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ON FINITE REGULAR RINGS

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Several new properties are derived for von Neumann finite rings. A comparison is made of the properties of von Neumann finite *regular* rings and unit regular rings, and necessary and sufficient conditions are given for a matrix ring over a regular ring to be respectively von Neumann finite or unit regular. The converse of a theorem of Henriksen is proven, namely that if $R_{n \times n}$, the $n \times n$ matrix ring over ring R , is unit regular, then so is the ring R . It is shown that if $R_{2 \times 2}$ is finite regular then $a \in R$ is unit regular if and only if there is $x \in R$ such that $R = aR + x(a^\circ)$, where a° denotes the right annihilator of a in R .

1. Introduction. In [13], Henriksen posed the question whether a finite regular ring is unit regular. This was subsequently proven in part by Ehrlich [4] for a particular class of regular rings. An example of finite regular rings which are not unit regular was recently given by Bergman (1974) (see Handelman [8]). In his paper [8], Handelman showed that a regular ring R is unit regular if and only if, for any finitely generated projective right R -modules A, B , and C , $A \oplus B \cong A \oplus C$ implies $B \cong C$. He also characterized unit regular rings by perspectivity on the lattices of their principal right ideals. The purpose of this paper, however, is to characterize finite regular rings and to compare their properties with unit regular rings. Some of the results of the theory of generalized inverses [1] are used to show that in a regular ring, the properties of finiteness and unit-regularity each correspond to a suitable cancellation law for principal ideals. These cancellation laws are closely related to the substitution property of Fuchs [5], and the cancellation law of Ehrlich [4]. We shall use a result by Vidav [15] to show that if the matrix ring $R_{n \times n}$ is unit regular then so is ring R . Let us begin by defining our concepts and by stating some useful general results. A ring R is called *regular* if for all $a \in R$, there is a solution $a^- \in R$ to the equation $axa = a$. The element a^- is called as inner or 1-inverse of a [1]. Similarly, any solution to $axa = a, xax = x$ is called a reflexive or 1-2 inverse of a , and will be denoted by a^+ . For example, a^-aa^- is always such a solution. An element $a \in R$ is said to have a group inverse $a^\# \in R$ if it is a group member, i.e., a belongs to some multiplicative group of R . Necessary and sufficient conditions for $a^\#$ to exist are that $a^2R = aR$, $Ra^2 = Ra$, or $axa = a, xax = x, ax = xa$, for some $x \in R$ [10]. A regular ring R with unity is called *unit regular*

[3] if every $a \in R$ has a unit inner inverse $a^- \in R$. R is called von Neumann or Dedekind *finite*, finite for short, if it contains a unity 1 and $ab = 1 \Rightarrow ba = 1$. It is easily seen that unit-regular rings are finite. An idempotent $e^2 = e \in R$ is said to be finite if eRe is finite. The ring of $n \times n$ matrices over R will be denoted by $R_{n \times n}$, while R^n denotes the free module of $n \times 1$ columns over R . Isomorphisms will be denoted by \cong , similarity by \approx , and *internal* and *external* direct sums by $+$ and \oplus respectively. As usual, the right and left annihilator of a will be denoted by $a^o = \{x \in R; ax = 0\}$ and ${}^o a = \{x \in R; xa = 0\}$ respectively, while the Jacobson radical is given by $\mathcal{J}(R)$. Throughout this paper all rings are assumed to have unity 1.

For two idempotents $e, f \in R$, $e \sim f$ means that $e = p\hat{p}$, $f = \hat{p}p$ for some $p \in eRf$ and $\hat{p} \in fRe$, while $e \leq f$ denotes the well-known [14] ordering for compatible idempotents defined by $e = ef = fe$, or equivalently by $eRe \subseteq fRf$.

We shall make continued use of the following facts which hold for idempotents e and f in any ring R with unity [14].

- (i) $e \sim f \iff eR \cong fR$ as right R -modules
 $\iff Re \cong Rf$ as left R -modules.
- (1.1) (ii) $e \approx f \iff e \sim f$ and $1 - e \sim 1 - f$.
- (iii) $eR = fR \implies e \approx f$.
- (iv) $e = pfq$, p, q invertible $\implies e \approx f$.

In addition we shall use the result that

LEMMA 1. If R is a ring, and A, B, C, D are matrices in $R_{n \times n}$, then the following are equivalent in pairs.

- (α) (i) $AR_{n \times n} \oplus BR_{n \times n} \cong CR_{n \times n} \oplus DR_{n \times n}$ as $R_{n \times n}$ -modules.
(ii) $AR^n \oplus BR^n \cong CR^n \oplus DR^n$ as R -modules.
- (β) (i) $AR_{n \times n} + BR_{n \times n} \cong CR_{n \times n} + DR_{n \times n}$ as $R_{n \times n}$ -modules.
(ii) $AR^n + BR^n \cong CR^n + DR^n$ as R -modules.

