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A ring R with identity is a right *PCQI*-ring (*PCI*-ring) if every cyclic right R -module $C \neq R$ is quasi-injective (injective). Left *PCQI*-rings (*PCI*-rings) are similarly defined. Among others the following results are proved: (1) A right *PCQI*-ring is either prime or semi-perfect. (2) A nonprime nonlocal ring is a right *PCQI*-ring iff every cyclic right R -module is quasi-injective or $R \cong \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$, where D is a division ring. In particular, a nonprime nonlocal right *PCQI*-ring is also a left *PCQI*-ring. (3) A local right *PCQI*-ring with maximal ideal M is a right valuation ring or $M^2 = (0)$. (4) A prime local right *PCQI*-ring is a right valuation domain. (5) A right *PCQI*-domain is a right Öre-domain. Faith proved (5) for right *PCI*-domains. If R is commutative then some of the main results of Klatt and Levy on pre-self-injective rings follow as a special case of these results.

Since, in a commutative Dedekind domain D , for each nonzero ideal A , D/A is a self-injective ring, or equivalently D/A is a quasi-injective D -module, every commutative Dedekind domain is a *PCQI*-ring. An example of a *PCQI*-ring which is not a Dedekind domain is given in Levy [14]. Commutative *PCQI*-rings are precisely the pre-self-injective rings characterized by Klatt and Levy [11]. *PCI*-rings have recently been investigated by Faith [4]. Right self-injective right *PCQI*-rings are *qc*-rings which have been studied by Ahsan [1] and Koehler [13].

1. Definitions and preliminaries. Throughout all modules are unitary and right unless specified. An R -module X is called injective relative to an R -module M if for each short exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ the sequence $0 \rightarrow \text{Hom}_R(M/N, X) \rightarrow \text{Hom}_R(M, X) \rightarrow \text{Hom}_R(N, X) \rightarrow 0$ is exact. X is called quasi-injective if X is injective relative to itself. Any R -module injective relative to all R -modules is called injective. Relative projectivity is defined dually.

A ring R is called a right q -ring if each of its right ideals is quasi-injective (see Jain, Mohamed, and Singh [9]). For more results, see [7], [8], [13], [15]. Dually, a ring R is called a right q^* -ring if each cyclic right R -module is quasi-projective (see Koehler [12]).

A ring R is right *qc*-ring if each cyclic right R -module is quasi-injective (see Ahsan [1]). A well-known result of Osofsky [16] states

that R is semisimple artinian iff each cyclic R -module is injective. Koehler [13] showed that R is a right qc -ring iff R is a finite direct sum of rings each of which is semisimple artinian or a rank o duo maximal valuation ring. As a consequence, every qc -ring is both a q -ring and q^* -ring.

In this paper the classes of rings initially called q -rings, q^* -rings, and qc -rings have been called Q -rings, Q^* -rings, and QC -rings respectively.

Let $J(R)$ denote the radical of a ring R . R is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents modulo $J(R)$ can be lifted to R . If R is semiperfect, then there exists a finite maximal family of primitive orthogonal idempotents $\{e_i\}_{1 \leq i \leq n}$ such that $R = \bigoplus \sum_{i=1}^n e_i R$.

R is called a local ring if it has a unique maximal right ideal which must be the radical $J(R)$.

R is a right valuation ring if the set of all right ideals is linearly ordered. R is a maximal valuation ring if every family of pairwise solvable congruences of the form $x \equiv x_\alpha \pmod{A_\alpha}$ has a simultaneous solution where $x_\alpha \in R$ and each A_α is an ideal in R . R is called an almost maximal valuation ring if each of its proper homomorphic images is a maximal valuation ring.

A ring is right duo if every right ideal is two-sided. A ring R has rank O if every prime ideal is a maximal ideal. By duo rings or valuation rings, we shall mean both right and left.

3. General results.

SUBLEMMA 1. *Let I be a right ideal in a ring R such that $R/I \cong R$. Then $R = I \oplus J$, where J is a right ideal, and thus $I = eR$, $e = e^2 \in R$.*

Proof. $R/I \cong R$ implies R/I is projective, and hence I is a direct summand of R .

PROPOSITION 2. *Let R be a right PCQI-ring. If I is a right ideal of R such that $R/I \cong R$, then I is contained in every nonzero two-sided ideal of R .*

Proof. Let S be a nonzero two-sided ideal of R . Then R/S is a qc -ring, hence is semiperfect. Let $f: R/I \rightarrow R$ be an isomorphism. Since $1 + I$ generates R/I , $\bar{R} = xR$, where $x = f(1 + I)$. Then $I = \text{ann } x = \{r \in R \mid xr = 0\}$. So there exists $y \in R$ such that $xy = 1$. Since R/S is semiperfect, $(x + S)(y + S) = 1 + S = (y + S)(x + S)$. Then $1 - yx \in S$. Let $a \in I$, i.e., $xa = 0$. Then $(1 - yx)a = a - yxa = a$, hence $a \in S$. So $I \subseteq S$.

