Pacific Journal of Mathematics

PARTIAL REGULARITY OF SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

VLADIMIR SCHEFFER

Vol. 66, No. 2 December 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 66, No. 2, 1976

PARTIAL REGULARITY OF SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

VLADIMIR SCHEFFER

At the first instant of time when a viscous incompressible fluid flow with finite kinetic energy in three space becomes singular, the singularities in space are concentrated on a closed set whose one dimensional Hausdorff measure is finite.

§1. Introduction. Let $v: R^3 \times R^+ \to R^3$ (where $R^+ = \{t \in R: t > 0\}$ represents time) be a weak solution to the Navier–Stokes equations of incompressible viscous fluid flow in 3 dimensional euclidean space with finite initial kinetic energy and viscosity equal to 1. Our definition of weak solution coincides with Leray's definition of "solution turbulente" [**4**, pp. 240, 241, 235]. In that paper, Leray showed that weak solutions always exist for prescribed initial conditions with finite energy. He also proved the following regularity theorem:

LERAY'S THEOREM. *There exists a finite or countable sequence J_0, J_1, $J_2, \cdots$ such that $J_q \subset R^+$, $J_0 = \{t: t > a\}$ for some a, J_q is an open interval for $q > 0$, the J_q are disjointed, the Lebesgue measure of $R^+ - \bigcup_{q \geqq 0} J_q$ is zero, v can be modified on a set of Lebesgue measure zero so that its restriction to each $R^3 \times J_q$ becomes smooth, and*

$$\sum_{q>0} (\text{length}\,(J_q))^{1/2}$$

is finite.

It is not known whether there exist v with singularities ($J_0 = R^+$ is a possibility). The purpose of this paper is to prove the following theorem on the nature of possible singularities of v. We assume that v has been modified to be smooth on each $R^3 \times J_q$.

THEOREM 1. *Let t_0 be the right endpoint of an interval J_q with $q > 0$. Then there exists a closed set $S \subset R^3$ such that v can be extended to a continuous function on*

$$(R^3 \times J_q) \cup ((R^3 - S) \times \{t_0\})$$

and the 1 dimensional Hausdorff measure of S is finite.

The definition of Hausdorff measure can be found in [**2**, p. 171]. We note in passing that Leray's theorem yields

THEOREM 2. *The* 1/2 *dimensional Hausdorff measure of* $R^+ - \bigcup_{q \geqq 0} J_q$ *is zero.*

There is a proof of Theorem 2 in [**7**]. Research on the Hausdorff dimension of singularities of fluid flow was started by Mandelbrot [**5**]. The conclusion of Theorem 1 resembles the partial regularity results in [**1**, IV. 13 (6), p. 126].

Leray's theorem has been generalized by M. Shinbrot and S. Kaniel to flows on a bounded domain [**8**]. I do not know whether Theorem 1 generalizes to that case.

NOTATION. We set $(a, b) = \{t\colon a < t < b\}$, $[a, b) = \{t\colon a \leqq t < b\}$, and so on for $(a, b]$ and $[a, b]$. If f is a function defined on a subset of $R^3 \times R$ then $f_{,i}$, $f_{,ij}$, etc. are the partial derivatives $(\partial/\partial x_i)f$, $(\partial^2/\partial x_i \partial x_j)f$, etc. where x_1, x_2, x_3 are the coordinates of R^3. The partial derivative with respect to the R variable is denoted by $f_{,t}$. We set $D^0 f = f$, $D^1 f = Df = (f_{,1}, f_{,2}, f_{,3})$, $D^2 f = (f_{,ij})$ for $i, j \in \{1, 2, 3\}$, and so forth for $D^n f$. We let $|D^n f(x, t)|$ be the euclidean norm. If, in addition, f has range R^3 then f_i is the corresponding component of f for $i = 1, 2, 3$. In that case we set $\operatorname{div}(f) = \Sigma_{i=1}^3 f_{i,i}$. The summation convention for repeated indices is used throughout, e.g. $\operatorname{div}(f) = f_{i,i}$. If f is a function defined on a subset of R^3 then $Df(x)$ and $|Df(x)|$ are the gradient and its norm.

An *absolute constant* is a finite positive constant that does not depend on any of the parameters in this paper. The symbol C will always denote an absolute constant, and the value of C may change from one line to the next (e.g. $2C \leqq C$). The symbols $C_1, C_2, C_3, \cdots$ are not treated in this way, and their value does not change in the course of the paper.

We begin to prove Theorem 1. Let $\phi\colon R^3 \times \{t\colon t < 0\} \to R^+$ be defined by

$$\phi(x, t) = (2\sqrt{\pi})^{-3}(-t)^{-3/2} \exp(|x|^2/(4t)). \tag{1.1}$$

Since ϕ is just the fundamental solution to the heat equation running backwards in time, it satisfies

$$\phi_{,ii} = -\phi_{,t} \tag{1.2}$$

and

$$\lim_{\epsilon \downarrow 0} \int_{R^3} f(y, t-\epsilon)\phi(y - x, -\epsilon)\,dy = f(x, t)$$

if f is continuous at (x, t) and $\int_{R^3} |f(y, s)|^2 dy$ is bounded as a function of s. We also define ψ: $R^3 \times \{t: t < 0\} \to R^+$ by

$$\psi(x, t) = -(4\pi)^{-1} \int_{R^3} \phi(y, t) |y - x|^{-1} dy. \tag{1.3}$$

This Newtonian potential of ϕ satisfies the Poisson equation

$$\psi_{,ii} = \phi. \tag{1.4}$$

We have the estimates

$$\begin{aligned} |D^n \phi(x, t)| &\leqq E_n (|x|^2 - t)^{-(n+3)/2}, \\ |D^n \psi(x, t)| &\leqq E_n (|x|^2 - t)^{-(n+1)/2} \end{aligned} \tag{1.5}$$

where E_n is an absolute constant for each n.

