mathbb{R}^n)$, where $\mathcal{D}'_s(\mathbb{R}^n) = \{S \in \mathcal{D}'(\mathbb{R}^n) \mid \text{supp } S \text{ is finite}\}$;

(4) The following relations hold for the Dirac δ_{x_0} distribution, concentrated in $x_0 \in \mathbb{R}^n$:

$$(x - x_0)' \cdot D^q \delta_{x_0} = 0 \in A_{p, \lambda}, \quad \forall p \in N^n, \quad \lambda \in \Lambda,$$

if $q, r \in N^n, r \geq p + e, r \geq q + e$, where $e = (1, \dots, 1) \in N^n$.

In the present paper, within the one dimensional case $n = 1$, necessary or sufficient conditions are given for $T \in A_{p, \lambda}$, in order to be a solution of one of the equations $x^m \cdot T = 0 \in A_{p, \lambda}$ and $x^m \cdot T = S \in A_{p, \lambda}$, with $m \in N, m \geq 1$.

2. Notations. Several classes of sequences of complex valued smooth functions (see [5] and [6]) will be needed.

(1) $\mathcal{W} = N \rightarrow \mathcal{C}^\infty(\mathbb{R}^1)$; if $s \in \mathcal{W}, \nu \in N, x \in \mathbb{R}^1$, then $s(\nu) \in \mathcal{C}^\infty(\mathbb{R}^1), s(\nu)(x) \in C^1$; for $\psi \in \mathcal{C}^\infty(\mathbb{R}^1)$ denote $u(\psi) \in \mathcal{W}$, where $u(\psi)(\nu) = \psi, \forall \nu \in N$; \mathcal{W} is in a natural way an associative, commutative algebra (the vector spaces and algebras are considered over the field C^1 of

complex numbers), with the unit element $u(1)$ and zero element $u(0)$; thus, $\mathcal{O} = \{u(0)\}$ is the null space in \mathcal{W} ;

(2) $D: \mathcal{W} \rightarrow \mathcal{W}$ is defined by $(Ds)(\nu)(x) = (Ds(\nu))(x)$, $\forall s \in \mathcal{W}$, $\nu \in N$, $x \in R^1$; for given $x_0 \in R^1$, define $\tau_{x_0}: \mathcal{W} \rightarrow \mathcal{W}$ by $(\tau_{x_0}s)(\nu)(x) = s(\nu)(x - x_0)$, $\forall s \in \mathcal{W}$, $\nu \in N$, $x \in R^1$;

(3) $\mathcal{U} = \{u(\psi) \mid \psi \in \mathcal{C}^\infty(R^1)\}$;

(4) \mathcal{S}_0 is the set of $s \in \mathcal{W}$, weakly convergent in $\mathcal{D}'(R^1)$; \mathcal{V}_0 is the kernel of the linear surjection:

$$\mathcal{S}_0 \ni s \rightarrow \langle s, \cdot \rangle \in \mathcal{D}'(R^1),$$

where

$$\langle s, \psi \rangle = \lim_{\nu \rightarrow \infty} \int_{R^1} s(\nu)(x) \psi(x) dx, \quad \forall \psi \in \mathcal{D}(R^1);$$

One of the basic ideas in the construction of the associative and commutative distribution multiplication in [5] and [6], is the way the weakly convergent sequences of smooth functions representing the Dirac δ distribution are chosen:

(5) \mathcal{X}_δ^0 is the set of $s \in \mathcal{S}_0$, satisfying the conditions:

$$(5.1) \quad \langle s, \cdot \rangle = \delta,$$

$$(5.2) \quad \forall \epsilon > 0: \exists \nu_\epsilon \in N: \forall \nu \in N, \\ \nu \geq \nu_\epsilon, x \in R^1, |x| \geq \epsilon: s(\nu)(x) = 0$$

$$(5.3) \quad \forall p \in N: \exists \nu_p \in N: \forall \nu \in N, \\ \nu \geq \nu_p: W(s(\nu), \dots, s(\nu + p))(0) \neq 0.$$

where $W(\psi_1, \dots, \psi_m)(x)$, $x \in R^1$, denotes the Wronskian function of $\psi_1, \dots, \psi_m \in \mathcal{C}^\infty(R^1)$.