PROPOSITION 3. *Let R be a right PCQI-ring. Then either R is a prime ring or R is semiperfect with nil radical.*

Proof. Suppose R is not prime, and $P \neq 0$ is a prime ideal. Then R/P is a qc-ring, and hence a q -ring. So R/P is simple artinian [9]. Thus P is maximal, hence primitive. So the Jacobson radical is nil.

Since R is not prime, there exist nonzero ideals A, B such that $AB = 0$. Since R is a right PCQI-ring, R/A and R/B are semiperfect, hence each of them has finitely many prime ideals. Since every prime ideal of R contains A or B , it follows that R has finitely many prime ideals as well. Thus $R/J(R)$ is semisimple artinian, and since $J(R)$ is nil, R is semiperfect.

4. Nonlocal semiperfect PCQI-rings. By Proposition 3, all nonprime right PCQI-rings are semiperfect, so the results of this section hold for the class of nonprime nonlocal right PCQI-rings. The case of local right PCQI-rings is discussed in the next section.

LEMMA 4. *Let R be a semiperfect ring. Then R/A is a proper cyclic right R -module, for all nonzero right ideals A .*

Proof. There exists a positive integer n such that R is a direct sum of n indecomposable right R -modules, and R cannot be expressed as a direct sum of more than n right R -modules. Now, if $R/A \cong R$, then, by Lemma 1, $R = A \oplus B$ and $B \cong R$. So $A = (0)$, proving the lemma.

Let R be a nonlocal semiperfect ring, and let $\{e_i\}_{1 \leq i \leq n}$ be a maximal set of primitive orthogonal idempotents in R . Then $R = \bigoplus_{i=1}^n e_i R$ and $n \geq 2$. Throughout this section, e_i 's will denote primitive idempotents. We shall often use a well-known fact that if $A \oplus B$ is a quasi-injective module then any monomorphism $A \rightarrow B$ splits.

LEMMA 5. *Let R be a semiperfect nonlocal right PCQI-ring. If $\sigma \in \text{Hom}_R(e_i R, e_j R)$ such that $\sigma \neq 0$, where $i \neq j$, then $\ker \sigma = (0)$.*

Proof. Suppose $\ker \sigma \neq (0)$, where $0 \neq \sigma \in \text{Hom}_R(e_i R, e_j R)$, $i \neq j$. Then $R/\ker \sigma \cong \bigoplus_{\substack{k=1 \\ k \neq i}}^n e_k R \times \text{Im } \sigma$, and $R/\ker \sigma$ is quasi-injective. Since $\text{Im } \sigma \subseteq e_j R$, the inclusion map $i: \text{Im } \sigma \rightarrow \bigoplus_{\substack{k=1 \\ k \neq i}}^n e_k R$ is a monomorphism. Since $R/\ker \sigma$ is quasi-injective, the inclusion map splits. So $\text{Im } \sigma$ is a direct summand of $e_j R$, hence $\text{Im } \sigma = e_j R$. Since $e_j R$ is projective, $\sigma: e_i R \rightarrow e_j R$ splits. Thus $\ker \sigma = (0)$.

LEMMA 6. *Let R be a semiperfect nonlocal right PCQI-ring with decomposition $\bigoplus \sum_{i=1}^n e_i R$, where $n > 2$. Then $\text{Hom}_R(e_i R, e_j R) \neq 0$ iff $e_i R \cong e_j R$, i.e., $e_j R e_i \neq 0$ iff $e_i R \cong e_j R$.*

Proof. Let $\sigma \in \text{Hom}_R(e_i R, e_j R)$ such that $\sigma \neq 0$. By Lemma 5, $\ker \sigma = 0$. Since $n > 2$, $e_i R \oplus e_j R \cong R / \bigoplus_{\substack{k=1 \\ k \neq i, j}}^n e_k R$ is quasi-injective. Then σ splits, and $0 \neq \text{Im } \sigma$ is a direct summand of $e_j R$. So $\text{Im } \sigma = e_j R$, and σ is an isomorphism. The converse is trivial.