Two consequences of the definition of weak solution are:

$$\begin{aligned} &\int_{R^3} |v(x, t)|^2 dx \leqq C_1 \quad \text{if} \quad t \in \bigcup_{q \geqq 0} J_q \\ &\int_{R^3 \times R^+} |Dv|^2 \leqq C_1 \end{aligned} \tag{1.6}$$

for some $C_1 < \infty$, and

$$\operatorname{div}(v)(x, t) = 0 \quad \text{if} \quad t \in \bigcup_{q \geqq 0} J_q. \tag{1.7}$$

A third consequence is the following lemma:

LEMMA 1.1. *If $[t_1, t_2] \subset J_q$ then for $i \in \{1, 2, 3\}$ and $x \in R^3$ we have*

$$\begin{aligned} &v_i(x, t_2) \\ &\quad = \int_{R^3} v_i(y, t_1) \phi(y - x, t_1 - t_2) dy \\ &\quad + \int_{t_1}^{t_2} \int_{R^3} v_j(y, t) v_i(y, t) \phi_{,j}(y - x, t - t_2) dy dt \\ &\quad - \int_{t_1}^{t_2} \int_{R^3} v_j(y, t) v_k(y, t) \psi_{,ijk}(y - x, t - t_2) dy dt. \end{aligned} \tag{1.8}$$

Proof. We fix $i \in \{1,2,3\}$ and $x \in R^3$. Let $f\colon R^3 \times \{t\colon t < t_2\} \to R^3$ be given by

$$(1.9)\qquad \begin{aligned} f_j(y,t) &= \phi(y-x,t-t_2) - \psi_{,ij}(y-x,t-t_2) \quad \text{if} \quad j = i,\\ f_j(y,t) &= -\psi_{,ij}(y-x,t-t_2) \quad \text{if} \quad j \neq i. \end{aligned}$$

We were careful not to write $\psi_{,ii}$ in the first identity of (1.9) because there is no summation over the index i. Using (1.4) we obtain

$$(1.10)\qquad \begin{aligned} \operatorname{div}(f)(y,t) &= \phi_{,i}(y-x,t-t_2) - \psi_{,ijj}(y-x,t-t_2)\\ &= \phi_{,i}(y-x,t-t_2) - \phi_{,i}(y-x,t-t_2) = 0. \end{aligned}$$

Now take $0 < \epsilon < t_2 - t_1$. The definition of weak solution, (1.10), and the good behavior of f on $R^3 \times [t_1, t_2 - \epsilon]$ allow us to write (see (1.6))

$$(1.11)\qquad \begin{aligned} &\int_{R^3} v_j(y,t_2-\epsilon) f_j(y,t_2-\epsilon)\,dy\\ &\quad - \int_{R^3} v_j(y,t_1) f_j(y,t_1)\,dy\\ &= \int_{R^3\times[t_1,t_2-\epsilon]} (v_j)(f_{j,kk} + f_{j,t})\\ &\quad - \int_{R^3\times[t_1,t_2-\epsilon]} v_k v_{j,k} f_j. \end{aligned}$$

Integration by parts with respect to the x_j and x_k variables, (1.6), and (1.7) yield

$$(1.12)\qquad \begin{aligned} &\int_{R^3} v_j(y,t_2-\epsilon)\psi_{,ij}(y-x,-\epsilon)\,dy = 0,\\ &\int_{R^3} v_j(y,t_1)\psi_{,ij}(y-x,t_1-t_2)\,dy = 0,\\ &\int_{t_1}^{t_2-\epsilon}\int_{R^3} v_j(y,t)(\psi_{,ijkk}(y-x,t-t_2)\\ &\qquad + \psi_{,ijt}(y-x,t-t_2))\,dy\,dt = 0,\\ &\int_{R^3\times[t_1,t_2-\epsilon]} v_k v_{j,k} f_j\\ &\qquad = -\int_{R^3\times[t_1,t_2-\epsilon]} v_k v_j f_{j,k}. \end{aligned}$$

Identities (1.9), (1.11), (1.12), (1.2) yield

$$
\begin{aligned}
&\int_{R^3} v_i(y, t_2-\epsilon)\phi(y-x,-\epsilon)dy \\
&\qquad -\int_{R^3} v_i(y, t_1)\phi(y-x, t_1-t_2)dy \\
&\qquad =\int_{t_1}^{t_2-\epsilon}\int_{R^3} v_i(y,t)(\phi_{,kk}(y-x,t-t_2) \\
&\qquad + \phi_{,t}(y-x,t-t_2))dydt \qquad (1.13)\\
&\qquad +\int_{R^3\times[t_1,t_2-\epsilon]} v_k v_j f_{j,k} \\
&\qquad =0+\int_{t_1}^{t_2-\epsilon}\int_{R^3} v_k(y,t)v_i(y,t)\phi_{,k}(y-x,t-t_2)dydt \\
&\qquad -\int_{t_1}^{t_2-\epsilon}\int_{R^3} v_k(y,t)v_j(y,t)\psi_{,ijk}(y-x,t-t_2)dydt.
\end{aligned}
$$

Now (1.13), (1.6), and (1.2) yield the conclusion of the lemma.

