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**ON CERTAIN ALGEBRAIC INTEGERS AND APPROXIMATION
BY RATIONAL FUNCTIONS WITH INTEGRAL COEFFICIENTS**

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ON CERTAIN ALGEBRAIC INTEGERS AND APPROXIMATION BY RATIONAL FUNCTIONS WITH INTEGRAL COEFFICIENTS

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Let A be a finite set of integers $\{a_1, a_2, \dots, a_l\}$ and (possibly) ∞ . Let X be a nonempty closed subset of $C \cup \{\infty\}$, the field of complex numbers together with ∞ , under the topology of the Riemann sphere. Suppose that X is symmetric with respect to the field of real numbers R (i.e. if $z \in X$ then $\bar{z} \in X$) and disjoint from A . We are interested in the following two problems:

I. Under what conditions do there exist, for each neighborhood N of X , infinitely many algebraic numbers θ such that $1/(\theta - a_1), 1/(\theta - a_2), \dots, 1/(\theta - a_l)$ are algebraic integers and, if $\infty \in A$, θ is itself an algebraic integer, such that all of the (algebraic) conjugates of θ lie in N ?

II. If X has empty interior and connected complement, then the polynomials are dense in the ring of continuous functions of X . What is the uniform closure of the polynomials with integral coefficients in $1/(x - a_1), 1/(x - a_2), \dots, 1/(x - a_l)$, and if $\infty \in A$, x itself?

Problem I was investigated by Raphael Robinson [10]; however instead of requiring the $1/(\theta - a_i)$ to be algebraic integers, he required that the $b_i/(\theta - a_i)$ be algebraic integers, where the b_i are integers satisfying $(a_i - a_j) | b_i$ for each $j \neq i$. Our methods are similar to those of Robinson; there are, however, significant differences.

Throughout the remainder of this paper, A will denote a nonempty finite set consisting of real numbers a_1, a_2, \dots, a_l and (possibly) ∞ . We assume that $|a_i - a_j| \geq 1$ if $i \neq j$. In §§ 2, 3, 4, we shall assume that the a_i are integers. If $\infty \in A$, we shall sometimes denote it by a_0 . By a *symmetric closed* (SC) A -set X , we shall mean a nonempty closed subset of the Riemann sphere, symmetric with respect to the x -axis, satisfying $A \cap X = \emptyset$.

If $P(z)$ is a polynomial, we shall denote the leading coefficient of $P(z)$ by $P(\infty)$.

1. Classification of SC A -sets. A rational function with real coefficients $\varphi(z)$ is said to be an A -function if it is regular except possibly for poles at $a_i \in A$. Such a function can be written uniquely in the form $P(z)/D(z)$ where $P(z)$ is a polynomial, $D(z) = \prod_{i=1}^l (z - a_i)^{r_i}$ where the $r_i \geq 0$ and $P(a_i) \neq 0$ when $r_i > 0$, for $1 \leq i \leq l$. If

$\infty \notin A$ put $r = \sum_{i=1}^l r_i$, while if $\infty \in A$, put $r = \max(\deg P(z), \sum_{i=1}^l r_i)$ and $r_0 = r - \sum_{i=1}^l r_i$. Thus, in either case, r is the number of poles (counting multiplicity) of $\varphi(z)$. We call $\{r_1, r_2, \dots, r_l\}$ (or if $\infty \in A$, $\{r_0, r_1, r_2, \dots, r_l\}$) the *degree sequence* of $\varphi(z)$ (with respect to A). We shall say that the A -function $\varphi(z)$ is an *upper* A -function if all r_i are positive and $|P(a)| \geq 1$ for each $a \in A$. (Recall that by our convention $P(\infty)$ is the leading coefficient of $P(z)$.) We shall say that the A -function $\varphi(z)$ is a *lower* A -function if all r_i are positive and $0 < |P(a)| \leq 1$ for all $a \in A$. We shall say that the A -function $\varphi(z)$ is a *normal* A -function if it is both upper and lower; i.e. if all r_i are positive and $|P(a)| = 1$ for all $a \in A$. We say that the A -function $\varphi(z)$ is an *integral* A -function if $P(z)$ has integral coefficients.

An SC A -set X is said to be *A-small* if there exists an upper A -function $\varphi(z)$ with $\|\varphi\|_X < 1$. (Here and throughout $\|\cdot\|_X$ denotes the sup norm over X .) The set X is said to be *A-large* if for each neighborhood N of X there exists a lower A -function $\varphi(z)$ satisfying $\{z: |\varphi(z)| = 1\} \subset N$ and $X \subset \{z: |\varphi(z)| < 1\}$. Note that if $A = \{\infty\}$ then an A -small set is simply a set with transfinite diameter < 1 and an A -large set is one with transfinite diameter ≥ 1 [3, Theorem I].

THEOREM 1.1. *Suppose A' is a non-empty subset of A . No SC A -set X is both A -large and A' -small.*

Proof. Suppose X is both A -large and A' -small. Let $f(z)$ be an upper A' -function with $\rho = \|f(z)\|_X < 1$. Choose σ satisfying $\rho < \sigma < 1$. The set $N_\sigma = \{z: |f(z)| < \sigma\}$ is an open neighborhood of X . Since X is A -large there exists a lower A -function $g(z)$ such that $\{z: |g(z)| = 1\} \subset N_\sigma$. Then, for any z , $|g(z)| = 1$ implies $|f(z)| < \sigma < 1$. Now suppose that $\infty \in A$; the proof is similar and simpler if $\infty \notin A$. Let $\{r_0, r_1, \dots, r_l\}$ be the degree sequence of f (with respect to A) and let $\{s_0, s_1, \dots, s_l\}$ be the degree sequence of g . Clearly all s_i are > 0 . Choose h so that $r_h/s_h = \max_j (r_j/s_j)$; r_h is > 0 . Put $g_1(z) = g(z)^{r_h}$ and $f_1(z) = f(z)^{s_h}$. The degree sequence of f_1 is \leq (componentwise) the degree sequence of g_1 , with equality at the h^{th} component. Put $u(z) = f_1(z)/g_1(z)$; $u(z)$ is regular for all z for which $g_1(z) \neq 0$; in particular $u(z)$ is regular in $D = \{z: |g_1(z)| > 1\}$. On $|g_1(z)| = 1$, the boundary of D , $|f_1(z)| < \sigma^{s_h} < 1$, hence $|u(z)| < 1$; by the maximum principle this holds for all $z \in D$. But at $z = a_h \in D$, $|u(z)| \geq 1$, since $f_1(z)$ is an upper A' -function and $g_1(z)$ is a lower A -function. This contradiction completes the proof.