PROPOSITION 7. *Let R be a semiperfect nonlocal right PCQI-ring with decomposition $R = \bigoplus \sum_{i=1}^n e_i R$, where $n > 2$. Then R is a qc-ring.*

Proof. For each i , $e_i R \cong R / \bigoplus_{\substack{k=1 \\ k \neq i}}^n e_k R$. So $e_i R$ is quasi-injective, for each i . Let A_i be the sum of all those $e_i R$ which are isomorphic to each other. Then $R = \bigoplus \sum_{i=1}^p A_i$. We claim that A_i is a two-sided ideal of R , for each i . Clearly A_i is a right ideal. Consider $e_j R$ such that $e_j R \not\subseteq A_i$. Define $f: e_i R \rightarrow e_j R$, where $e_i R \subseteq A_i$, by $f(e_i r) = e_j x e_i r$, for $x \in R$. Then $f \in \text{Hom}_R(e_i R, e_j R)$. Since $e_i R$ and $e_j R$ are not isomorphic, $f = 0$ by Lemma 6. So, for $e_j R \not\subseteq A_i$, $e_j R A_i = 0$. So $R A_i \subseteq A_i$. Since A_i is a finite direct sum of isomorphic quasi-injective right ideals, A_i is quasi-injective, hence a qc-ring. Thus, by Koehler [13], R is a qc-ring.

PROPOSITION 8. *Let R be a semiperfect right PCQI-ring such that $R = e_1 R \oplus e_2 R$. If $e_1 R \cong e_2 R$, then R is a qc-ring.*

Proof. Now $e_1 R \cong e_2 R$ and $R/e_2 R \cong R/e_1 R$, hence $e_2 R$ and $e_1 R$ are quasi-injective. Since $e_1 R \cong e_2 R$, $R = e_1 R \oplus e_2 R$ is quasi-injective, hence right self-injective. So R is a qc-ring.

PROPOSITION 9. *Let R be a semiperfect right PCQI-ring such that $R = e_1 R \oplus e_2 R$. If $e_1 R e_2 = 0$ and $e_2 R e_1 = 0$, then R is a qc-ring.*

Proof. If $e_1 R e_2 = 0$ and $e_2 R e_1 = 0$, then $e_1 R$ and $e_2 R$ are two-sided ideals of R . Thus $e_1 R \cong R/e_2 R$ and $e_2 R \cong R/e_1 R$ are qc-rings. Then $R = e_1 R \oplus e_2 R$ is a qc-ring.

PROPOSITION 10. *Let R be a semiperfect right PCQI-ring such that $R = e_1 R \oplus e_2 R$. If $e_1 R e_2 \neq 0$ and $e_2 R e_1 \neq 0$, then R is a qc-ring.*

Proof. $e_1 R e_2 \neq 0$ and $e_2 R e_1 \neq 0$ imply that there exist nonzero homomorphisms, hence monomorphisms by Lemma 5, from $e_1 R$ to $e_2 R$ and from $e_2 R$ to $e_1 R$. Thus, by Bumby [2], $e_1 R \cong e_2 R$, and Proposition 8 yields the result.

PROPOSITION 11. *Let $R = e_1R \oplus e_2R$ be a semiperfect right PCQI-ring where $e_1R \not\cong e_2R$ and exactly one of e_1Re_2 or e_2Re_1 is zero. Then R is nonprime with nil radical.*

Proof. It follows from the fact that if $e_1Re_2 \neq 0$, then e_1Re_2 is a nilpotent ideal.

THEOREM 12. *Let R be a nonlocal right PCQI-ring. Then R is semiperfect iff R is nonprime or simple artinian.*

Proof. Necessity follows by Proposition 3, and sufficiency follows from Proposition 7-11 and Koehler's characterization of qc-rings [13] (cf. definitions and preliminaries).

THEOREM 13. *Let R be a semiperfect nonlocal ring. Then R is a right PCQI-ring iff either (i) $R = \bigoplus_{i=1}^n R_i$, where R_i is semisimple artinian or a rank 0 duo maximal valuation ring or (ii) $R = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$, where D is a division ring.*