For $a \in R^3$ and $0<r<\infty$ we set

$$B(a,r)=\{x \in R^3 : |x-a| \leqq r\}. \tag{1.14}$$

If X is a set and $f: X \to R$ is a function we write

$$\sup(f, X) = \text{supremum}\{f(x): x \in X\}. \tag{1.15}$$

LEMMA 1.2. *Let $f: B(a,r) \to R$ be a smooth function and let $B(b, r/4) \subset B(a,r)$. Then*

$$\int_{B(a,r)} |f|^2 \leqq Cr^2\left(\int_{B(a,r)} |Df|^2\right) + Cr^3 \sup(|f|^2, B(b, r/4)).$$

Proof. Let $\mathscr{L}$ be the set of lines L passing through b. Let μ be the rotation invariant Radon measure on $\mathscr{L}$ that satisfies $\mu(\mathscr{L})=1$. For each $L \in \mathscr{L}$ the fundamental theorem of calculus yields

$$
\begin{aligned}
&\int_{B(a,r)\cap L} |f|^2 \\
&\qquad \leqq Cr^2\left(\int_{(B(a,r)-B(b,r/4))\cap L} |Df|^2\right) \\
&\qquad\qquad + C\sup(|f|^2, B(b,r/4)\cap L)r.
\end{aligned}
$$

Hence

$$\begin{aligned}\int_{B(a,r)} |f|^2 &\leqq Cr^2 \int_{\mathscr{L}} \left(\int_{B(a,r)\cap L} |f|^2 \right) d\mu \\ &\leqq Cr^4 \int_{\mathscr{L}} \left(\int_{(B(a,r)-B(b,r/4))\cap L} |Df|^2 \right) d\mu \\ &\quad + Cr^3 \sup(|f|^2, B(b,r/4)) \\ &\leqq Cr^2 \left(\int_{B(a,r)-B(b,r/4)} |Df|^2 \right) \\ &\quad + Cr^3 \sup(|f|^2, B(b,r/4)).\end{aligned}$$

2. The basic estimate. Throughout this section we fix $0 < d_0 < (\text{length}\,(J_q))^{1/2}$, where J_q is the interval in the hypotheses of Theorem 1, and we fix $x_0 \in R^3$. We define $u\colon R^3 \times [-1,0) \to R^3$ by

$$u(x,t) = d_0 v(x_0 + d_0 x, t_0 + d_0^2 t), \tag{2.1}$$

where t_0 is the right endpoint of J_q as in Theorem 1, and observe that u satisfies the Navier–Stokes equations with viscosity 1 in the same way as v. Therefore Lemma 1.1 allows us to use the identity

$$\begin{aligned}u_i(x,t) &= \int_{R^3} u_\iota(y,-1)\phi'(y,-1)dy \\ &\quad + \left(\int_{R^3\times[-1,t\}} u_j u_i \phi'_{,j} \right) \\ &\quad - \int_{R^3\times[-1,t]} u_j u_k \psi'_{,ijk}\end{aligned} \tag{2.2}$$

for $-1 < t < 0$, where

$$\phi'(y,s) = \phi(y-x, s-t),\ \psi'(y,s) = \psi(y-x, s-t). \tag{2.3}$$

We also set

$$\begin{aligned}A_p &= \{(y,s) \in R^3 \times R : |y| \leqq 1 - 2^{-p}, 2^{-2p} - 1 \leqq s < 0\} \\ B_p &= \{(y,s) \in R^3 \times R : 1 - 2^{1-p} \leqq |y| \leqq 1 + 2^{1-p}, -1 \leqq s \leqq 0\} \\ C_t &= \{(y,s) \in R^3 \times R : -1 \leqq s \leqq t\} \\ D &= \{(y,s) \in R^3 \times R : |y| \geqq 3/2, -1 \leqq s \leqq 0\} \\ E &= \{y \in R^3 : |y| \geqq 3/2\} \\ F &= \{y \in R^3 : |y| \leqq 2\}\end{aligned} \tag{2.4}$$

for $p = 1, 2, 3, \cdots$ and $-1 < t < 0$. In addition we set

$$A_0 = \varnothing, \quad B_{-2} = B_{-1} = B_0 = B_1. \tag{2.5}$$

LEMMA 2.1. *There exist absolute constants C_2, C_3 such that*

$$\begin{aligned} |u(x,t)| \leqq {} & C_3(t+1)^{-1/2}\int_{R^3} |u(y,-1)|^2(1+|y|)^{-4}dy \\ & + C_3(t+1)^{-3/2}\int_{C_t} |u(y,s)|^2(1+|y|)^{-4}dyds \\ & + C_3(t+1)^{-1/2}\int_F |Du(y,-1)|^2 dy \\ & + C_3(t+1)^{-3/2}\left(\int_{B_1\cap C_t} |Du|^2\right) \\ & + C_3\left(\sum_{p=1}^{n+1} 2^{2p}\int_{B_p} |Du|^2\right) \\ & + C_2\left(\sum_{p=1}^{n+3} 2^{-p}\sup(|u|^2, A_p\cap C_t)\right) + C_2^{-1}2^{-12} \end{aligned} \tag{2.6}$$

holds if $(x,t) \in A_{n+1} - A_n$ for $n \geqq 0$.

Proof. We fix $(x,t) \in A_{n+1} - A_n$ and define ϕ', ψ' as in (2.3). We set

$$G_p = \{(y,s) \in R^3 \times R : |y - x| \leqq 2^{1-p}, t - 2^{-2p} \leqq s \leqq t\} \tag{2.7}$$

for integers $p \geqq 2$. We have

$$G_{n+4} \subset G_{n+3} \subset A_{n+2} \cap C_t. \tag{2.8}$$

The integer m is defined by the relation

$$2^{4-2(m-1)} > t + 1 \geqq 2^{4-2m}. \tag{2.9}$$

The requirement $(x,t) \in A_{n+1}$, (2.9), and $t + 1 < 1$ yield

$$3 \leqq m \leqq n + 3, G_p \subset C_t \quad \text{for} \quad p \geqq m. \tag{2.10}$$

For $p \in \{2, 3, 4, \cdots\}$ the point $x_p \in R^3$ is defined as follows: If $x \neq 0$ then $x_p = x - 3 \cdot 2^{-1-p}|x|^{-1}x$, and if $x = 0$ we choose x_p so that $|x_p| = 3 \cdot 2^{-1-p}$ holds. We then set

$$H_p = \{(y, s): |y - x_p| \leqq 2^{-1-p}, t - 2^{-2p} \leqq s \leqq t\}.$$

Then $H_p \subset G_p$ holds and (2.9), (2.10), and $|x| < 1$ yield

$$H_p \subset A_p \cap C_t \quad \text{for} \quad p \geqq m. \tag{2.11}$$

We set $C'_s = R^3 \times \{s\}$. For $s \in [t - 2^{-2p}, t]$ Lemma 1.2 yields

$$\begin{aligned} &\int_{G_p \cap C'_s} |u|^2 \\ &\leqq C2^{-2p}\left(\int_{G_p \cap C'_s} |Du|^2\right) + C2^{-3p} \sup(|u|^2, H_p \cap C'_s). \end{aligned} \tag{2.12}$$

