Pacific Journal of Mathematics

A COMMUTATIVITY THEOREM FOR NON-ASSOCIATIVE ALGEBRAS OVER A PRINCIPAL IDEAL DOMAIN

JIANG LUH AND MOHAN S. PUTCHA

Vol. 68, No. 2 April 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol 68 No 2, 1977

A COMMUTATIVITY THEOREM FOR NON-ASSOCIATIVE ALGEBRAS OVER A PRINCIPAL IDEAL DOMAIN

JIANG LUH AND MOHAN S. PUTCHA

Let A be an algebra (not necessarily associative) over a principal ideal domain R such that for all $a, b \in A$, there exist $\alpha, \beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha ab = \beta ba$. It is shown that A is commutative.

Throughout this paper N will denote the set of natural numbers and Z^+ the set of positive integers. A will denote an algebra with identity 1 over a Principal Ideal Domain R. If $a, b \in A$ then $[a, b] = ab - ba$. If $\alpha, \beta \in R$, then (α, β) denotes the greatest common divisor of α and β. If $a \in A$, then the *order* of a, $o(a)$ is the generator of the ideal $I = \{\alpha \mid a \in R, \alpha a = 0\}$ of R. $o(a)$ is unique up to associates. As a generalization of concepts in **[1]**, **[2]**, **[3]**, **[4]**, **[5]** we consider the following:

(*) For all $a, b \in A$, there exist $\alpha, \beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha ab = \beta ba$.

We will show that if A satisfies (*), then A is commutative. This generalizes [**3**; Theorem 3.5].

LEMMA 1. *Let p be a prime in R, $m \in Z^+$ such that $p^m A = (0)$. If A satisfies (*), then A is commutative.*

Proof. Let C denote the center of A. Let $x \in A$, $o(x) = p^k$, $k \in N$. We prove by induction on k that $x \in C$. If $k = 0$, then $x = 0$. So let $k > 0$. Let $y \in A$. First we show

$$[x, y] \neq 0 \quad \text{implies} \quad [yx, y] = 0. \tag{1}$$

If $yx = 0$, this is trivial. So let $yx \neq 0$. Now for some $\alpha_1, \alpha_2 \in R$,

$$\begin{aligned} &\alpha_1 xy = \alpha_2 yx, (\alpha_1, \alpha_2) = 1 \\ &\beta_1(x+1)y = \beta_2 y(x+1), (\beta_1 \beta_2) = 1. \end{aligned} \tag{2}$$

So $\alpha_1\beta_1(x+1)y = \alpha_1\beta_2 y(x+1)$. Thus substituting the above, we get

$$(\alpha_2\beta_1 - \alpha_1\beta_2)yx = (\alpha_1\beta_2 - \alpha_1\beta_1)y. \tag{3}$$

We claim that $(\alpha_2\beta_1-\alpha_1\beta_2)yx\neq 0$. For otherwise $(\alpha_1\beta_2-\alpha_1\beta_1)y=0$. Since $y\neq 0$, we get $p\mid\alpha_1\beta_2-\alpha_1\beta_1$.

Also $(\alpha_1\beta_2-\alpha_1\beta_1)yx=0$. Since $(\alpha_2\beta_1-\alpha_1\beta_2)yx=0$, we get $(\alpha_2-\alpha_1)\beta_1yx=0$. Since $yx\neq 0$, $p\mid\beta_1(\alpha_2-\alpha_1)$. So

$$p\mid\alpha_1(\beta_2-\beta_1),\, p\mid\beta_1(\alpha_2-\alpha_1).$$

Case 1. $p\nmid\alpha_1$. Then since $\alpha_1(\beta_2-\beta_1)y=0$, we get $(\beta_2-\beta_1)y=0$. So by (2), $\beta_1[x,y]=0=\beta_2[x,y]$. Since $[x,y]\neq 0$, we get $p\mid\beta_1$, $p\mid\beta_2$, contradicting (2).