Proof. Let R be a right PCQI-ring. By Propositions 7-10, R is a qc-ring unless $R = e_1R \oplus e_2R$, where e_1R and e_2R are not isomorphic and exactly one of e_1Re_2 or e_2Re_1 is zero, say $e_1Re_2 \neq 0$ and $e_2Re_1 = 0$. If R is a QC-ring, we get (i) by Koehler [13]. Otherwise, we have $R \cong \begin{pmatrix} e_1Re_1 & e_1Re_2 \\ 0 & e_2Re_2 \end{pmatrix}$. We claim that e_1Re_1 and e_2Re_2 are isomorphic division rings and $M = e_1Re_2$ is a (D, D) -bimodule such that $\dim_D M = 1 = \dim M_D$, where $D \cong e_1Re_1 \cong e_2Re_2$. Clearly e_1Re_2 is nilpotent ideal and since it is nonzero, R is not prime. So, by Proposition 3, the radical N of R is a nil ideal. Thus e_2Ne_2 is nil. We claim that $e_2Ne_2 = 0$. Let $e_2xe_2 \in e_2Ne_2$. Define $\sigma: e_2R \rightarrow e_2R$ by $\sigma(e_2y) = e_2xe_2y$. Then $\sigma \in \text{Hom}_R(e_2R, e_2R)$, and since e_2xe_2 is nilpotent, σ is not a monomorphism. So $\ker \sigma \neq (0)$. Since $\text{Hom}_R(e_2R, e_1R) \neq 0$, there exists an embedding $\eta: e_2R \rightarrow e_1R$. Now $\eta\sigma: e_2R \rightarrow e_1R$, and since $\ker \sigma \neq (0)$, $\ker \eta\sigma \neq (0)$. By Lemma 5, $\eta\sigma = 0$. Since η is a monomorphism, we have $\sigma = 0$. Thus $e_2xe_2 = 0$, and $e_2Ne_2 = 0$. So e_2Re_2 is a division ring. Further $e_2Re_2 = e_2R$ since $e_2Re_1 = (0)$. Thus $e_2N = 0$, and e_2R is a minimal right ideal. Now e_1R is uniform because it is quasi-injective and indecomposable. Since $0 \neq e_1Re_2R$ is the sum of the images of all R -homomorphisms of e_2R into e_1R , the fact that e_2R is minimal and e_1R is uniform yields that e_1Re_2R itself is the unique minimal right subideal of e_1R , is isomorphic to e_2R , and is contained in every nonzero right subideal of e_1R . We claim that $e_1Ne_1 = 0$. Let $0 \neq e_1xe_1 \in e_1Ne_1$. Since N is nil, e_1xe_1 is nilpotent. Then $\sigma: e_1R \rightarrow e_1R$ defined by $\sigma(e_1r) = e_1xe_1r$ is an endo-

morphism of e_1R with $\ker \sigma \neq (0)$. Let $A = \ker \sigma$. Then $e_1Re_2R \subset A$, and we have $e_1xe_1Re_2 = (0)$. On the other hand, $e_1Re_2R \subseteq e_1xe_1R$ yields that $e_1xe_1Re_2 \neq (0)$. This is a contradiction. Hence $e_1Ne_1 = (0)$, and e_1Re_1 is a division ring. Now using the fact that $\text{Hom}_R(e_1R, e_1R)$ is a division ring and that e_1R is quasi-injective, it follows that every member of $\text{Hom}(e_1Re_2R, e_1Re_2R)$ admits a unique extension to an endomorphism of e_1R . Further, every endomorphism of e_1R maps e_1Re_2R into itself since e_1Re_2R is the unique minimal subideal of e_1R . Thus $\text{Hom}(e_1Re_2R, e_1Re_2R) \cong \text{Hom}(e_1R, e_1R)$. Since $e_1Re_2R \cong e_2R$, we obtain $e_1Re_1 \cong e_2Re_2$.

Now $e_1N = e_1Ne_2$ because $e_1Ne_1 = (0)$. Since $e_1Re_2R \subseteq e_1N$, we get $e_1N = e_1Re_2 = e_1Re_2R$. Thus $M = e_1Re_2$ is a one-dimensional right vector space over $D = e_2Re_2$. We show that M is also a one-dimensional left e_1Re_1 -space. Let $X = \begin{pmatrix} e_1Re_1 & M \\ 0 & 0 \end{pmatrix} \cong R/A$, where $A = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$. Then X is quasi-injective. Let $0 \neq x \in M$, and let $y \in M$. Consider $\sigma: \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ defined by $\sigma \begin{pmatrix} 0 & xc \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & yc \\ 0 & 0 \end{pmatrix}$, for $c \in D$. Then σ is an R -endomorphism, so it can be extended to an endomorphism η of X . Let $\eta \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Then we have $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = \sigma \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \eta \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ax \\ 0 & 0 \end{pmatrix}$. Thus $y = ax$, so $M = e_1Re_1x$. So M is a one-dimensional left vector space over e_1Re_1 . Thus, for each $d \in e_1Re_1$, there exists a unique $d' \in e_2Re_2$ such that $dx = xd'$. Define $\theta: e_1Re_1 \rightarrow e_2Re_2$ by $\theta(d) = d'$. Then θ is an isomorphism, and we may identify d and d' . Then $\eta: \begin{pmatrix} D & D \\ 0 & D \end{pmatrix} \rightarrow \begin{pmatrix} D & M \\ 0 & D \end{pmatrix}$ defined by $\eta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & bx \\ 0 & c \end{pmatrix}$ is an isomorphism.