Integration of (2.12) with respect to s and (2.11) yield

$$\int_{G_p} |u|^2 \leqq C2^{-2p}\left(\int_{G_p} |Du|^2\right) + C2^{-5p} \sup(|u|^2, A_p \cap C_t) \text{ if } p \geqq m. \tag{2.13}$$

Observing $G_{m+1} \subset G_m \subset B_1$, $B_1 \cup D = C_0$, $D \cap G_m = \emptyset$, we let f_1, f_2, f_3 be smooth functions from C_t into $[0, 1]$ such that $f_1 + f_2 + f_3 = 1$, $f_1(y, s) = 1$ for $(y, s) \notin B_1$, $f_1(y, s) = 0$ for $(y, s) \notin D$, $f_2(y, s) = 0$ for $(y, s) \notin B_1$, $f_2(y, s) = 0$ for $(y, s) \in G_{m+1}$, $f_2(y, s) = 1$ for $(y, s) \notin D \cup G_m$, $|Df_2(y, s)| \leqq C$ for $(y, s) \in D \cap B_1$, $|Df_2(y, s)| \leqq C2^m$ for $(y, s) \in G_m - G_{m+1}$, $f_3(y, s) = 0$ for $(y, s) \notin G_m$ and $f_3(y, s) = 1$ for $(y, s) \in G_{m+1}$ (note that f_j is defined *only on* C_t). Using (1.5) and $x \in A_{n+1}$ we obtain

$$\begin{aligned} &\left|\int_{C_t} u_j u_i \phi'_{,j} f_1\right| + \left|\int_{C_t} u_j u_k \psi'_{,ijk} f_1\right| \\ &\leqq C\int_{D \cap C_t} |u(y, s)|^2 |y|^{-4} dy ds. \end{aligned} \tag{2.14}$$

We use integration by parts, (1.7), (1.5), the inequality $ab \leqq \epsilon a^2/2 + \epsilon^{-1}b^2/2$, (2.13), and (2.9) to estimate

$$\begin{aligned} &\left|\int_{C_t} u_j u_i \phi'_{,j} f_2\right| + \left|\int_{C_t} u_j u_k \psi'_{,ijk} f_2\right| \\ &\leqq \left|\int_{C_t} u_j u_{i,j} \phi' f_2\right| + \left|\int_{C_t} u_j u_i \phi' f_{2,j}\right| \\ &\quad + \left|\int_{C_t} u_j u_{k,j} \psi'_{,ik} f_2\right| + \left|\int_{C_t} u_j u_k \psi'_{,ik} f_{2,j}\right| \\ &\leqq C\left(\int_{(B_1 \cap C_t) - G_{m+1}} |u||Du|(|\phi'| + |D^2\psi'|)\right) \end{aligned}$$

$$
\begin{aligned}
&+C\left(\int_{D\cap B_1\cap C_t}|u|^2(|\phi'|+|D^2\psi'|)\right)\\
&+C\int_{G_m-G_{m+1}}|u|^2(|\phi'|+|D^2\psi'|)2^m\\
(2.15)\qquad &\leqq C\left(\int_{B_1\cap C_t}|u|\,|Du|\,2^{3m}\right)+C\left(\int_{B_1\cap C_t}|u|^2\right)+C\int_{G_m}|u|^2 2^{4m}\\
&\leqq C2^{3m}\left(\int_{B_1\cap C_t}|u|^2\right)+C2^{3m}\left(\int_{B_1\cap C_t}|Du|^2\right)\\
&+C2^{2m}\left(\int_{G_m}|Du|^2\right)+C2^{-m}\sup(|u|^2,A_m\cap C_t)\\
&\leqq C(t+1)^{-3/2}\left(\int_{B_1\cap C_t}|u|^2\right)\\
&+C(t+1)^{-3/2}\left(\int_{B_1\cap C_t}|Du|^2\right)\\
&+C2^{2m}\left(\int_{G_m}|Du|^2\right)+C2^{-m}\sup(|u|^2,A_m\cap C_t).
\end{aligned}
$$

We use (2.10), (1.5), (2.13), (2.8), and (2.10) to estimate

$$
\begin{aligned}
&\left|\int_{C_t}u_ju_i\phi'_{,j}f_3\right|+\left|\int_{C_t}u_ju_k\psi'_{,ijk}f_3\right|\\
&\leqq C\int_{G_m}|u|^2(|D\phi'|+|D^3\psi'|)\\
&\leqq C\left(\sum_{p=m}^{n+3}\int_{G_p-G_{p+1}}|u|^2(|D\phi'|+|D^3\psi'|)\right)\\
&+C\int_{G_{n+4}}|u|^2(|D\phi'|+|D^3\psi'|)\\
(2.16)\qquad &\leqq C\left(\sum_{p=m}^{n+3}2^{4p}\int_{G_p}|u|^2\right)\\
&+C\left(\int_{G_{n+4}}|D\phi'|+|D^3\psi'|\right)\sup(|u|^2,G_{n+4})\\
&\leqq C\left(\sum_{p=m}^{n+3}2^{2p}\int_{G_p}|Du|^2\right)+C\left(\sum_{p=m}^{n+3}2^{-p}\sup(|u|^2,A_p\cap C_t)\right)\\
&+C2^{-n}\sup(|u|^2,A_{n+2}\cap C_t)\\
&\leqq C\left(\sum_{p=m}^{n+3}2^{2p}\int_{G_p}|Du|^2\right)+C\left(\sum_{p=1}^{n+3}2^{-p}\sup(|u|^2,A_p\cap C_t)\right).
\end{aligned}
$$