Indeed, if $\phi \begin{bmatrix} AX \\ BY \end{bmatrix} = \phi \begin{bmatrix} A \\ 0 \end{bmatrix} X + \phi \begin{bmatrix} 0 \\ B \end{bmatrix} Y$ is the given isomorphism in (α i) then the mapping χ , given by

$$\chi \begin{bmatrix} Ax \\ By \end{bmatrix} = \phi \begin{bmatrix} A \\ 0 \end{bmatrix} x + \phi \begin{bmatrix} 0 \\ B \end{bmatrix} y$$

will be a desired isomorphism for (α ii), while conversely if $\chi \begin{bmatrix} Ax \\ By \end{bmatrix} = \chi \begin{bmatrix} A \\ 0 \end{bmatrix} x + \chi \begin{bmatrix} 0 \\ B \end{bmatrix} y$ is the isomorphism given by (α ii) then the map ϕ defined by

$$\phi \begin{bmatrix} A[x_1, \dots, x_n] \\ B[y_1, \dots, y_n] \end{bmatrix} = \begin{bmatrix} \chi \begin{bmatrix} Ax_1 \\ By_1 \end{bmatrix}, \dots, \chi \begin{bmatrix} Ax_n \\ By_n \end{bmatrix} \end{bmatrix}$$

is one of the desired $R_{n \times n}$ isomorphisms for (αi) . Similar maps can be defined for case (β) . In particular, if E and F are idempotents in $R_{n \times n}$, then

$$(1.2) \quad E \sim F \iff ER_{n \times n} \cong FR_{n \times n} \iff ER^n \cong FR^n.$$

In addition, if for example, $E = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$ and $F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}$ then

$$(1.3) \quad E \sim F \iff \begin{bmatrix} e_1 R \\ e_2 R \end{bmatrix} \cong \begin{bmatrix} f_1 R \\ f_2 R \end{bmatrix} \iff e_1 R \oplus e_2 R \cong f_1 R \oplus f_2 R.$$

The dual result for left modules is obvious. Lastly, we note that in any ring R ,

$$(1.4) \quad aR = bR \implies Ra \cong Rb.$$

Indeed, if $ax = b$ and $a = by$, then the mapping $ra \mapsto rax$ is a desired isomorphism. Let us now turn to some useful results concerning finite rings.

2. Finite rings.

THEOREM 1. *Let R be a ring with unity 1 and suppose that e and f are arbitrary idempotents in R . The following are equivalent.*

- (i) R is finite
- (ii) $eR \subseteq fR, e \sim f \implies eR = fR$.
- (iii) $Re \subseteq Rf, e \sim f \implies Re = Rf$.
- (iv) $e \sim 1 \implies e = 1$.

Proof. (i) \implies (ii). Let $eR \subseteq fR$ and $e \sim f$. Then $e = fe$ and $e = p\hat{p}$, $f = \hat{p}p$ for some $p \in eRf$, $\hat{p} \in fRe$.

Consider now

$$\begin{aligned} x &= \hat{p} + (1 - f)(1 - e) = \hat{p} + 1 - f, \quad y = p + (1 - e)(1 - f) \\ &= p + 1 - e - f + ef. \end{aligned}$$

Then $xy = 1$ and hence, since R is finite, $y = x^{-1}$.

But now $fy = fp + f - fe - f + fef = fp - e + ef \in feRf + eR = eRf + eR \subseteq eR$. Thus $f \in eRy^{-1} = eR$ and so $fR = eR$. We may again replace $e \sim f$ by either $eR \subseteq fR$ or $Re \subseteq Rf$.

(ii) \implies (iv). Let $e \sim 1$ and $eR \subseteq R$. By (ii), $eR = R$, which implies

that $1 = ex$ for some x , or $e = ex = 1$.

(iv) \Rightarrow (i). Let $ab = 1$ and set $e = ba = e^2$. Then $e \sim 1$, because $e = p\hat{p}$, $1 = \hat{p}p$ where $p = b$ and $\hat{p} = a$. In fact, $p \in eR1 = baR$ as $p = bab = b$ and $\hat{p} \in 1Re = Rba$ as $\hat{p} = aba = a$. Hence by (iii) $e = 1$ and so $ba = 1$ as desired. Part (iii) follows by left-right symmetry.

REMARKS 1. The last part should be compared with the result of Vidav [15], and Fuchs [5], which states that a regular ring is unit regular if and only if $e \sim f \Rightarrow e \approx f$.

2. The second part is best possible in that neither e nor f may fail to be idempotent. Indeed, if R is finite then $aR \cong R \Rightarrow aR = R$, as seen from the example of the ring of integers with $a = 2$. On the other hand, a ring R with the property that $aR \cong R \Rightarrow aR = R$ must be finite, yet need not be regular as seen from the following counterexample.