(The author would like to thank the referee for providing this elegant short proof; the original was much longer and more complicated.) We shall need the following.

LEMMA 1.2. Suppose $B = (\beta_{ij})$ is a matrix with real entries whose off-diagonal elements are nonnegative. Then either (a) there exists a nonzero vector $x \geq 0$ such that $Bx \geq 0$, or else (b) B is invertible and B^{-1} is ≤ 0 .

Proof. Choose μ so that $B + \mu I$ is ≥ 0 and let λ be the largest eigenvalue of $B + \mu I$. By an extension of the Perron-Frobenius Theorem [4, Chapter XIII, Theorem 3, p. 66], $-(B + \mu I)^{-1}$ exists if $\mu > \lambda$ and when that is so is ≥ 0 , while if $\mu \leq \lambda$, then $B + \mu I$ has a nonnegative eigenvector x satisfying $(B + \mu I)x = \lambda x$ or $Bx = (\lambda - \mu)x \geq 0$.

The following is closely related to the main result of § 4 of [11].

THEOREM 1.3. Let X be an SC A -set. Then either X is A -large or there exists a nonempty subset A' of A such that X is A' -small. If X is A -large then for every neighborhood N of X there exists a normal A -function $\varphi(z)$ and $R > 1$ such that $\{z: |\varphi(z)| = R\} \subset N$ and $X \subset \{z: |\varphi(z)| < R\}$. Finally, if all finite $a_i \in A$ are rational, we may choose $\varphi(z)$ so that its numerator has rational coefficients.

Proof. We shall prove this when $\infty \in A$. The case when $\infty \notin A$ is simpler. The complement of X in the Riemann sphere is a union of components. Let $C_0, C_1, C_2, \dots, C_s$ be those components which have a nonempty intersection with A , and suppose they are numbered so that $\infty \in C_0$. Put $A_k = A \cap C_k$ and put $I_k = \{i: a_i \in C_k\}$. Denote by X_k the complement of C_k (in the Riemann sphere). Let N_k be a neighborhood of X_k disjoint from A . Suppose that $a_j \in A_k$. By Theorem G of [3], there exist polynomials $f_j(z)$ with real coefficients such that

$$(1) \quad \begin{cases} |X_k \subset \{z: |f_j(1/(z - a_j))| < 1\} \text{ if } j > 0 \\ |X_0 \subset \{z: |f_0(z)| < 1\} \end{cases}$$

$$(2) \quad \begin{cases} \{z: |f_j(1/(z - a_j))| \leq 1\} \subset N_k \text{ if } j > 0 \\ \{z: |f_0(z)| \leq 1\} \subset N_0 \end{cases}.$$

Since each $N_k \cap A$ is empty, $|f_0(a_j)| > 1$ and $|f_k(1/(a_j - a_k))| > 1$ for all $k > 0$ and $j \neq k$. By replacing each f_j by a positive integral power of itself, if necessary, we may assume that the f_j all have the same degree, say d , and that $d > l$. We are going to construct a function $\varphi(z)$ of the form

$$(3) \quad \begin{aligned} \varphi(z) = & f_0(z)^{t_0} \prod_{j \in I'_0} f_j(1/(z - a_j))^{t_j} \\ & + \sum_{k=1}^s \prod_{j \in I_k} f_j(1/(z - a_j))^{t_j}, \end{aligned}$$

where here, and throughout this proof, $I'_0 = I_0 - \{0\}$. We can write $\varphi(z)$ in the form

$$\varphi(z) = \frac{P(z)}{\prod_{j=1}^l (z - a_j)^{dt_j}},$$

where $P(z)$ is a polynomial of degree $d(t_0 + t_1 + \cdots + t_l)$, and is explicitly given by

$$(4) \quad \begin{aligned} P(z) = & f_0(z)^{t_0} \prod_{j \in I'_0} [(z - a_j)^d f_j(1/(z - a_j))]^{t_j} \cdot \prod_{j \in I_0} (z - a_j)^{dt_j} \\ & + \sum_{k=1}^s \prod_{j \in I_k} [(z - a_j)^d f_j(1/(z - a_j))]^{t_j} \cdot \prod_{\substack{j \notin I_k \\ j \neq 0}} (z - a_j)^{dt_j}. \end{aligned}$$

Then

$$\begin{aligned} P(\infty) &= f_0(\infty)^{t_0} \prod_{j \in I'_0} f_j(0)^{t_j}, \\ P(a_i) &= f_0(a_i)^{t_0} \prod_{\substack{j \in I'_0 \\ j \neq i}} [(a_i - a_j)^d f_j(1/(a_i - a_j))]^{t_j} \\ &\quad \times f_i(\infty)^{t_i} \cdot \prod_{j \in I_0} (a_i - a_j)^{dt_j} \quad \text{if } i \in I'_0, \\ P(a_i) &= \prod_{\substack{j \in I_k \\ j \neq i}} [(a_i - a_j)^d f_j(1/(a_i - a_j))]^{t_j} \\ &\quad \times f_i(\infty)^{t_i} \cdot \prod_{\substack{j \notin I_k \\ j \neq 0}} (a_i - a_j)^{dt_j} \quad \text{if } i \in I_k, k \neq 0. \end{aligned}$$

Put

$$\begin{aligned} \beta_{00} &= \log |f_0(\infty)|, \\ \beta_{0j} &= \begin{cases} \log |f_j(0)| & \text{if } j \in I'_0 \\ 0 & \text{if } j \in I_0 \end{cases}, \\ \beta_{i0} &= \begin{cases} \log |f_0(a_i)| & \text{if } i \in I'_0 \\ 0 & \text{if } i \in I_0 \end{cases}; \end{aligned}$$

if $i, j > 0$ then put

$$\beta_{ij} = \begin{cases} \log |(a_i - a_j)^d f_j(1/(a_i - a_j))| & \text{if } i \sim j, i \neq j \\ \log |f_j(\infty)| & \text{if } i = j \\ \log |(a_i - a_j)^d| & \text{if } i \not\sim j \end{cases}.$$