Conversely, if R satisfies (i), then, by Koehler [13], R is a QC -ring, hence a $PCQI$ -ring. If R satisfies (ii), then straightforward computation shows that R is a right $PCQI$ -ring.

Since every right QC -ring is a left QC -ring and $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ is also a left $PCQI$ -ring, we get the following corollary.

COROLLARY. *A nonlocal semiperfect right $PCQI$ -ring is also a left $PCQI$ -ring.*

5. Local $PCQI$ -rings. Theorem 13 and Theorems 14, 15, and 16 which follow generalize Klatt and Levy's [11] theorems for commutative pre-self-injective rings which are not domains. Throughout this section M will denote the unique maximal right ideal of a local ring R . M is then the Jacobson radical of R , and R/M is a division ring.

THEOREM 14. *Let R be a local right $PCQI$ -ring with maximal ideal M . Then either R is a right valuation ring or $M^2 = (0)$ and M_R has composition length 2.*

Proof. First note that for all nonzero right ideals A , R/A is indecomposable quasi-injective and hence uniform. Now we show that all nonzero right ideals are either minimal or essential. Let A, B be nonzero right ideals such that $A \cap B = (0)$. We claim that A is minimal. Let C be a nonzero right ideal properly contained in A . Then R/C is quasi-injective and not uniform since $A/C \cap (B + C)/C = 0$. This is a contradiction, so A is minimal. Similarly, B is minimal. In particular, it follows that any maximal independent family of minimal right ideals can contain at most two members.

If $\text{Soc } R_R = (0)$, then all nonzero right ideals are essential. Let A, B be two nonzero right ideals. If neither $A \subseteq B$ nor $B \subseteq A$, then $R/A \cap B$ is quasi-injective but not uniform since $A/(A \cap B) \cap B/(A \cap B) = (\bar{0})$. As before, this is a contradiction. So either $A \subseteq B$ or $B \subseteq A$.

If $\text{Soc } R_R$ consists of a unique minimal right ideal then it is clear that R is a right valuation ring.

Finally, suppose $\text{Soc } R_R = A \oplus B$, where A, B are minimal right ideals. Then R cannot be prime. Let $x \in M$, and consider xR . If xR is not minimal, then xR is quasi-injective and decomposable. Then $xR = A \oplus B$. In any case, for all $x \in M$, $x \in \text{Soc } R_R$. This implies that $M^2 = (0)$, and the composition length of M is 2, completing the proof.

The next two theorems give the structure of non-prime local right PCQI-rings. Prime local PCQI-rings are discussed in the next section.

THEOREM 15. *For a nonprime right valuation ring R , the following are equivalent:*

- (i) *R is a right PCQI-ring.*
- (ii) *R is a right duo almost maximal valuation ring of rank 0 such that any left ideal containing a nonzero right ideal is two-sided.*

Proof. (i) \Rightarrow (ii). Since R is not prime, M is nil by Proposition 3. So, if xR is a nontrivial principal right ideal of R , xR is quasi-injective. Since xR is essential in R , the injective hull of xR is the same as that of R . Hence, by Johnson and Wong [10], $RxR \subseteq xR$. So xR is a two-sided ideal of R . Thus R is a right duo ring. Since each proper homomorphic image of a PCQI-ring is a QC-ring, the proof of (i) \Rightarrow (ii) as well as that of (ii) \Rightarrow (i) is completed by a theorem of Koehler [13].

THEOREM 16. *For a local ring R with $M^2 = (0)$ and the composition length of M_R equal to 2, the following are equivalent:*

(i) R is a right PCQI-ring.

(ii) For each nonzero right ideal A in R and for each $m_1, m_2 \notin A$, the congruence $xm_1 \equiv m_2 \pmod{A}$ has a solution, $x = \alpha$, such that $\alpha A \subset A$.

Proof. Under the hypothesis the only nonzero right ideals A of R different from M and R are minimal right ideals, and M/A is a simple right R -module.

(i) \Rightarrow (2) Let A be a nontrivial right ideal in R , and let $m_1, m_2 \in R$ such that $m_1, m_2 \notin A$. Then $\bar{m}_1 R = M/A = \bar{m}_2 R$, and the mapping $\sigma: M/A \rightarrow M/A$ which sends $\bar{m}_1 r$ to $\bar{m}_2 r$ is a well-defined R -homomorphism. Since R/A is quasi-injective, σ can be lifted to $\sigma^* \in \text{Hom}_R(R/A, R/A)$. Let $\sigma^*(\bar{1}) = \bar{\alpha}$. Then $\bar{\alpha} m_1 = \bar{m}_2$. Hence $xm_1 \equiv m_2 \pmod{A}$ has a solution $x = \alpha$. Clearly $\alpha A \subset A$.