Combining (2.14), (2.15), (2.16), (2.10), $0 < t+1 < 1$, and $f_1 + f_2 + f_3 = 1$ we obtain

$$
\begin{aligned}
&\left| \int_{C_t} u_j u_i \phi'_{,j} \right| + \left| \int_{C_t} u_j u_k \psi'_{,ijk} \right| \\
&\leqq C(t+1)^{-3/2} \int_{C_t} |u(y,s)|^2 (1+|y|)^{-4} dy ds \\
&+ C(t+1)^{-3/2} \left(\int_{B_1 \cap C_t} |Du|^2 \right) \\
&+ C \left(\sum_{p=m}^{n+3} 2^{2p} \int_{G_p} |Du|^2 \right) \\
&+ C \left(\sum_{p=1}^{n+3} 2^{-p} \sup (|u|^2, A_p \cap C_t) \right).
\end{aligned} \tag{2.17}
$$

Since $(x,t) \notin A_n$, we know that either (I) $|x| \geqq 1 - 2^{-n}$ or (II) $t+1 \leqq 2^{-2n}$ holds. If (I) is satisfied then $G_p \subset B_{p-4}$ for $m \leqq p \leqq n+3$ (see (2.4), (2.5), (2.7), (2.10), and use $(x,t) \in A_{n+1}$) and hence (see (2.5))

$$
\sum_{p=m}^{n+3} 2^{2p} \int_{G_p} |Du|^2 \leqq C \sum_{p=1}^{n+1} 2^{2p} \int_{B_p} |Du|^2 \tag{2.18}
$$

if (I) holds. If, on the other hand, (II) holds then (2.9) yields $m \geqq n+2$ and hence (2.9), (2.10), and (2.7) yield

$$
\sum_{p=m}^{n+3} 2^{2p} \int_{G_p} |Du|^2 \leqq C(t+1)^{-1} \int_{B_1 \cap C_t} |Du|^2 \tag{2.19}
$$

if (II) holds. Hence (2.18), (2.19), and $0 < t+1 < 1$ yield

$$
\sum_{p=m}^{n+3} 2^{2p} \int_{G_p} |Du|^2 \leqq C \left(\sum_{p=1}^{n+1} 2^{2p} \int_{B_p} |Du|^2 \right) + C(t+1)^{-3/2} \int_{B_1 \cap C_t} |Du|^2. \tag{2.20}
$$

Let g_1, g_2 be smooth functions from R^3 into $[0,1]$ such that (see (2.4)) $g_1 + g_2 = 1$, $g_1 = 1$ outside F, $g_2 = 1$ outside E, $|Dg_1| \leqq C$, and $|Dg_2| \leqq C$. Using (1.1) (*not* (1.5)) we estimate

$$
\left| \int_{R^3} u_i(y,-1) \phi'(y,-1) g_1(y) dy \right| \leqq C \int_E |u(y,-1)| \, |y|^{-4} dy. \tag{2.21}
$$

We use the inequality

$$
\int_{R^3} |f|^6 \leqq C \left(\int_{R^3} |Df|^2 \right)^3,
$$

valid for smooth functions $f\colon R^3 \to R$ with compact support [**3**, p. 12], Hölder's inequality, and (1.1) to compute

$$
\begin{aligned}
&\left| \int_{R^3} u_i(y,-1)\phi'(y,-1)g_2(y)\,dy \right| \\
&\leqq \int_{R^3} |g_2(y)u(y,-1)|\,|\phi'(y,-1)|\,dy \\
&\leqq \left(\int_{R^3} |g_2(y)u(y,-1)|^6 dy\right)^{1/6} \left(\int_F |\phi'(y,-1)|^{6/5} dy\right)^{5/6} \\
&\leqq C\left(\int_{R^3} (|Dg_2(y)|\,|u(y,-1)| \right. \\
&\qquad \left. + |g_2(y)|\,|Du(y,-1)|)^2 dy\right)^{1/2} (t+1)^{-1/4} \\
&\leqq C(t+1)^{-1/4}\left(\int_F |u(y,-1)|^2 dy\right)^{1/2} \\
&\qquad + C(t+1)^{-1/4}\left(\int_F |Du(y,-1)|^2 dy\right)^{1/2}
\end{aligned}
\tag{2.22}
$$

Now we combine (2.17), (2.20), (2.21), (2.22), $g_1 + g_2 = 1$, and (2.2) to write

$$
\begin{aligned}
&|u(x,t)| \\
&\leqq C_2\left(\int_E |u(y,-1)|\,|y|^{-4} dy\right) \\
&\qquad + C_2(t+1)^{-1/4}\left(\int_F |u(y,-1)|^2 dy\right)^{1/2} \\
&\qquad + C_2(t+1)^{-1/4}\left(\int_F |Du(y,-1)|^2 dy\right)^{1/2} \\
&\qquad + C_2(t+1)^{-3/2}\left(\int_{C_t} |u(y,s)|^2 (1+|y|)^{-4} dy\,ds\right) \\
&\qquad + C_2(t+1)^{-3/2}\left(\int_{B_1 \cap C_t} |Du|^2\right) \\
&\qquad + C_2\left(\sum_{p=1}^{n+1} 2^{2p} \int_{B_p} |Du|^2\right) \\
&\qquad + C_2\left(\sum_{p=1}^{n+3} 2^{-p} \sup(|u|^2, A_p \cap C_t)\right),
\end{aligned}
\tag{2.23}
$$

where C_2 is fixed (see §1). For $\epsilon > 0$ we can use the inequality $ab \leqq \epsilon a^2/2 + \epsilon^{-1} b^2/2$ to write

$$
\begin{aligned}
&\int_E |u(y,-1)|\,|y|^{-4}dy \\
(2.24)\qquad &= \int_E (|u(y,-1)|\,|y|^{-2})(|y|^{-2})dy \\
&\leqq (\epsilon^{-1}/2)\left(\int_E |u(y,-1)|^2|y|^{-4}dy\right) + (\epsilon/2)\left(\int_E |y|^{-4}dy\right)
\end{aligned}
$$

and, for $w = u$ or $w = Du$,

$$
\begin{aligned}
&(t+1)^{-1/4}\left(\int_F |w(y,-1)|^2dy\right)^{1/2} \\
(2.25)\qquad &\leqq (\epsilon^{-1}/2)(t+1)^{-1/2}\left(\int_F |w(y,-1)|^2dy\right) + \epsilon/2.
\end{aligned}
$$