The condition (5.3), called “*strong local presence of s in $x = 0$* ” and replaced in [6] by a weaker form, plays a central role in the associative, commutative distribution multiplication presented in [5] and [6].

(6) for $p \in \bar{N}$, denote by $\mathcal{V}_{\delta,p}^0$ the set of $v \in \mathcal{V}_0$, satisfying the above condition (5.2), as well as

$$(6.1) \quad \forall q \in N, q \leq p: \exists \nu_q \in N: \forall \nu \in N: \nu \geq \nu_q \Rightarrow D^q v(\nu)(0) = 0;$$

$$(7) \quad \mathcal{S}_\delta^0 = \{s \in \mathcal{S}_0 \mid \text{supp } \langle s, \cdot \rangle \subset \{0\}\};$$

(8) $\mathcal{V}_{\delta,p}$, with $p \in \bar{N}$, and \mathcal{S}_δ are the vector subspaces generated in \mathcal{W} by $\bigcup_{x \in R^1} \tau_x \mathcal{V}_{\delta,p}^0$, respectively $\bigcup_{x \in R^1} \tau_x \mathcal{S}_\delta^0$;

$$(9) \quad \mathcal{X}_\delta = \bigcup_{x \in R^1} \tau_x \mathcal{X}_\delta^0;$$

(10) for $\Sigma = (s_x \mid x \in R^1) \in \mathcal{X}_\delta$, denote by $\mathcal{S}(\Sigma)$ the vector subspace generated in \mathcal{S}_0 by the sequences $D^p s_x$, with $x \in R^1$, $p \in N$.

And now, the definition of the associative, commutative algebras

$(A_{p,\lambda} | p \in \bar{N}, \lambda \in \Lambda)$, where Λ is the set of all $\lambda = (\Sigma, \mathcal{S}_1)$ with $\Sigma \in \mathcal{Z}_\delta$ and \mathcal{S}_1 vector subspace in \mathcal{S}_0 , such that $(\mathcal{U} + \mathcal{S}_\delta) \cap \mathcal{S}_1 = \mathcal{O}$ and $\mathcal{S}_0 = \mathcal{U} + \mathcal{S}_\delta + \mathcal{S}_1$.

Suppose $p \in \bar{N}$, $\lambda = (\Sigma, \mathcal{S}_1) \in \Lambda$ and denote

$$(11) \quad \mathcal{S}_{p,\lambda} = \mathcal{V}_{\delta,p} \oplus \mathcal{U} \oplus \mathcal{S}(\Sigma) \oplus \mathcal{S}_1;$$

(12) $\mathcal{A}_{p,\lambda}$ the smallest subalgebra in \mathcal{W} , containing $\mathcal{S}_{p,\lambda}$ and invariant of the mapping $D: \mathcal{W} \rightarrow \mathcal{W}$;

$$(13) \quad \mathcal{I}_{p,\lambda} \text{ the vector subspace generated in } \mathcal{W} \text{ by } \mathcal{V}_{\delta,p} \cdot \mathcal{A}_{p,\lambda}.$$

Then (see [5] and [6])

$$(1) \quad A_{p,\lambda} = \mathcal{A}_{p,\lambda} / \mathcal{I}_{p,\lambda},$$

$$(2) \quad D: A_{p+1,\lambda} \rightarrow A_{p,\lambda} \text{ is given by}$$

$$D(t + \mathcal{I}_{p+1,\lambda}) = Dt + \mathcal{I}_{p,\lambda}, \quad \forall t \in \mathcal{A}_{p+1,\lambda}.$$

3. Multiplication by $1/x^m$, $m = 1, 2, \dots$ It is shown (see Corollary 2) that in the algebras $A_{p,\lambda}$, the multiplication by $1/x^m$ does not represent the division by x^m .