Here $i \sim j$ means i and j are in the same I_k and $i \not\sim j$ means that this is not so. Since $|a_i - a_j| \geq 1$ if $i \neq j$, $i, j > 0$, we see that if $i \neq j$ then $\beta_{ij} \geq 0$; moreover if A lies in the one component C_0 and $i \neq j$, $\beta_{ij} > 0$. We have that

$$(5) \quad \log |P(a_i)| = \sum_{j=0}^l \beta_{ij} t_j.$$

We now apply Lemma 1.2 to the matrix $B = (\beta_{ij})$. If case (a) holds, then there exist real $t_0, t_1, \dots, t_l \geq 0$, not all 0, such that all of the sums $\sum_j \beta_{ij} t_j$ are ≥ 0 . Let A' be the union of those A_k for which there exist $j \in I_k$ such that $t_j > 0$. Put $I' = \{i: a_i \in A'\}$. By replacing each f_i by λf_i , where $\lambda > 1$ is small enough that (1) and (2) are still satisfied, we increase β_{ij} when $i \sim j$. Hence if $i \in I'$, we increase at least one coefficient of a positive t_j in the linear form $\sum_j \beta_{ij} t_j$. Thus we may assume that the linear forms $\sum_j \beta_{ij} t_j$ are positive when $i \in I'$. By modifying the positive t_j for which $j \in I'$ slightly to make them positive rationals and then multiplying through by a common denominator, we may assume the t_j are positive integers, and $\sum_j \beta_{ij} t_j > 0$ when $i \in I'$. We can multiply the t_j by such a large positive integer that if $i \in I_k$, $i \neq 0$, then $|f_i(1/(z - a_i))^{t_i}|$ is $< 1/(s + 1)$ for all $z \in X_k$ and is > 1 for all z outside of N_k . Similarly we will have $|f_0(z)^{t_0}| < 1/(s + 1)$ for $z \in X_0$ and $|f_0(z)^{t_0}| > 1$ for $z \notin N_0$. Now, construct φ as in (3) substituting A' for A and using the same N_k and f_j . Then φ is an upper A' -function and it is easy to see that $X \subset \{z: |\varphi(z)| < 1\}$ so that X is A' -small.

Next suppose that case (b) of Lemma 1.2 holds. Then $B^{-1} \leq 0$. Put

$$(6) \quad t = (t_0, t_1, \dots, t_l)^* = B^{-1}(-1, -1, \dots, -1)^*.$$

Then $\sum_j \beta_{ij} t_j = -1$ and each component t_j of t is > 0 , for clearly $t_j \geq 0$ and if $t_j = 0$, then the j^{th} row of B^{-1} would be 0, which is not possible. There is a unique polynomial $g_0(z)$ of degree $\leq l - 1$ such that $z^d + g_0(z) = 0$ for $z = a_1, a_2, \dots, a_l$. Since $d > l$ the polynomial $f_0(z) + \delta_0(z^d + g_0(z))$ has leading coefficient $f_0(\infty) + \delta_0$ and takes the same values at $z = a_1, a_2, \dots, a_l$ as $f_0(z)$. Thus replacing $f_0(z)$ by $f_0(z) + \delta_0(z^d + g_0(z))$ would change β_{00} but none of the other β_{i0} . If δ_0 is small enough then (1) and (2) would remain satisfied. Similar comments apply to f_1, f_2, \dots, f_l . Thus there exists $\varepsilon > 0$ such that each f_j can be modified in such a way that β_{ij} is unchanged if $i \neq j$, while β_{jj} varies over an interval of length 2ε , and at the same time (1) and (2) remain valid. Choose positive rational t'_j so close to t_j , $0 \leq j \leq l$, that $|\sum_{i=0}^l \beta_{ij}(t_j - t'_j)/t'_i| < \varepsilon$ for $0 \leq i \leq l$.

Now put $\beta'_{ij} = \beta_{ij}$ if $i \neq j$ and choose β'_{ii} so that $\sum_{j=0}^l \beta'_{ij} t'_j = -1$

for $0 \leq i \leq l$. Then $|\beta'_{ii} - \beta_{ii}| = |\sum_{j=0}^l \beta_{ij}(t_j - t'_j)/t'_i| < \varepsilon$ for $0 \leq i \leq l$. Now modify the f_i slightly so that the β_{ij} are replaced by the β'_{ij} and the t_i by the t'_i , still preserving (1) and (2). Thus, after this replacement we may assume that the t_i are all positive rational numbers. Now multiply the t_j by such a large positive integer n that they become integers and such that if $i \in I_k$, $i \neq 0$, then $|f_i(1/(z - a_i))^{t_i}|$ is $< 1/(s + 1)$ for $z \in X_k$ and > 2 for $z \notin N_k$. Similarly $|f_0(z)|^{t_0}$ is $< 1/(s + 1)$ for $z \in X_0$ and > 2 for $z \notin N_0$. Then φ as defined in (3) is a lower A -function and all of the $|P(a_j)|$ are equal to $1/e^n$. By replacing φ by φ^2 , we obtain a lower A -function φ with $P(a_j) = 1/e^{2n}$ for $0 \leq j \leq l$. For $z \notin N_k$ all but one of the terms in (3) have absolute value $< 1/(s + 1)$ while the remaining term has value > 2 . Thus $|\varphi(z)|$ is > 1 outside of each N_k . If $z \in X$, however, then each term in (3) has absolute value $< 1/(s + 1)$ and $|\varphi(z)| < 1$. Let Y be the union of X and those components of the complement of X which are disjoint from A ; i.e., Y is obtained from X by filling in those holes which contain no a_i . If N is any neighborhood of X , then there exist neighborhoods N_k of X_k such that $\bigcap_k N_k \subset N \cup Y$. The φ , as modified above and corresponding to this choice of the N_k is lower, $X \subset \{z: |\varphi(z)| < 1\}$, and $\{z: |\varphi(z)| = 1\} \subset N$. Thus X is A -large. Then $\varphi_1(z) = \varphi(z)e^{2n}$ is A -normal and $X \subset \{z: |\varphi_1(z)| < e^{2n}\}$ and $\{z: |\varphi_1(z)| = e^{2n}\} \subset N$.