Since $\int_E |y|^{-4}dy$ is finite and C_2 is fixed, we can choose $\epsilon > 0$ so that

$$
(2.26)\qquad C_2\left((\epsilon/2)\left(\int_E |y|^{-4}dy\right) + \epsilon\right) \leqq C_2^{-1}2^{-12}
$$

holds. Now (2.23), (2.24), (2.25), (2.26), and $0 < t+1 < 1$ yield (2.6).

LEMMA 2.2. *There exists an absolute constant $\epsilon > 0$ such that the following holds: If the conditions*

$$
\begin{aligned}
&(t+1)^{-1}\int_{C_t} |u(y,s)|^2(1+|y|)^{-4}dyds \leqq \epsilon, \\
(2.27)\qquad &(t+1)^{-1}\int_{B_1\cap C_t} |Du|^2 \leqq \epsilon, \\
&2^p\int_{B_p} |Du|^2 \leqq \epsilon
\end{aligned}
$$

are satisfied for all $t \in (-1,0)$ and $p \in \{1,2,3,\cdots\}$ then u can be extended continuously to the closure of A_1 in $R^3 \times R$.

Proof. We choose $\epsilon > 0$ so that

$$
(2.28)\qquad (12)\,C_3\epsilon \leqq C_2^{-1}2^{-12}
$$

holds (see Lemma 2.1). Let $f\colon \bigcup_{n=1}^{\infty} A_n \to R^+$ be a continuous function satisfying

$$C_2^{-1}2^{n-10} \leqq f(x,t) \leqq C_2^{-1}2^{n-7} \quad \text{if} \quad (x,t) \in A_{n+1} - A_n, \tag{2.29}$$

where $n \geqq 0$ (see (2.5)). We wish to show that (2.27) implies

$$|u(x,t)| \leqq f(x,t) \quad \text{for all} \quad (x,t) \in \bigcup_{n=1}^{\infty} A_n. \tag{2.30}$$

Assume, to the contrary, that (2.27) holds but (2.30) does not. Since u is continuous on $R^3 \times [-1, 0)$ (see first paragraph of §2) and the continuous function $f(x,t)$ tends to ∞ as (x,t) tends to

$$\{(x,-1): |x| \leqq 1\} \cup \{(x,t): |x| = 1, -1 \leqq t < 0\},$$

there must exist $(x,t) \in \bigcup_{n=1}^{\infty} A_n$ such that (2.31) and (2.32) hold:

$$|u(x,t)| = f(x,t) \tag{2.31}$$

$$|u(y,s)| \leqq f(y,s) \quad \text{if} \quad (y,s) \in \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad s \leqq t. \tag{2.32}$$

Taking the limit as t tends to -1 in (2.27) and using Fatou's lemma we obtain (recall (2.4))

$$\begin{aligned} &\int_{R^3} |u(y,-1)|^2 (1+|y|)^{-4} dy \leqq \epsilon, \\ &\int_F |Du(y,-1)|^2 dy \leqq \epsilon. \end{aligned} \tag{2.33}$$

We define n by the condition $(x,t) \in A_{n+1} - A_n$ and use Lemma 2.1, (2.33), (2.27), (2.32), the inequality $t+1 \geqq 2^{-2(n+1)}$ (which follows from $(x,t) \in A_{n+1}$), (2.29), (2.28), and $n \geqq 0$ to write

$$\begin{aligned} &|u(x,t)| \\ &\leqq 4C_3(t+1)^{-1/2}\epsilon + C_3\left(\sum_{p=1}^{n+1} 2^p \epsilon\right) \\ &\quad + C_2\left(\sum_{p=1}^{n+3} 2^{-p} \sup(f^2, A_p \cap C_t)\right) + C_2^{-1}2^{-12} \\ &\leqq C_3 2^{n+3}\epsilon + C_3 2^{n+2}\epsilon + C_2\left(\sum_{p=1}^{n+3} 2^{-p}(C_2^{-1}2^{p-8})^2\right) + C_2^{-1}2^{-12} \\ &\leqq C_2^{-1}2^{n-12} + C_2^{-1}2^{n-12} + C_2^{-1}2^{-12} \\ &\leqq (3/4)C_2^{-1}2^{n-10} \leqq (3/4) f(x,t). \end{aligned} \tag{2.34}$$

However, (2.34) contradicts (2.31) since $|u(x,t)| = f(x,t)$ is positive. Hence (2.27) implies (2.30).