Case 2. $p\mid\alpha_1$. Then $p\nmid\alpha_2$ and so $p\nmid\alpha_2-\alpha_1$. Thus $p\mid\beta_1$. So $p\nmid\beta_2$, $p\nmid\beta_2-\beta_1$. Since $\alpha_1(\beta_2-\beta_1)y=0$ we get $\alpha_1y=0$. So $\alpha_1xy=0$. By (2), $\alpha_2yx=0$. Since $yx\neq 0$, we get $p\mid\alpha_2$, a contradiction.

Hence by (3)

$$(\alpha_2\beta_1-\alpha_1\beta_2)yx\neq 0.$$

In particular

$$\alpha_2\beta_1-\alpha_1\beta_2\neq 0.$$

So

$$\alpha_2\beta_1-\alpha_1\beta_2=p^t\delta,\ t\in N,\ \delta\in R,\ (\delta,p)=1.$$

If $t\geqq k$, then $(\alpha_2\beta_1-\alpha_1\beta_2)yx=0$, a contradiction. So $t<k$. Hence

$$p^{k-t}(\alpha_1\beta_2-\alpha_1\beta_1)y=p^{k-t}p^t\delta yx=0.$$

Let $o(y)=p^i$, $i\in N$. If $i<k$, then $y\in C$, a contradiction. So $i\geqq k$. Hence

$$p^k\mid p^i\mid p^{k-t}(\alpha_1\beta_2-\alpha_1\beta_1).$$

So $p^t\mid\alpha_2\beta_2-\alpha_1\beta_1$ and $\alpha_1\beta_2-\alpha_1\beta_1=p^t\gamma$, $\gamma\in R$. Then $p^t\delta yx=p^t\gamma y$. Hence $p^t(\delta yx-\gamma y)=0$. By induction hypothesis, $\delta yx-\gamma y\in C$. So $[\delta yx-\gamma y,y]=0$. Thus $\delta[yx,y]=0$. Since $(\delta,p)=1$, $[yx,y]=0$. This establishes (1).

Now let $u\in A$ and suppose $[x,u]\neq 0$. Then also $[x,u+1]\neq 0$. By (1), $[ux,u]=0=[(u+1)x,u]$. So $[x,u]=0$, a contradiction. So $x\in C$ and the lemma is proved.

LEMMA 2. *Suppose A satisfies* (*). *Let $a,b\in A,o(b)=0$. If $ba=0$, then $ab=0$.*

Proof. Suppose $ab \neq 0$. Then there exist $\beta_1, \beta_2, \gamma_1, \gamma_2 \in R$ such that

$$\begin{aligned} &\beta_1(a+1)b = \beta_2 b(a+1),\ (\beta_1, \beta_2) = 1, \\ &\gamma_1 a(b+1) = \gamma_2 b(a+1),\ (\gamma_1, \gamma_2) = 1. \end{aligned} \tag{4}$$

So

$$\beta_1 ab = (\beta_2 - \beta_1)b \quad \text{and} \quad (\gamma_2 - \gamma_1)a = \gamma_1 ab. \tag{5}$$

If $\beta_2 = \beta_1$, then β_1, β_2 are units and by (5) $ab = ba = 0$, a contradiction. So $\beta_2 - \beta_1 \neq 0$. Similarly $\gamma_2 - \gamma_1 \neq 0$. Since $o(b) = 0$, we get by (5) that $o(ab) = 0$. So $o(a) = 0$. Hence by (5), $\beta_1 \neq 0$, $\gamma_1 \neq 0$. Also by (5) $[\beta_1 ab, b] = 0$.

So

$$\begin{aligned} (\gamma_2 - \gamma_1)\beta_1 ab &= \gamma_1\beta_1(ab)b \\ &= \gamma_1\beta_1 b(ab) \\ &= \beta_1(\gamma_2 - \gamma_1)ba \\ &= 0. \end{aligned}$$

So $o(ab) \neq 0$, a contradiction. This proves the lemma.