Finally, suppose the a_i are rational, and $\varphi_1(z) = P_1(z)/D(z)$; then $P_1(a_i) = 1$, $0 \leq i \leq l$. We can choose a polynomial $C(z)$ of degree $< \deg(P_1(z)) - l$ and with arbitrarily small coefficients such that $P_2(z) = P_1(z) + C(z) \prod_{i=1}^l (z - a_i)$ has all coefficients of terms of degree $\geq l$ rational. Since $P_2(a_i) = 1$, $0 \leq i \leq l$, the remaining coefficients are rational. If $C(z)$ is small enough then $\varphi_2(z) = P_2(z)/D(z)$ meets the requirements of the theorem.

REMARK 1.4. The A' in the above theorem is a union of some of the A_k . In particular if all a_i lie in one component of the complement of X , then either X is A -large or X is A -small.

We shall need the following theorem in § 4.

THEOREM 1.5. *A finite SC A -set X is A -small.*

Proof. By standard interpolation theory results, there exists a monic polynomial P which vanishes at each element of X and is 1 at each finite element of A . We may choose P to have degree $\geq l + 1$, and then $\varphi(z) = P(z)/\prod_{i=1}^l (z - a_i)$ is a normal A -function which has absolute value < 1 on X .

THEOREM 1.6. *If X is an A -small set, then there exists a normal A -function $\varphi(z)$ such that $\|\varphi(z)\|_X < 1$.*

Proof. We shall prove this in the case when $\infty \in A$. The case when $\infty \notin A$ is simpler. By definition there exists an upper A -function $Q(z)/D(z)$ such that $\|Q(z)/D(z)\|_X < 1$. Suppose $D(z) = \prod_{j=1}^l (z - a_j)^{r_j}$. Since z and each of the functions $1/(z - a_j)$ is bounded on X , there exists an integer $n \geq 1$ so large that $\|Q(z)^n/D(z)^{n-1}\|_X < 1/(l+1)$ and $\|Q(z)^n/((z - a_j)^{d_j} D(z)^{n-1})\|_X < 1/(l+1)$ for $1 \leq j \leq l$. Now put

$$P(z) = (\alpha_0 D(z) + \sum_{j=1}^l \alpha_j D(z)/(z - a_j)^{r_j}) Q(z)^n$$

where the α_i will be chosen later. Then $P(\infty) = \alpha_0 Q(\infty)^n$ and $P(a_i) = \alpha_i \prod_{j \neq i} (a_i - a_j)^{r_j} Q(a_i)^n$. Thus there exist unique choices for the a_i so that $P(\infty) = 1$ and all $P(a_i) = 1$, and the α_i will have absolute value ≤ 1 . Put $\varphi(z) = P(z)/D(z)^n$; $\varphi(z)$ is a normal A -function and

$$\begin{aligned} \|\varphi(z)\|_X &\leq |\alpha_0| \cdot \|Q(z)^n/D(z)^{n-1}\|_X \\ &\quad + \sum_{j=1}^l |\alpha_j| \cdot \|Q(z)^n/((z - a_j)^{r_j} D(z)^{n-1})\|_X \\ &< 1. \end{aligned}$$

If $N(z)$ is a nonconstant polynomial, then any power series $u(z)$ can be written uniquely in the form

$$u(z) = \sum_{i=0}^{\infty} c_i(z) N(z)^i$$

where the $c_i(z)$ are polynomials of degree $< \deg(N(z))$. This is the special case, $A = \{\infty\}$, of the next lemma. To extend to general sets A , we must replace $N(z)$ by a rational function which has poles at each $a_i \in A$, and allow the $c_i(z)$ to be rational functions with poles of bounded order at each $a_i \in A$. In the following lemma, $N(z)$ is replaced by $N(z)/D(z)$ and the $c_i(z)$ by the $c_i(z)/D(z)$.

LEMMA 1.7. *Suppose $\infty \in A$ and $D(z) = \prod_{i=1}^l (z - a_i)^{r_i}$ where the r_i are > 0 . Suppose $N(z)$ is a polynomial, relatively prime to $D(z)$, of degree $r = \sum_{i=0}^l r_i$ where r_0 is > 0 . If $u(z)$ is an A -function satisfying $u(a_i) \neq 0$, $0 \leq i \leq r$, we can write uniquely*

$$u(z) = \sum_{i=0}^n c_i(z) N(z)^i / D(z)^{i+1}$$

where n is the least integer ≥ 0 such that

$$-\text{ord}_{\infty} u(z) \leq (n+1)r_0 - 1$$

and

$$-\text{ord}_{a_i} u(z) \leq (n+1)r_i \quad \text{for } 1 \leq i \leq l;$$

and where the $c_i(z)$ are polynomials of degree $< r$ and $c_n(z)$ is not 0.

Suppose $\infty \notin A$ and $D(z) = \prod_{i=1}^l (z - a_i)^{r_i}$, where the r_i are > 0 . Suppose $N(z)$ is a polynomial, relatively prime to $D(z)$ and of degree $\leq r = \sum_{i=1}^l r_i$. If $u(z)$ is an A -function satisfying $u(a_i) \neq 0$, $1 \leq i \leq r$ and vanishing at ∞ , we can write, uniquely,

$$u(z) = \sum_{i=0}^n c_i(z) N(z)^i / D(z)^{i+1}$$

where n is the least integer ≥ 0 such that

$$-\text{ord}_{a_i} u(z) \leq (n+1)r_i \quad \text{for } 1 \leq i \leq l;$$

and where the $c_i(z)$ are polynomials of degree $< r$, and $c_n(z)$ is not 0.