We set $A = B(0, 1/4) \times [-3/16, 0)$ (see (1.14)). From (2.30) and (2.29) we conclude that $|u|$ is bounded on A_2. Hence the integrability of $D\phi$ and $D^3\psi$ on A (see (1.5)), the boundedness of $D\phi$, $D^3\psi$ outside A, (1.6) and (1.1) allow us to extend the domain of definition of u to include the closure of A_1 by substitution of $t = 0$ in (2.2). The above integrability property allows us to construct infinite sequences of continuous functions ${}^m f_j$ and ${}^m g_{ijk}$ for $m = 1, 2, 3, \cdots$ and $i, j, k \in \{1, 2, 3\}$ such that the restrictions of ${}^m f_j$ and ${}^m g_{ijk}$ to A converge as $m \to \infty$ to $\phi_{,j}$ and $\psi_{,ijk}$, respectively, in the L^1 norm; and such that ${}^m f_j$, ${}^m g_{ijk}$ coincide with $\phi_{,j}$, $\psi_{,ijk}$ outside A. We use (1.1), (1.5), (1.6) to define

$$
\begin{aligned}
{}^m u_i(x,t) = \int_{R^3} & u_i(y,-1)\phi'(y,-1)dy \\
&+ \int_{R^3 \times [-1,t]} (u_j u_i ({}^m f'_j) - u_j u_k ({}^m g'_{ijk}))
\end{aligned}
$$

for $-1 < t \leqq 0$, where ϕ' is as in (2.3), ${}^m f'_j(y,s) = {}^m f_j(y - x, s - t)$, ${}^m g'_{ijk}(y,s) = {}^m g_{ijk}(y - x, s - t)$. The statements in this paragraph and (2.2) imply that ${}^m u$ converges to u uniformly on the closure of A_1. The conclusion of the lemma follows because each ${}^m u$ is continuous.

3. The basic estimate and Hausdorff measure. As before, J_q is the interval in Theorem 1, and its right endpoint is t_0. We recall (1.14) and we define $S(a,r) = \{x \in R^3 : |x - a| = r\}$ for $a \in R^3$. The integral of f over $S(a,r)$ with respect to area measure will be denoted $\int_{S(a,r)} f(x)dx$ for simplicity.

LEMMA 3.1. *There exists an absolute constant $\delta > 0$ such that the following holds: If $x_0 \in R^3$, $0 < d < (\text{length}(J_q))^{1/2}$, and condition*

$$
\begin{aligned}
& d^{-2} \int_{t_0 - d^2}^{t_0} \int_{R^3} |v(x,t)|^2 (1 + |x - x_0|/d)^{-4} dx dt \\
(3.1) \qquad & \\
& \quad + \int_{t_0 - d^2}^{t_0} \int_{B(x_0, 2d)} |Dv(x,t)|^2 dx dt \leqq \delta d
\end{aligned}
$$

is satisfied then v can be extended continuously to $(R^3 \times J_q) \cup (V \times \{t_0\})$, where V is a neighborhood of x_0 in R^3.

Proof. We fix $x_0 \in R^3$ and $0 < d < \text{length}(J_q)^{1/2}$, and define functions k_1, k_2: $R \to \{t \in R : t \geqq 0\}$ by (see first paragraph of §3)

$$(3.2)\quad \begin{aligned} k_1(t) &= d^{-2}\int_{R^3} |v(x,t)|^2(1+|x-x_0|/d)^{-4}dx \\ &\quad + \int_{B(x_0,2d)} |Dv(x,t)|^2 dx \quad \text{if} \quad t \in (t_0 - d^2, t_0), \\ k_2(r) &= \int_{t_0-d^2}^{t_0}\int_{S(x_0,r)} |Dv(x,t)|^2 dx\,dt \quad \text{if} \quad r \in (0, 2d), \\ k_1(t) &= 0 = k_2(r) \quad \text{if} \quad t \notin (t_0 - d^2, t_0) \quad \text{and} \quad r \notin (0, 2d). \end{aligned}$$

We let Mk_ι be the cubic Hardy–Littlewood maximal function of k_ι [**9**, *p.* 53]. That is,

$$(3.3)\qquad Mk_\iota(a) = \sup\{(2b)^{-1}\int_{a-b}^{a+b} k_\iota(c)dc : 0 < b < \infty\}.$$

We let $\| \ \|_1$ denote the L^1 norm and $| \ |$ denote Lebesgue measure. The Hardy–Littlewood theorem for L^1 [**9**, (3.5) on p. 55] implies that (3.4) holds for some absolute constant C_4:

$$(3.4)\quad \begin{aligned} &|A| \leqq d^2/8 \quad \text{where} \quad A = \{t : Mk_1(t) > C_4(d^2/8)^{-1}\|k_1\|_1\}, \\ &|B| \leqq d/8 \quad \text{where} \quad B = \{r : Mk_2(r) > C_4(d/8)^{-1}\|k_2\|_1\}. \end{aligned}$$

We have $|\{e \in [d/2, d] : t_0 - e^2 \in A\}| \leqq d^{-1}|A| \leqq d/8$. This and (3.4) imply the existence of $d_0 \in [d/2, d]$ such that $t_0 - d_0^2 \notin A$ and $d_0 \notin B$. Now (3.2), (3.3), and (3.4) yield

$$(3.5)\quad \begin{aligned} &(2b)^{-1}\int_{t_0-d_0^2}^{t_0-d_0^2+b} d^{-2}\int_{R^3} |v(x,t)|^2(1+|x-x_0|/d)^{-4}dx\,dt \\ &\quad + (2b)^{-1}\int_{t_0-d_0^2}^{t_0-d_0^2+b}\int_{B(x_0,2d)} |Dv(x,t)|^2 dx\,dt \\ &\quad \leqq 8C_4 d^{-2}\|k_1\|_1 \quad \text{for} \quad 0 < b < d_0^2, \end{aligned}$$

$$(3.6)\quad \begin{aligned} &(2b)^{-1}\int_{t_0-d^2}^{t_0}\int_{d_0-b \leqq |x-x_0| \leqq d_0+b} |Dv(x,t)|^2 dx\,dt \\ &\quad \leqq 8C_4 d^{-1}\|k_2\|_1 \quad \text{for} \quad 0 < b \leqq d_0. \end{aligned}$$