(ii) \Rightarrow (i) We only need to prove that if A is a nontrivial right ideal of R and $\sigma: M/A \rightarrow R/A$, is a nonzero R -homomorphism, then σ can be extended to an R -homomorphism $\sigma^*: R/A \rightarrow R/A$. Let $m \in M$, where $m \notin A$. Then $M/A = \bar{m}R$. Also, $\sigma(M/A) = M/A$. Let $\sigma(\bar{m}) = \bar{m}r$. Since $M^2 = (0)$, $r \notin M$. So r is invertible, and $mr \notin A$. Let $\alpha \in R$ be chosen such that $\alpha m \equiv mr \pmod{A}$, and $\alpha A \subseteq A$. Then $\sigma^*(\bar{r}) = \bar{\alpha}R$ is well-defined, and it extends σ , completing the proof.

The example which follows shows that a local right PCQI-ring is not necessarily a left PCQI-ring.

EXAMPLE. Let F be a field which has a monomorphism $\rho: F \rightarrow F$ such that $[F: \rho(F)] > 2$. Take x to be an indeterminate over F . Make $V = xF$ into a right vector space over F in a natural way. Let $R = \{(\alpha, x\beta) \mid \alpha, \beta \in F\}$. Define

$$(\alpha_1, x\beta_1) + (\alpha_2, x\beta_2) = (\alpha_1 + \alpha_2, x\beta_1 + x\beta_2)$$

and

$$(\alpha_1, x\beta_1)(\alpha_2, x\beta_2) = (\alpha_1\alpha_2, x(\rho(\alpha_1)\beta_2 + \beta_1\alpha_2)).$$

Then R is a local ring with identity with the maximal ideal

$$M = \{(0, x\alpha) \mid \alpha \in F\}.$$

In fact, M is also a minimal right ideal and $M^2 = (0)$. Thus R is a right PCQI-ring. Further, if $\{\alpha_i\}_{i \in I}$ is a basis of F as a vector space over $\rho(F)$ then straightforward computations yield that $M = \bigoplus \sum R(0, x\alpha_i)$ as a direct sum of irreducible left R -modules $R(0, x\alpha_i)$. Since $\text{card } I > 2$, it follows by Theorem 14 that R is not a left PCQI-ring.

6. Prime local PCQI-rings.

THEOREM 17. *Let R be a prime local right PCQI-ring. Then R is a right valuation domain, hence right semihereditary.*

Proof. By Theorem 14, R is a right valuation ring. Let A denote the intersection of all nonzero two-sided ideals of R . The proof that R is a domain falls into three cases.

(i) $A = (0)$.

Let $x, y \in R$ such that $xy = 0$. Suppose $y \neq 0$. Then yR is a nonzero right ideal of R . Since R is right valuation and $A = (0)$, yR must contain a nonzero two-sided ideal of R . Further, each proper homomorphic image of R is a local QC-ring, hence a duo ring [13]. This implies that yR is two-sided. Hence $x = 0$, and R is an integral domain.

(ii) $A \neq (0)$ and $A \neq M$.

Under these hypotheses, A cannot be a prime ideal. So there exist $x, y \in R$ such that $xRy \subseteq A$, $x \notin A$ and $y \notin A$. Since R is right valuation, $A \subseteq xR$ and $A \subseteq yR$. So both xR and yR are two-sided ideals. For definiteness, let $xR \subseteq yR$. Then $(xR)^2 \subseteq (xR)(yR) \subseteq AR = A$ gives that $(xR)^2 = A$ by the minimality of A . Also $A = A^2$, hence $(xR)^2 = (xR)^4$. It follows that $x^2R = x^4R$. Then $x^2 = x^4r$, for some $r \in R$, and $x^2(1 - x^2r) = 0$. So $x^2 = 0$. Thus $A = (0)$, and this case cannot occur.

(iii) $A = M$.

Let $S \subset R$, and let $r(S)$ denote the right annihilator of S in R . Let $Z(R) = \{x \in R \mid r(x) \text{ is an essential right ideal}\}$. Then $Z(R)$ is an ideal in R called the right singular ideal.

Since R is a right valuation ring, R is immediately a domain if $Z(R) = (0)$.

So assume that $Z(R) \neq (0)$. Then $Z(R) = M$, and each element in M is a right zero divisor. So $x \in M$ implies that xR is proper cyclic, hence quasi-injective. Also xR is an essential right ideal in R . By Johnson and Wong [10], $RxR \subseteq xR$. Hence xR is two-sided. So R is a prime right duo ring, and it follows that R is a domain.