Proof. We give the proof for the case $\infty \in A$, and it is by induction on n . The result is clear when $n = 0$, for then $D(z)u(z)$ is a polynomial of degree $< r$. If $n \geq 1$, choose the polynomial $c_0(z)$ of degree $< r$ and $\equiv u(z)D(z) \pmod{N(z)}$; then the polynomial $D(z)^n(u(z)D(z) - c_0(z))$ is divisible by $N(z)$. Note that this is the unique choice for $c_0(z)$. Then

$$-\text{ord}_{\infty} (u(z)D(z) - c_0(z))/N(z) \leq nr_0 - 1$$

and

$$-\text{ord}_{a_i} (u(z)D(z) - c_0(z))/N(z) \leq nr_i$$

for $1 \leq i \leq l$. Thus, inductively, we have, uniquely,

$$(u(z) - c_0(z)/D(z)) \frac{D(z)}{N(z)} = \sum_{i=0}^{n-1} c_{i+1}(z) \frac{N(z)^i}{D(z)^{i+1}}$$

and then

$$u(z) = \sum_{i=0}^n c_i(z) N(z)^i / D(z)^{i+1}.$$

LEMMA 1.8. Suppose $\infty \in A$ and X is an SC A -set. Suppose $g(z) = P(z)/D(z)$ is a normal A -function where $D(z) = \prod_{i=1}^l (z - a_i)^{r_i}$ and $P(z)$ has degree $r = \sum_{i=1}^l r_i$. Put $\lambda = \|g(z)\|_X^{1/r}$. Then there exists $M > 0$ and for each integer $n \geq 0$ an A -function $\theta_n(z)$ such that when $\theta_n(z)$ is expanded according to Lemma 1.7, with $N(z) = z^{r_0}D(z) + 1$,

$$\theta_n(z) = \sum_{i=0}^s d_i(z) N(z)^i / D(z)^{i+1},$$

then $n = rs + t$, where $0 \leq t < r$, $d_s(z)$ is a monic polynomial of degree t , $\|\theta_n(z)\|_X < M\lambda^n$, and $|\theta_n(z)| < M|g(z)|^{1+n/r}$ when $|g(z)| > 1$.

Suppose $\infty \notin A$ and X is an SC A -set. Suppose $g(z) = P(z)/D(z)$ is a normal A -function where $D(z) = \prod_{i=1}^l (z - a_i)^{r_i}$ and $P(z)$ has degree $< r = \sum_{i=1}^l r_i$. Put $\lambda = \|g(z)\|_X^{1/r}$. Then there exists $M > 0$ and for each integer $n \geq 0$ an A -function $\theta_n(z)$ such that when $\theta_n(z)$ is expanded according to Lemma 1.7, using $N(z) = 1$,

$$\theta_n(z) = \sum_{i=0}^s d_i(z) / D(z)^{i+1},$$

then $n = rs + t$ where $0 \leq t < r$ and $d_s(z)$ is a monic polynomial of degree t , $\|\theta_n(z)\|_X < M\lambda^n$, and $|\theta_n(z)| < M|g(z)|^{1+n/r}$ when $|g(z)| > 1$.

Proof. Suppose first that $\infty \in A$. Expand $g(z)^m$ by Lemma 1.7:

$$g(z)^m = \sum_{i=0}^m c_i(z) N(z)^i / D(z)^{i+1}.$$

It is easy to verify that $c_m(z) = D(z)$, hence is monic of degree $\sum_{i=1}^l r_i$. Then $g(z)^m$ will serve for $\theta_{(m+1)r-r_0}$. The functions

$$z\theta_{(m+1)r-r_0}, z^2\theta_{(m+1)r-r_0}, \dots, z^{r_0-1}\theta_{(m+1)r-r_0}$$

will serve for $\theta_{(m+1)r-r_0+1}, \theta_{(m+1)r-r_0+2}, \dots, \theta_{(m+1)r-1}$, respectively. The functions $\theta_{(m+1)r-r_0}/(z - a_i), \theta_{(m+1)r-r_0}/(z - a_i)^2, \dots, \theta_{(m+1)r-r_0}(z)/(z - a_i)^{r_1}$ will serve for $\theta_{(m+1)r-r_0-1}, \theta_{(m+1)r-r_0-2}, \dots, \theta_{(m+1)r-r_0-r_1}$, respectively. Continuing in this way, dividing next by $(z - a_1)^{r_1}(z - a_2)$, then $(z - a_1)^{r_1}(z - a_2)^2, \dots$, and so forth will give the remaining functions. Since all of the functions $z, 1/(z - a_1), \dots, 1/(z - a_l)$ are bounded on X and $z^{r_0}/g(z), 1/((z - a_1)^{r_1}g(z)), \dots, 1/((z - a_l)^{r_l}g(z))$ are bounded when $|g(z)| > 1$, there exists $M > 0$ as required for the Lemma. If $\infty \notin A$, use the above procedure with $r_0 = 0$, omitting z and $z^{r_0}/g(z)$ when defining M .

2. Classification of A -sets—Integral A . In this and succeeding sections we assume that the $a_i \in A$ are integers and strengthen the results of § 1.

THEOREM 2.1. *If X is A -small there exists an integral, normal A -function $\varphi(z)$ such that $\|\varphi(z)\|_X < 1$.*

Proof. We give the proof in the case that $\infty \in A$. There exists an A -normal function $P(z)/D(z)$, where $D(z) = \prod_{i=1}^l (z - a_i)^{r_i}$ and $\|P(z)/D(z)\|_X < 1$. Suppose $P(z)$ has degree $r = \sum_{i=0}^l r_i$ and put

$N(z) = z^{r_0}D(z) + 1$. Choose $m > 0$. For any $n > m$, the function $(P(z)/D(z))^n(1 + 1/((z - a_1)(z - a_2) \cdots (z - a_l)))$ is A -normal and by Lemma 1.7 can be written in the form $\sum_{i=0}^n c_i^{(n)}(z)N(z)^i/D(z)^{i+1}$. It is easy to verify that $c_n^{(n)}(z) = D(z)(1 + 1/((z - a_1)(z - a_2) \cdots (z - a_l)))$. We can successively add $\varepsilon_1\theta_{nr-1}(z)$, $\varepsilon_2\theta_{nr-2}(z)$, \dots , $\varepsilon_{nr-mr}\theta_{mr}(z)$, where the $\theta_i(z)$ are the functions defined in Lemma 1.8 and the ε_i are real numbers in the interval $[-1/2, 1/2]$, so as to obtain a function

$$(7) \quad h_n(z) = \sum_{i=0}^n d_i^{(n)}(z)N(z)^i/D(z)^{i+1}$$

where $d_n^{(n)}(z) = c_n^{(n)}(z)$ and $d_m^{(n)}(z)$, $d_{m+1}^{(n)}(z)$, \dots , $d_{n-1}^{(n)}(z)$ have integral coefficients. Furthermore, with M and λ as defined in Lemma 1.8,