THEOREM 1. Suppose $T \in A_{p,\lambda}$, with given $p \in \bar{N}$, $\lambda \in \Lambda$. Suppose $\psi \in \mathcal{C}^\infty(R^1)$ such that for a certain $m \in \bar{N}$

$$D^q \psi(0) = 0, \quad \forall q \in N, \quad q \leq m.$$

If there exists $\chi \in \mathcal{C}^\infty(R^1)$ such that $\psi \cdot T = \chi$ in $A_{p,\lambda}$, then:

$$D^q \chi(0) = 0, \quad \forall q \in N, \quad q \leq \min\{p, m\}.$$

Proof. Assume $T = t + \mathcal{I}_{p,\lambda}$, with $t \in \mathcal{A}_{p,\lambda}$. Then $\psi \cdot T = \chi$ in $A_{p,\lambda}$ implies $u(\chi) = u(\psi) \cdot t + w$, with $w \in \mathcal{I}_{p,\lambda}$. Therefore,

$$\forall q \in N, q \leq p: \exists \nu_q \in N: \forall \nu \in N, \nu \geq \nu_q: D^q w(\nu)(0) = 0.$$

Since $\chi = \psi \cdot t(\nu) + w(\nu)$, $\forall \nu \in N$, the proof is completed.

COROLLARY 1. Suppose $T \in A_{p,\lambda}$, with given $p \in \bar{N}$, $\lambda \in \Lambda$.

If $\psi \in \mathcal{C}^\infty(R^1)$ such that $\psi(0) \neq 0$, then, $x^m \cdot T \neq \psi$ in $A_{p,\lambda}$, $\forall m \in N$, $m \geq 1$.

COROLLARY 2. If $m \in N$, $m \geq 1$, then, $x^m \cdot (1/x^m) \neq 1$, in each of the algebras $A_{p,\lambda}$, $p \in \bar{N}$, $\lambda \in \Lambda$.

4. Division by x^m , $m = 1, 2, \dots$ First, in Theorem 2, a

sufficient condition is given for $T \in A_{p,\lambda}$, in order to be a solution of the equation $x^m \cdot T = 0 \in A_{p,\lambda}$, where $m \in N$, $m \geq 1$.

For $p \in \bar{N}$ and $\lambda \in \Lambda$, denote by $B_{p,\lambda}^0$ all the elements $T \in A_{p,\lambda}$ of the form $T = t + \mathcal{J}_{p,\lambda}$, where $t \in \mathcal{A}_{p,\lambda} \cap \mathcal{V}_0$ and satisfies also (5.2) in §2.

PROPOSITION 1. *Suppose given $p \in \bar{N}$, $\lambda \in \Lambda$ and $\psi \in \mathcal{C}^\infty(R^1)$, such that, for a certain $q \in \bar{N}$, $q \geq p$:*

$$D^r \psi(0) = 0, \quad \forall r \in N, \quad r \leq q.$$

Then, $\psi \cdot B_{p,\lambda}^0 = \{0\} \subset A_{p,\lambda}$.

Proof. Assume $T \in B_{p,\lambda}^0$ and $T = t + \mathcal{J}_{p,\lambda}$, with $t \in \mathcal{A}_{p,\lambda} \cap \mathcal{V}_0$ and satisfying (5.2) in §2. Then, $\psi \cdot T = u(\psi) \cdot t + \mathcal{J}_{p,\lambda}$. But, obviously, $u(\psi) \cdot t \in \mathcal{V}_{\delta,q}^0 \subset \mathcal{V}_{\delta,p}^0 \subset \mathcal{J}_{p,\lambda}$, hence, $T = 0 \in A_{p,\lambda}$.

THEOREM 2. *Suppose given $p \in N$, $\lambda \in \Lambda$ and $m \in N$, $m \geq 1$. Then, any*

$$T_0 = \sum_{0 \leq i \leq k} x^{r_i} \cdot T_{1i} \cdot T_{2i} + \sum_{0 \leq j \leq h} x^{q_j} \cdot D^{p_j} \delta \cdot T_{3j},$$

with $k, h, r_i, q_j, p_j \in N$, $r_i > p - m$,

$q_j > \max\{p, p_j\} - m$,

and $T_{1i} \in B_{p,\lambda}^0$, $T_{2i}, T_{3j} \in A_{p,\lambda}$,

will be a solution in $A_{p,\lambda}$ of the equation $x^m \cdot T = 0$.

Proof. According to Proposition 1, $x^m \cdot x^{r_i} \cdot T_{1i} = x^{m+r_i} \cdot T_{1i} = 0 \in A_{p,\lambda}$, since $m + r_i > p$. According to (4) in §1 (see also 3) in Theorem 6, §8 [5]), $x^m \cdot x^{q_j} \cdot D^{p_j} \delta = x^{m+q_j} \cdot D^{p_j} \delta = 0 \in A_{p,\lambda}$, since $m + q_j > \max\{p, p_j\}$.