7. PCQI-domains. In this section we discuss right PCQI-rings which are integral domains and prove that these are right Öre-domains. This generalizes the result of Faith [4]. Our proof, in this case, though it runs on the same lines as that of Faith, does not use Faith's result.

PROPOSITION 18. *Let R be a right PCQI-domain, and let I be a nonessential right ideal of R . Then R/I is an injective right R -*

module containing a copy of R .

Proof. Since I is nonessential, there exists a nonzero right ideal J in R such that $I \cap J = 0$. Let $a \in J$ such that $a \neq 0$. Then $aR \cap I \subseteq J \cap I = 0$. Consider $r(a + I) = \{x \in R \mid ax \in I\}$. Clearly $r(a + I) = 0$. So R/I contains a copy of R . Since R/I is also quasi-injective, this implies that R/I is injective by [17].

For a right R -module A , let \hat{A} denote the injective hull of A .

PROPOSITION 19. *Let R be a right PCQI-domain which is not a right Öre-domain. Then \hat{R} is finitely presented.*

Proof. Let $a \in R$ such that $a \neq 0$ and aR is not essential. Then R/aR is injective. Since R/aR contains a copy of R and is injective, R/aR contains a copy of \hat{R} . Then $R/aR = Y/aR \oplus X/aR$, where $X/aR \cong \hat{R}$. Now Y/aR is cyclic. So $Y = aR + bR$, for some $b \in R$, and the short exact sequence $0 \rightarrow Y \rightarrow R \rightarrow R/Y \cong X/aR \cong \hat{R} \rightarrow 0$ shows that \hat{R} is finitely presented.

THEOREM 20. *A right PCQI-domain R is a right Öre-domain.*

Proof. Let R be a right PCQI-domain. Suppose R is not a right Öre-domain. Then, as in Proposition 19, there exists $a \in R$ such that $R/aR = Y/aR \oplus X/aR$, where $X/aR \cong \hat{R} \cong R/Y$ and $Y = aR + bR$. We also get that $R = X + Y$, where $X \cap Y = aR$. This yields an exact sequence $0 \rightarrow aR \rightarrow X \times Y \rightarrow R \rightarrow 0$ which splits. So $X \times Y \cong aR \times R \cong R \times R$. This implies that $Y = aR + bR$ is a finitely generated projective right ideal. Since $\hat{R} \cong R/Y$, $0 \rightarrow Y \rightarrow R \rightarrow \hat{R} \rightarrow 0$ is exact. Then $Y \otimes_R \hat{R} \rightarrow R \otimes_R \hat{R} \rightarrow \hat{R} \otimes_R \hat{R} \rightarrow 0$ is exact. Also, a finitely generated projective R -module is essentially finitely related. So, by Cateforis ([3], Proposition 1.7), $(aR + bR) \otimes_R \hat{R}$ is projective as an \hat{R} -module. Then $Y \otimes_R \hat{R}$ is a direct summand of a free \hat{R} -module. Now $Z(\hat{R}_{\hat{R}}) = 0$, hence $Z(Y \otimes_R \hat{R}) = 0$ because $Y \otimes_R \hat{R}$ is a direct summand of a free \hat{R} -module. Now consider $Y \otimes_R \hat{R} \xrightarrow{i} R \otimes_R \hat{R} \rightarrow \hat{R} \otimes_R \hat{R} \rightarrow 0$. Again, by Cateforis ([3], Lemma 1.8), $\ker i = Z(Y \otimes_R \hat{R}) = 0$. So $0 \rightarrow Y \otimes_R \hat{R} \xrightarrow{i} R \otimes_R \hat{R} \rightarrow \hat{R} \otimes_R \hat{R} \rightarrow 0$ is exact. Since $R \otimes_R \hat{R} \cong \hat{R}$, let $f: R \otimes_R \hat{R} \rightarrow \hat{R}$ be the canonical isomorphism. Then $fi: Y \otimes_R \hat{R} \rightarrow \hat{R}$ is a monomorphism, and $Y \otimes_R \hat{R} \cong Y\hat{R}$. Since Y is finitely generated, $Y\hat{R}$ is a finitely generated right ideal of \hat{R} . So $Y\hat{R} = e\hat{R}$, where $e^2 = e$. Thus we have the following exact sequence: $0 \rightarrow e\hat{R} \rightarrow \hat{R} \rightarrow \hat{R} \otimes_R \hat{R} \rightarrow 0$, and $\hat{R} \otimes_R \hat{R} \cong \hat{R}/e\hat{R} = (1 - e)\hat{R}$. Hence $\hat{R} \otimes_R \hat{R}$ is isomorphic to a direct summand of \hat{R} . Since $Z(\hat{R}_R) = 0$, $Z(\hat{R} \otimes_R \hat{R}) = 0$. Since $\hat{R} = xR$, for some $x \in \hat{R}$, the

kernel of the canonical map $f: \hat{R} \otimes_R \hat{R} \rightarrow \hat{R}$ defined by $f(a \otimes b) = ab$ is contained in $Z(\hat{R} \otimes_R \hat{R})$ and hence must be zero. Since f is surjective, f is an isomorphism. By Silver ([18], Proposition 1.1), there exists an epimorphism in the category of rings from R to \hat{R} .