$$\|h_n(z)\|_X < M' \|P(z)/D(z)\|_X^n + M(\lambda^{mr} + \lambda^{mr+1} + \dots + \lambda^{nr-1}) < M''\lambda^{mr}$$

where $M' = \|(1 + 1/((z - a_1)(z - a_2) \cdots (z - a_l)))\|_X$ and $M'' = \max(M, M')/(1 - \lambda)$. We can choose m so large that $M''\lambda^{mr} < 1/3$. For each $n > m$, we obtain such a function $h_n(z)$ and in the expansion (7), all of the $d_i^{(n)}(z)$, except those with $i < m$, have integral coefficients. We can find $n_2 > n_1 > m$ so that all of the coefficients of the $d_i^{(n_2)}(z) - d_i^{(n_1)}(z)$, for $0 \leq i \leq m - 1$, are extremely small modulo 1.

When this is the case put $\varphi(z) = \sum_{i \geq 0} e_i(z)N(z)^i/D(z)^{i+1}$ where $e_i(z)$ is the polynomial with integral coefficients nearest to $d_i^{(n_2)}(z) - d_i^{(n_1)}(z)$; here we put $d_i^{(n_1)}(z) = 0$ when $i > n_1$. If n_1 and n_2 were chosen appropriately, $\varphi(z)$ will satisfy $\|\varphi(z)\|_X < 1$ and since $e_{n_2}(z) = (1 + 1/((z - a_1)(z - a_2) \cdots (z - a_l)))D(z)$, $\varphi(z)$ is normal.

THEOREM 2.2. *Suppose X is A -large. Then for each neighborhood N of X there exists an integral normal A -function $\varphi(z)$ and an integer $S > 1$ such that $\{z: |\varphi(z)| = S\} \subset N$ and $X \subset \{z: |\varphi(z)| < S\}$.*

Proof. We give the proof for the case that $\infty \in A$. By Theorem 1.3, there exists a normal A -function $g(z)$ with rational coefficients and $R > 1$ such that $\{z: |g(z)| = R\} \subset N$ and $X \subset \{z: |g(z)| < R\}$. We can write $g(z) = N(z)/D(z) + c(z)/(hD(z))$ where, as usual, $D(z) = \prod_{i=1}^l (z - a_i)^{r_i}$, $N(z) = z^{r_0}D(z) + 1$, $r = \sum_{i=1}^l r_i$, $c(z)$ is a polynomial of degree $< r$ with integral coefficients satisfying $c(a_i) = 0$ for $i \leq l$, and h is a positive integer. We can write

$$\begin{aligned} g(z)^n &= (N(z) + c(z)/h)^n/D(z)^n \\ &= \sum_{i=0}^{n-1} N(z)^{n-i} c(z)^i \binom{n}{i} / (h^i D(z)^n) \\ &\quad + \sum_{i=m}^n N(z)^{n-i} c(z)^i \binom{n}{i} / (h^i D(z)^n), \end{aligned}$$

where $m < n$ will be chosen later in this proof.

When the first sum is written as a rational function in z with denominator $D(z)^n$, each coefficient of a power of z in the numerator will be a polynomial in n with rational coefficients. Since the polynomial $\binom{n}{i}$ in n is divisible by n for each $i > 0$, the numerator polynomial will have integral coefficients when n is divisible by a certain fixed integer n_0 .

Since $c(z)$ has degree $< r$, the second sum has a pole at ∞ of order $\leq (n-m)r + m(r-1) - n(r-r_0) = nr_0 - m$. Since $c(z)$ vanishes at each a_i , the second sum has a pole at a_i of order $\leq nr_i - m$. By Lemma 1.7, the second sum can be written in the form

$$\sum_{i=0}^k b_i(z) N(z)^i / D(z)^{i+1}$$

where k is the least integer ≥ 0 satisfying

$$k+1 \geq n - (m-1)/r_0$$

$$k+1 \geq n - m/r_i; \quad 1 \leq i \leq l.$$

Put $j = (k+1)r - 1$. Let $\theta_0(z)$, $\theta_1(z)$, $\theta_2(z)$, \dots be the functions constructed in Lemma 1.8 using $P(z)/D(z) = g(z)$. By adding successively $\varepsilon_j \theta_j(z)$, $\varepsilon_{j-1} \theta_{j-1}(z)$, \dots where the ε_i are chosen appropriately from the interval $[-1/2, 1/2)$, to $g(z)^n$ we obtain an integral normal A -function $f_n(z)$. Choose R_1 and R_2 close to R with $1 < R_1 < R < R_2$ such that $X \subset \{z: |g(z)| < R_1\}$ and $\{z: R_1 \leq |g(z)| \leq R_2\} \subset N$. Then $f_n(z)$ differs from $g(z)^n$ in the set $\{z: R_1 < |g(z)|\}$ by less than $M|g(z)|(|g(z)|^{j/r} + |g(z)|^{(j-1)/r} + \dots + 1)$ or by less than $M'|g(z)|^{2+j/r}$ where $M' = M/(R_1^{1/r} - 1)$. Hence if n/j is large enough, $f_n(z)$ does not vanish when $|g_n(z)| \geq R_2$. Similarly, if $|g(z)| \leq R_1$, then $f_n(z)$ differs from $g(z)^n$ by $\leq M'' R_1^{j/r}$. Thus by the maximal principal, if $|g(z)| \geq R_2$, $|f_n(z)| \geq (1-\delta)R_2^n$ and if $|g(z)| \leq R_1$, $|f_n(z)| \leq (1+\delta)R_1^n$, where $\delta > 0$ can be made arbitrarily close to 0 by choosing m large. If n is large enough and divisible by n_0 there will be an integer S in the interval $((1+\delta)R_1^n, (1-\delta)R_2^n)$; putting $\varphi(z) = f_n(z)$ completes the proof.

3. *A-integers.* An algebraic number θ is said to be an *A-integer* if $1/(\theta - a_i)$ is an algebraic integer for each $a_i \in A$ and θ is an algebraic integer if $\infty \in A$.