It results the following sufficient condition on $T \in A_{p,\lambda}$, solution of the equation $x^m \cdot T = S \in A_{p,\lambda}$.

COROLLARY 3. *Suppose $S \in A_{p,\lambda}$, with $p \in N$, $\lambda \in \Lambda$ given and $m \in N$, $m \geq 1$.*

If T_1 is any solution in $A_{p,\lambda}$ of the equation $x^m \cdot T = S$ and T_0 is given as in Theorem 2, then $T = T_1 + T_0$ will be again a solution of that equation.

Before a necessary condition is given on $T \in A_{p,\lambda}$, solution of the equation $x^m \cdot T = 0 \in A_{p,\lambda}$, the notion of *support* of the elements in $A_{p,\lambda}$ will be defined.

Suppose $T \in A_{p, \lambda}$, with $p \in \bar{N}$, $\lambda \in \Lambda$ given and $E \subset R^1$. Then,

(1) T vanishes on E , only if $T = t + \mathcal{J}_{p, \lambda}$, with $t \in \mathcal{A}_{p, \lambda}$, such that $t(\nu)(x) = 0$, $\forall \nu \in N$, $\nu \geq \nu_0$, $x \in E$.

(2) T strictly vanishes on E , only if T vanishes on a certain open set $G \subset R^1$, containing E .

(3) T is supported by E , only if for every open set $G \subset R^1$, containing E , one can write $T = t + \mathcal{J}_{p, \lambda}$, with $t \in \mathcal{A}_{p, \lambda}$, such that $\text{supp } t(\nu) \subset G$, $\forall \nu \in N$, $\nu \geq \nu_0$.

The support of T is defined as the closed set

$$\text{supp } T = R^1 \setminus \{x \in R^1 \mid T \text{ strictly vanishes on } \{x\}\}.$$

Obviously, for the distributions in $\mathcal{C}^\infty(R^1) \oplus \mathcal{D}'_\delta(R^1)$, the above notion of support is identical with the usual one for distributions.

PROPOSITION 2. Suppose $x_0 \in R^1$ and $q \in N$, then, $D^q \delta_{x_0} \in A_{p, \lambda}$, for $p \in \bar{N}$, $\lambda \in \Lambda$, and

- (1) $D^q \delta_{x_0}$ is supported by $\{x_0\}$ and $\text{supp } D^q \delta_{x_0} = \{x_0\}$,
- (2) if $E \subset R^1$ and $x_0 \notin \text{closure } E$, then $D^q \delta_{x_0}$ strictly vanishes on E ,
- (3) $D^q \delta_{x_0}$ does not vanish on $R^1 \setminus \{x_0\}$,
- (4) $D^q \delta_{x_0}$ does not vanish on $\{x_0\}$.

Proof. (1), (2) and (3) follow easily.

(4) Assume $\lambda = (\Sigma, \mathcal{J}_1)$ and $\Sigma = (s_x \mid x \in R^1)$, then, $D^q \delta_{x_0} = D^q s_{x_0} + \mathcal{J}_{p, \lambda}$ and $s_{x_0} \in \tau_{x_0} \mathcal{Z}_{\delta}^0$. Suppose, $D^q \delta_{x_0}$ vanishes on $\{x_0\}$, then, there exists $t \in \mathcal{A}_{p, \lambda}$, such that $t - D^q s_{x_0} \in \mathcal{J}_{p, \lambda}$ and $t(\nu)(x_0) = 0$, $\forall \nu \in N$, $\nu \geq \nu_0$. Denoting $v = t - D^q s_{x_0}$, the relation $v \in \mathcal{J}_{p, \lambda}$ implies $v(\nu)(x_0) = 0$, $\forall \nu \in N$, $\nu \geq \nu_1$. Therefore, it results

$$D^q s_{x_0}(\nu)(x_0) = t(\nu)(x_0) - v(\nu)(x_0) = 0, \quad \forall \nu \in N, \quad \nu \geq \nu_2.$$

But, that relation implies $W(s_{x_0}(\nu), \dots, s_{x_0}(\nu + q))(x_0) = 0$, $\forall \nu \in N$, $\nu \geq \nu_2$, which contradicts the assumption $s_{x_0} \in \tau_{x_0} \mathcal{Z}_{\delta}^0$.