Let M be a right \hat{R} -module which is quasi-injective as a right R -module. We claim that M is quasi-injective as a right \hat{R} -module. Let $0 \rightarrow A_{\hat{R}} \rightarrow M_{\hat{R}} \rightarrow B_{\hat{R}} \rightarrow 0$ be exact. Consider $0 \rightarrow \text{Hom}_{\hat{R}}(B_{\hat{R}}, M_{\hat{R}}) \rightarrow \text{Hom}_{\hat{R}}(M_{\hat{R}}, M_{\hat{R}}) \rightarrow \text{Hom}_{\hat{R}}(A_{\hat{R}}, M_{\hat{R}})$. By Silver ([18], Corollary 1.3), $\text{Hom}_{\hat{R}}(M_{\hat{R}}, M_{\hat{R}}) \cong \text{Hom}_R(N, N^1)$, where N, N^1 are right \hat{R} -modules. Also $0 \rightarrow \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(M, M) \rightarrow \text{Hom}_R(A, M) \rightarrow 0$ is exact since M_R is quasi-injective. Thus $0 \rightarrow \text{Hom}_{\hat{R}}(B, M) \rightarrow \text{Hom}_{\hat{R}}(M, M) \rightarrow \text{Hom}_R(A, M) \rightarrow 0$ is exact. So $M_{\hat{R}}$ is quasi-injective. Let K be a cyclic right R -module. Then K is a cyclic right R -module. Since R is a right *PCQI*-domain, K_R is quasi-injective. Thus $K_{\hat{R}}$ is quasi-injective. Since \hat{R} is right self-injective, \hat{R} is a *QC*-ring. So \hat{R} is semiperfect and simple, hence simple artinian. Thus \hat{R} is a division ring. This proves that R is a right Öre-domain.

We conclude by a remark that we have not studied arbitrary prime right *PCQI*-rings. This case remains open. Indeed, a characterization of right *PCQI*-domains has not yet been obtained.

REFERENCES

1. J. Ahsan, *Rings all of whose cyclic modules are quasi-injective*, Proc. London Math. Soc., **27** (1973), 425-439.
2. R. Bumby, *Modules which are isomorphic to submodules of each other*, Archiv. der Math., **16** (1965), 184-185.
3. V. C. Cateforis, *Flat regular quotient rings*, Trans. Amer. Math. Soc., **138** (1969), 241-249.
4. C. Faith, *When are proper cyclics injective*, Pacific J. Math., **45** (1973), 97-112.
5. ———, *Lectures on Injective Modules and Quotient Rings*, Springer-Verlag, Berlin, 1967.
6. M. Harada, *Note on Quasi-injective modules*, Osaka J. Math., **2** (1965), 351-356.
7. D. A. Hill, *Semi-perfect q -rings*, Math. Ann., **200** (1973), 113-121.
8. G. Ivanov, *Non-local rings whose ideals are all quasi-injective*, Bull. Australian Math. Soc., **6** (1972), 45-52.
9. S. K. Jain, S. Mohamed and Surjeet Singh, *Rings in which every right ideal is quasi-injective*, Pacific J. Math., **31** (1969), 73-79.
10. R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc., **36** (1961), 260-268.
11. G. B. Klatt and L. S. Levy, *Pre-self-injective rings*, Trans. Amer. Math. Soc., **137** (1969), 407-419.
12. A. Koehler, *Rings for which every cyclic module is quasi-projective*, Math. Ann., **189** (1970), 311-316.
13. ———, *Rings with quasi-injective cyclic modules*, Quar. J. Math. Oxford (2), **25** (1974), 51-55.
14. L. S. Levy, *Commutative rings whose homomorphic images are self-injective*, Pacific J. Math. **18** (1966), 149-153.

15. S. Mohamed, *q-rings with chain conditions*, J. London Math. Soc., **2** (1970), 453-460.
16. B. L. Osofsky, *Noncyclic injective modules*, Proc. Amer. Math. Soc., **19** (1968), 1383-1384.
17. E. deRobert, *Projectifs et injectifs relatifs*, C. R. Acad. Sci. Paris, Ser. A, **286** (1969), 361-364.
18. L. Silver, *Noncommutative localizations and applications*, J. Algebra, **7** (1967), 44-76.

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