LEMMA 3.1. *If $\varphi(z) = P(z)/D(z)$ is an integral normal A-function and θ is a complex number such that $\varphi(\theta) = \alpha$ is an algebraic*

integer, then θ is an A -integer.

Proof. The polynomial $P(z) - \alpha D(z)$ has algebraic integer coefficients and is satisfied by θ . If $\infty \in A$, then this polynomial is monic of degree r and hence θ is an algebraic integer. Since $P(a_i) - \alpha D(a_i) = P(a_i) = 1$, the polynomial with algebraic integer coefficients satisfied by $1/(\theta - a_i)$ is monic and $1/(\theta - a_i)$ is an algebraic integer.

LEMMA 3.2. *If $\varphi(z)$ is an integral A -function and θ is an A -integer then $\varphi(\theta)$ is an algebraic integer.*

Proof. We first show that the ring generated by the functions $1, 1/(z - a_1), 1/(z - a_2), \dots, 1/(z - a_l)$, and if $\infty \in A$, the function z , contains all integral A -functions. This is clear if $\infty \in A$, so suppose $\infty \notin A$. Suppose $P(z)/D(z)$ is an integral A -function, $D(z) = \sum_{i=1}^l (z - a_i)^{r_i}$, and $r = \sum_{i=1}^l r_i$. We proceed by induction on r . If $r = 0$, the result is clear. Otherwise some r_i , say r_1 , is > 0 . Then $P(z)/D(z) = (P(z) - P(a_1))/D(z) + P(a_1)/D(z)$. Clearly $P(a_1)/D(z)$ is in the ring and since $(z - a_1) \mid (P(z) - P(a_1))$, $(P(z) - P(a_1))/D(z)$ is in the ring by induction. Since each $1/(\theta - a_i)$ is an algebraic integer and if $\infty \in A$, θ is an algebraic integer, $\varphi(\theta)$ is an algebraic integer.

We now give the basic results of this section.

THEOREM 3.3. *Let X be a set which is not A -large. Then there exists a neighborhood of X which contains only finitely many complete conjugate sets of A -integers.*

Proof. By Theorems 1.3 and 2.1, A contains a nonempty subset A' for which there exists an integral A' -function $\varphi(z)$ such that $\|\varphi(z)\|_X < 1$. Put $N = \{z: |\varphi(z)| < 1\}$. If $\{\theta_1, \theta_2, \dots, \theta_m\}$ is a complete conjugate set of A -integers contained in N , then $\{\varphi(\theta_1), \varphi(\theta_2), \dots, \varphi(\theta_m)\}$ is a sequence of algebraic integers, consisting of repetitions of a complete conjugate set. Since each $\varphi(\theta_i)$ has absolute value < 1 , the norm of each is < 1 , hence 0. Thus each $\varphi(\theta_i) = 0$ and so the total number of θ_i is $\leq r$, the degree of the numerator of $\varphi(z)$.

THEOREM 3.4. *Let X be an A -large set. Then every neighborhood N of X contains infinitely many complete sets of conjugate A -integers.*

Proof. Let N be a neighborhood of X . By Theorem 2.2 there exists an integral normal A -function $\varphi(z)$ and an integer $S > 1$ such that $\{z: |\varphi(z)| = S\} \subset N$. The solutions to $\varphi(z)^n = S^n$ lie in N and

by Lemma 3.1 are A -integers.

It is probable that if X is an A -large subset of R then every real neighborhood of X contains infinitely many complete sets of conjugate A -integers. In the case $A = \{\infty\}$ and X is a finite union of closed intervals in R this was shown by Robinson in [7] and [8], and in the case X is a closed interval and $A = \{\infty, 0\}$ this was shown by Robinson in [9].

4. Approximation. Let X be an SC set with empty interior and such that each component of the complement of X in C contains an element of A . A complex valued function f on X is called *symmetric* if $f(\bar{x}) = \overline{f(x)}$ for all $x \in X$. We shall denote the ring of continuous symmetric functions on X by $C_s(X)$. A theorem of Mergelyan [6, Theorem 2.3] asserts that the A -functions are dense, in the uniform norm, in $C_s(X)$. We are interested in investigating the uniform closure of the integral A -functions in $C_s(X)$. For the case $A = \{\infty\}$ see [1] and [5]. If Y is an SC subset of X , we shall say that the symmetric function f is *matchable* on Y if there exists an integral A -function p such that $p(y) = f(y)$ for all $y \in Y$ and we shall say that f is *approximable* on Y if for each $\varepsilon > 0$ there exists an integral A -function p such that $\|p - f\|_Y < \varepsilon$.

THEOREM 4.1. *If X is A -large then the integral A -functions form a closed discrete subset of $C_s(X)$.*

Proof. Suppose φ_1 and φ_2 are integral A -functions with $\|\varphi_1 - \varphi_2\|_X < 1$. If $\varphi_1 \neq \varphi_2$ then $\varphi_1 - \varphi_2$ is an upper A' -function for some nonempty $A' \subset A$. But this implies that X is A' -small, contradicting Theorem 1.1.

Now define $J(X, A)$ to be the union of the complete sets of conjugate A -integers contained in X . Note that if X is not A -large then, by Theorem 3.3, $J(X, A)$ is finite.

THEOREM 4.2. *If X is A' -small for some non-empty $A' \subset A$ and each component of the complement of X contains an element of A' then $f \in C_s(X)$ is approximable on X if and only if it is matchable on $J(X, A)$.*

Proof. First observe that if φ is an integral A -function which satisfies $|\varphi(x)| < 1$ for each $x \in J(X, A)$, then $\varphi(x) = 0$ for each $x \in J(X, A)$. Indeed $J(X, A)$ is the disjoint union of complete sets of conjugate A -integers. Let x_1, x_2, \dots, x_r be one such complete set.

Then $\prod_{i=1}^r \varphi(x_i)$ is a rational integer with absolute value < 1 . Hence the product is 0, and so at least one of the $\varphi(x_i) = 0$, and since they are conjugate they are all 0, and φ vanishes on $J(X, A)$. Now suppose f is approximable on X and that $\|p_1 - f\|_X < 1/2$ and $\|p_2 - f\|_X < 1/2$. Then $\|p_1 - p_2\|_X < 1$. By what we proved above $p_1(x) = p_2(x)$ for all $x \in J(X, A)$. Since $\|p_2 - f\|_X$ can be chosen arbitrarily small, it follows that $f(x) = p_1(x)$ for all $x \in J(X, A)$; hence that f is matchable on $J(X, A)$.