REMARK. The property of the Dirac distributions that $D^q \delta_{x_0}$ does not vanish on $\{x_0\}$, $\forall x_0 \in R^1$, $q \in N$, is a direct consequence of the “condition of strong local presence” (see (5.3) in §2) and it is proper for the distribution multiplication presented in [5] and [6]. The “delta sequences” generally used (see [2]) do not necessarily prevent the vanishing of $D^q \delta_{x_0}$ on $\{x_0\}$.

THEOREM 3. Suppose $T \in A_{p, \lambda}$ with $p \in \bar{N}$, $\lambda \in \Lambda$ given.

If $x^m \cdot T = 0 \in A_{p, \lambda}$, for a certain $m \in N$, $m \geq 1$, then T is supported by $\{0\}$, hence $\text{supp } T \subset \{0\}$.

Proof. Assume $T = t + \mathcal{J}_{p, \lambda}$, with $t \in \mathcal{A}_{p, \lambda}$. Then $x^m \cdot T = 0 \in A_{p, \lambda}$ implies $u(x^m) \cdot t \in \mathcal{J}_{p, \lambda}$, therefore, according to the definition of $\mathcal{J}_{p, \lambda}$ (see (13), §2), it results

$$u(x^m) \cdot t = \sum_{0 \leq i \leq k} v_i \cdot a_i$$

with $k \in N$, $v_i \in \mathcal{V}_{\delta, p}$, $a_i \in \mathcal{A}_{p, \lambda}$.

Now, due to the definition $\mathcal{V}_{\delta, p}$ (see (8) and (6), §2), it follows that: $\forall i \in \{0, \dots, k\}: \exists X_i \subset R^1$, X_i finite: $v_i = \sum_{x \in X_i} v_{ix}$, where $v_{ix} \in \tau_x \mathcal{V}_{\delta, p}^0$.

Concluding, there exists $X \subset R^1$, X finite, such that

$$u(x^m) \cdot t = \sum_{x \in X} \sum_{0 \leq j \leq h} v_{xj} \cdot b_{xj} \quad \text{with} \quad h \in N, \quad v_{xj} \in \tau_x \mathcal{V}_{\delta, p}^0, \quad b_{xj} \in \mathcal{A}_{p, \lambda}.$$

It will be shown now, that in the above relation, one can consider $X = \{0\}$. Indeed, suppose $x_0 \in X \setminus \{0\}$, then $v_{x_0j} \in \tau_{x_0} \mathcal{V}_{\delta, p}^0$ with $0 \leq j \leq h$. The condition (5.2) in §2, results in the existence of $w_{x_0j} \in \mathcal{W}$, with $0 \leq j \leq h$, such that $v_{x_0j}(\nu)(x) = x^m \cdot w_{x_0j}(\nu)(x)$, $\forall 0 \leq j \leq h$, $x \in R^1$, $\nu \in N$, $\nu \geq \nu_0$. Moreover, $w_{x_0j} \in \tau_{x_0} \mathcal{V}_{\delta, p}^0$, $\forall 0 \leq j \leq h$, since $v_{x_0j} \in \tau_{x_0} \mathcal{V}_{\delta, p}^0$ with $0 \leq j \leq h$, and $x_0 \neq 0$.

Denoting

$$v = \sum_{\substack{x_0 \in X \\ x_0 \neq 0}} \sum_{0 \leq j \leq h} w_{x_0j} \cdot b_{x_0j}$$

it results $v \in \mathcal{J}_{p, \lambda}$, hence, $T = t_1 + \mathcal{J}_{p, \lambda}$, where $t_1 = t - v \in \mathcal{A}_{p, \lambda}$. But $u(x^m) \cdot t_1 = u(x^m) \cdot t - u(x^m) \cdot v = \sum_{0 \leq j \leq h} v_{0j} \cdot b_{0j}$.

Since v_{0j} , with $0 \leq j \leq h$, satisfy (5.2) in §2, it follows that $u(x^m) \cdot t_1$ and, therefore t_1 satisfy the same condition. Thus, $T = t_1 + \mathcal{J}_{p, \lambda}$ is supported by $\{0\}$, which obviously results in $\text{supp } T \subset \{0\}$.

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