Defining u by means of (2.1), using $d/2 \leqq d_0 \leqq d$, rewriting (3.5) and (3.6) in terms of u, and recalling (2.4), we obtain (3.7) and (3.8):

$$(t+1)^{-1}\int_{C_t} |u(y,s)|^2(1+|y|)^{-4}dyds + (t+1)^{-1}\int_{B_1\cap C_t} |Du(y,s)|^2 dyds \leqq Cd^{-1}\|k_1\|_1 \quad \text{for} \quad -1<t<0, \tag{3.7}$$

$$2^p\int_{B_p} |Du|^2 \leqq Cd^{-1}\|k_2\|_1 \quad \text{for} \quad p=1,2,3,\cdots. \tag{3.8}$$

From (3.2) we obtain

$$\|k_2\|_1 \leqq \|k_1\|_1 = d^{-2}\int_{t_0-d^2}^{t_0}\int_{R^3} |v(x,t)|^2(1+|x-x_0|/d)^{-4}dxdt + \int_{t_0-d^2}^{t_0}\int_{B(x_0,2d)} |Dv(x,t)|^2 dxdt. \tag{3.9}$$

Now (3.7), (3.8), and (3.9) imply the existence of an absolute constant $\delta > 0$ such that (3.1) yields (2.27). The conclusion of the lemma follows from Lemma 2.2.

We fix the constant δ in Lemma 3.1 and set

$$Q = \{(x_0, 2d) \in R^3 \times (0, 2\,(\text{length}\,(J_q))^{1/2})\text{: (3.1) does not hold}\}. \tag{3.10}$$

LEMMA 3.2. *There exists a finite constant N that depends only on C_1 (see (1.6)) such that the following holds: If*

(3.11) $0 < d < (\text{length}\,(J_q))^{1/2}, B \subset R^3, (b, 2d) \in Q$ if $b \in B$, $\{B(b, 2d)\text{: } b \in B\}$ *is a family of disjointed sets*

is satisfied then the number of points in B is at most N/d.

Proof. Let (3.11) hold. The disjointedness hypothesis implies that (3.12) holds for some absolute constant C_5:

$$\sum_{b\in B} (1+|x-b|/d)^{-4} \leqq C_5 \quad \text{for every} \quad x \in R^3. \tag{3.12}$$

Now (3.11), (3.10), (3.12), and (1.6) yield

δd (cardinality of B)

$$= \sum_{b\in B} \delta d$$

$$\leqq \sum_{b\in B} d^{-2}\int_{t_0-d^2}^{t_0}\int_{R^3} |v(x,t)|^2(1+|x-b|/d)^{-4}dxdt$$

$$+\sum_{b\in B}\int_{t_0-d^2}^{t_0}\int_{B(b,2d)} |Dv(x,t)|^2dxdt$$

$$\leqq C_5 d^{-2}\int_{t_0-d^2}^{t_0}\int_{R^3} |v(x,t)|^2dxdt$$

$$+\int_{t_0-d^2}^{t_0}\int_{R^3} |Dv(x,t)|^2dxdt \leqq C_5C_1 + C_1.$$

Hence we can set $N = (C_5C_1 + C_1)/\delta$.

The following lemma is a consequence of the Besicovich covering theorem [**2**, 2.8.14, 2.8.9].

LEMMA 3.3. *There exists an integral absolute constant K with the following property*: *If* $0 < d < \infty$ *and* $A \subset R^3$ *then there exist* $Y_k \subset A$ *for* $k = 1, 2, \cdots, K$ *such that* (I) *and* (II) *hold*:

(I) $\quad A \subset \cup \left\{B(y, 2d) \colon y \in \bigcup_{k=1}^{K} Y_k\right\}$

(II) *For each* k, $\{B(y, 2d)\colon y \in Y_k\}$ *is a family of disjointed sets.*

We can now finish the proof of Theorem 1. Let A be the set of points $x_0 \in R^3$ such that (3.1) fails to hold for every d satisfying $0 < d <$ (length $(J_q))^{1/2}$. Lemma 3.1 implies that there exists an open set $U \subset R^3$ such that $A \cup U = R^3$ and v can be extended to a continuous function on

$$(R^3 \times J_q) \cup (U \times \{t_0\}).$$

We set $S = R^3 - U$. Since $S \subset A$, all tht remains to show is that the 1 dimensional Hausdorff measure of A is at most $4KN$.

It suffices to show [**2**, p. 171] that for every $0 < d <$ (length $(J_q))^{1/2}$ there exists $Y \subset R^3$ such that

$$A \subset \cup\{B(y, 2d)\colon y \in Y\}$$

and

$$\sum_{y\in Y} \text{diameter}\,(B(y, 2d)) \leqq 4KN.$$

We apply Lemma 3.3 to find sets $Y_k \subset A$ satisfying (I) and (II). Lemma

3.2, (3.10), and the definition of A yield $\Sigma_{y \in Y_k}(4d) \leqq 4N$ for each k. Hence, setting $Y = \bigcup_{k=1}^{K} Y_k$, we obtain $\Sigma_{y \in Y}(4d) \leqq 4KN$. Theorem 1 is proved.

References

1. F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Memoirs of the American Mathematical Society 165, Providence, R. I., 1976.
2. H. Federer, *Geometric Measure Theory*, Springer–Verlag, New York, 1969.
3. O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, revised English edition, Gordon & Breach, New York, 1964.
4. J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math., **63** (1934), 193–248.
5. B. Mandelbrot, *Les Objets Fractals*, Flammarion, Paris, 1975.
6. V. Scheffer, *Géométrie fractale de la turbulence. Équations de Navier–Stokes et dimension de Hausdorff*, C. R. Acad. Sci. Paris, **282** (January 12, 1976), Série A 121–122.
7. ———, *Turbulence and Hausdorff dimension*, to appear in the proceedings of the conference on turbulence held at U. of Paris at Orsay in June, 1975; Lecture Notes in Mathematics, Springer–Verlag, New York.
8. M. Shinbrot, *Lectures on Fluid Mechanics*, Gordon & Breach, New York, 1973.
9. E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, 1971.

Received April 6, 1976.

STANFORD UNIVERSITY
STANFORD, CA 94305

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics
Vol. 66, No. 2 December, 1976