Assume $\infty \in A'$. The proof is similar when $\infty \notin A'$. Since X is A' -small, there exists a normal integral A' -function φ with $\|\varphi\|_X < 1$. Let K be the (finite) set of those zeros of φ contained in X . Since $\|\varphi\|_X < 1$, $|\varphi(x)| < 1$ for all $x \in J(X, A)$ and hence φ vanishes on $J(X, A)$. Thus $J(X, A) \subset K$.

Let m be a positive integer. By a standard extension of the Stone-Weierstrass theorem, the closed ideal generated by φ^m in $C_s(X)$ consists of all functions $g \in C_s(X)$ vanishing on K . By our assumption about X , the A' -functions are dense in $C_s(X)$. Thus if $\varepsilon > 0$ and $g \in C_s(X)$ vanishes on K , there exists an A' -function $h(x)$ such that $\|\varphi(x)^m h(x) - g(x)\|_X < \varepsilon$. By Lemma 1.7, we can write

$$\varphi(x)^m h(x) = \sum_{j=m}^n (h_i(x)/D(x)) \varphi(x)^j,$$

where $D(x) = \prod_{i=1}^{r'} (x - a_i)^{r_i}$ is the denominator of $\varphi(x)$ with $A' = \{a_0, a_1, \dots, a_{r'}\}$, and where the $h_i(x)$ are polynomials of degree $< r' = \sum_{i=0}^{r'} r_i$. Put $M = \sum_{i=0}^{r'-1} \|x^i/D(x)\|_X$. If $H_i(x)$ is the polynomial obtained from $h_i(x)$ by replacing each coefficient of $h_i(x)$ with its integral part, it is immediate that $\|(h_i(x) - H_i(x))/D(x)\|_X < M$. Put $p(x) = \sum_{j=m}^n (H_i(x)/D(x)) \varphi(x)^j$; $p(x)$ is an integral A -function and

$$\|p(x) - \varphi(x)^m h(x)\|_X \leq M \sum_{j=m}^n \|\varphi(x)\|_X^j$$

and hence

$$\|g(x) - p(x)\|_X \leq \varepsilon + M \cdot \|\varphi(x)\|_X^m / (1 - \|\varphi(x)\|_X).$$

Thus if m is sufficiently large, $\|g(x) - p(x)\|_X < 2\varepsilon$ and hence g is approximable on X . We have just shown that if g vanishes on K then g is approximable on X . If $\varepsilon > 0$ and $g \in C_s(S)$ satisfies $\|g\|_X < \varepsilon$, then it is easy to find $g_1 \in C_s(X)$ vanishing on K and satisfying $\|g - g_1\|_X < 2\varepsilon$. It is immediate that if g is approximable on K then it is approximable on X . Thus we must show that if g is matchable on $J(X, A)$ it is approximable on K . By replacing g by $g - p$ where p is an appropriate integral A -function, we may assume that g vanishes on $J(X, A)$. Now we must show

that if g vanishes on $J(X, A)$, then it is approximable on K . Choose $\theta \in K - J(X, A)$. Let $\theta = \theta_1, \theta_2, \dots, \theta_m$ be the conjugates of θ which are contained in K . Since $\theta \notin J(X, A)$, either θ is not an A -integer or θ has a conjugate outside of X . Suppose first that θ is not an A -integer. By Theorem 1.5, the set $\{\theta_1, \theta_2, \dots, \theta_m\}$ is A -small and hence there exists a normal, integral A -function p such that $|p(\theta_i)| < 1$ for $1 \leq i \leq m$. Since θ is not an A -integer, none of the $p(\theta_i)$ are 0. Next suppose that at least one conjugate is outside of X . Since m is less than the degree d of θ , there exist, by Minkowski's Theorem on linear forms, integers b_0, b_1, \dots, b_{d-1} not all 0 such that $|\sum_{j=0}^{d-1} b_j \theta_i^j| < 1$ for $1 \leq i \leq m$. If $p(x) = \sum_{j=0}^{d-1} b_j x^j$ then $p(\theta_i) \neq 0$, for the degree of $p(x)$ is less than the degree of θ . Thus in either case $p(x)$ is an integral A -function with $0 < |p(\theta_i)| < 1$ for $1 \leq i \leq m$.

By replacing p by $p^n h$ where n is a large enough integer and h is an appropriate integral A -function, we may assume in addition that p vanishes on all elements of K not conjugate to θ . Let p_1, p_2, \dots, p_s be the functions obtained for each set of conjugate A -integers in $K \sim J(X, A)$. If n is large enough, $\varphi = p_1^n + p_2^n + \dots + p_s^n$ will satisfy $0 < |\varphi(x)| < 1$ for $x \in K - J(X, A)$ and $\varphi(x) = 0$ for $x \in J(X, A)$. By the earlier part of the proof applied to K instead of X , any function in $C_s(K)$ which vanishes on $J(X, A)$ is approximable on K . By the earlier comments, the proof is complete.

We now give a characterization of $J(X, A)$.

THEOREM 4.3. *Suppose X is A' -small for some nonempty $A' \subset A$ and that each component of the complement of X contains an element of A' . There exists an integral A -function φ such that $\|\varphi(x)\| < 1$ and the zeros of φ in X form the set $J(X, A)$.*

Proof. Let $q(x)$ be an integral A -function whose zeros are the elements of $J(X, A)$. Choose $h \in C_s(X)$ satisfying, for all $x \in X$: (1) $\|h\|_x = 1$; (2) $h(x) = 1$ if $q(x) = 0$; (3) $|h(x)| < 1/(2|q(x)|)$ if $|q(x)| > 1/2$; (4) $h(x) \neq 0$. Such an h is matchable by 1 on $J(X, A)$, hence is approximable on X . Any sufficiently good approximation, say, the integral A -function g , satisfies, for all $x \in X$, (1) $\|g\| \leq 3/2$; (2) $|g(x)| < 2/(3|q(x)|)$ if $|q(x)| > 1/2$; (3) $g(x) \neq 0$. Put $\varphi = gq$ to complete the proof.

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