LEMMA 3. *Suppose A satisfies $(*)$. Let $b \in A$, $o(b) = 0$. Then $b \in C$, the center of A.*

Proof. Let $a \in A$. There exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ such that

$$\begin{aligned} &\alpha_1 ab = \alpha_2 ba,\ (\alpha_1, \alpha_2) = 1, \\ &\beta_1(a+1)b = \beta_2 b(a+1),\ (\beta_1, \beta_2) = 1. \end{aligned} \tag{6}$$

Multiplying the second equation by α_1 and substituting the first we obtain

$$b[(\alpha_2\beta_1 - \alpha_1\beta_2)a - (\alpha_1\beta_2 - \alpha_1\beta_1)\cdot 1] = 0.$$

By Lemma 2,

$$[(\alpha_2\beta_1 - \alpha_1\beta_2)a - (\alpha_1\beta_2 - \alpha_1\beta_1)\cdot 1]b = 0.$$

Let $\mu = \alpha_2\beta_1 - \alpha_1\beta_2$. Then $\alpha_1(\beta_2 - \beta_1)b = \mu ab = \mu ba$. By (6) $\alpha_1\mu ab = \alpha_2\mu ba = \alpha_2\mu ab$. So

$$(\alpha_2 - \alpha_1)\alpha_1(\beta_2 - \beta_1)b = 0.$$

Since $o(b) = 0$, we obtain by (6) that either $\alpha_1 = \alpha_2$ is a unit, $\beta_1 = \beta_2$ is a unit or else $\alpha_1 = 0$. The first two cases imply by (6) that $ab = ba$. So let $\alpha_1 = 0$. Then $\alpha_2 ba = 0$ and α_2 is a unit by (6). So $ba = 0$. By Lemma 2, $ab = 0$. Thus in any case $ab = ba$ and we are done.

THEOREM 4. *Suppose A satisfies* $(*)$. *Then A is commutative.*

Proof. Suppose A is not commutative. We will obtain a contradiction. There exists $x \in A$ such that $x \notin C$, the center of A. So $x + 1 \notin C$. By Lemma 3 $o(x) \neq 0$ and $o(x+1) \neq 0$. Hence $o(1) \neq 0$. Let $o(1) = d \neq 0$. Then d is not a unit and hence $d = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ for some primes $p_1, \cdots, p_t \in A$ and some positive integers $\alpha_1, \cdots, \alpha_t$. Let $A_i = \{a \mid a \in A, p_i^{\alpha_i} a = 0\}$. Then each A_i is a nonzero subalgebra of A and $A = A_1 \oplus \cdots \oplus A_t$. Being subalgebras of A, the A_i's also satisfy $(*)$. Being homomorphic images of A, all the A_i's have identity elements. By Lemma 1 each A_i and hence A is commutative, a contradiction. This proves the theorem.

REFERENCES

1. R. Coughlin and M. Rich, *On scalar dependent algebras*, Canad. J. Math., **24** (1972), 696–702.

2. R. Coughlin, E. Kleinfeld, and M. Rich, *Scalar dependent algebras*, Proc. Amer. Math. Soc., **39** (1973), 69–73.

3. K. Koh, J. Luh and M. S. Putcha, *On the associativity and commutativity of algebras over commutative rings*, Pacific J. Math., **63** (1976), 423–430.

4. M. Rich, *A commutativity theorem for algebras*, Amer. Math. Monthly, **82** (1975), 377–379.

5. J. C. K. Wang and J. Luh, *The structure of a certain class of rings*, Math. Japon., **20** (1975), 149–157.

Received October 5, 1976. The second author was partially supported by NSF Grant MCS 76–05784.

NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NC 27607

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $ 72.00 a year (6 Vols., 12 issues). Special rate: $ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics

Vol. 68, No. 2 April, 1977