EXAMPLE 1. Let $R_1 = \begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$, where $R = \mathbf{R}$ is the real field. Then clearly R_1 is not regular since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Suppose now that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} R_1 \cong R_1$. We claim that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} R_1 = R_1$. Let $\phi: \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} R_1 \rightarrow R_1$ denote the given isomorphism such that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \phi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} X\right) = \phi\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$. Now $a' \neq 0$ since $a' = 0$ would imply that

$$\phi\left(\begin{bmatrix} 0 & b' \\ 0 & c' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \phi\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

while $\phi\left(\begin{bmatrix} 0 & b' \\ 0 & c' \end{bmatrix}\right) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Next we claim that $c' \neq 0$. For if $c' = 0$, then $\phi\left(\begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b' \\ 0 & -a' \end{bmatrix}\right) = \phi\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$, while $\phi\left(\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}\right) \begin{bmatrix} 0 & b' \\ 0 & -a' \end{bmatrix} = \begin{bmatrix} 0 & b' \\ 0 & -a' \end{bmatrix}$, which implies that $a' = 0$, a contradiction. Thus $a' \neq 0 \neq c'$ and $\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$ is invertible. Hence

$$R_1 = \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} R_1 \subseteq \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} R_1 \subseteq R_1$$

as desired.

COROLLARY 1. If R is finite ring and if $e^2 = e \in R$, then the corner ring eRe is also finite.

Proof. Suppose that $(eae)(ebe) = e$. Then $eR = eaeR$ and $Re = Rebe$. Hence $Reae = Rebe(eae)$ and thus by (1.4) $eR = eaeR \cong ebe(eae)R \subseteq eR$, ensuring by Theorem 1, that $eR = (ebe)eaeR$. This implies that $e = ebe(eae)$, as desired. We may consequently order the compatible idempotents with the usual partial ordering relation " \leq ". The following corollary is equivalent to the one above.

COROLLARY 2. *If R is a finite ring and B is a subring with unity, then B is finite.*

Proof. Let R be a finite ring and let B be a subring of R with unity e . Suppose that $x, y \in B$ and $xy = e$. If $f = yx = f^2$, then $(fxf)(fyf) = f$ and thus since fRf is finite by Corollary 1, it follows that $y^2x^2 = f y f (f x f) = yx$. On premultiplication by x and postmultiplication by y this yields $yx = eyxe = e$. It is obvious that Corollary 2 implies Corollary 1. It should be noted here, that the latter result can also be proven directly, as pointed out to us by the referee.

COROLLARY 3. *If R is a finite ring and $e^2 = e \in R$, then the centralizer $C_e = \{x \in R \mid xe = ex\}$ is finite.*

COROLLARY 4. *If R is a finite ring and $e^2 = e \in R$, then eae is a unit in eRe if and only if $eaeR = eR$.*

Proof. If $eaebe = e$ then clearly $eR = eaeR$. Conversely, if $eR = eaeR$, then for some $x \in R$, $eaex = e = (eae)exe$. Since eRe is finite, it follows that $e = exe(eae)$ implying that eae is a unit in eRe .

We remark that this also follows from the fact that the maximal subgroup H_e containing e is the group of units in eRe and that $H_e = \{x \in R; xR = eR, Rx = Re\}$. Finiteness shows that $eaeR = eR \Leftrightarrow Re = Reae$.

We next obtain as a corollary the result by Kaplansky [14] p. 11, which says that an idempotent e is finite if it is not equivalent to a smaller idempotent.

COROLLARY 5. *Let R be a ring and let e, f be idempotents in R such that fRf is finite. Then*

$$eRe \subseteq fRf, e \sim f \implies e = f.$$

Proof. Since $e = ef = fe$, $eRf = f(eRf) \subseteq fRf$. Now eRf and fRf are isomorphic as right fRf -modules, since the map $erf \mapsto \hat{p}erf$,

with $e = p\hat{p}$, $f = \hat{p}p$, $p \in eRf = e(fRf)f$, $\hat{p} \in fRe = f(fRf)e$, is an example of such an isomorphism. Hence by Theorem 1, $eRf = fRf$, which ensures that $f = exf$ for some x or $e = ef = f$ as desired.

We remark that if R is a unit regular ring then $eRe \cong R \Rightarrow eRe = R$, as seen from the example where $R = \coprod_{i=1}^{\infty} R_i$, $R_i = D$ (a division ring) and $e = (0, 1, 1, \dots)$. The following is a partial converse to Corollary 1.

PROPOSITION 0 (Savage). *If R is a ring with unity 1 such that fRf is finite for every idempotent element $f \neq 1$, then R is finite.*

Proof. Replace B by R and e by 1 in the proof of Corollary 2.

PROPOSITION 1. *Suppose R is a ring with unity 1 and let $\mathcal{J}(R)$ denote its Jacobson radical. Then R is finite exactly when $R/\mathcal{J}(R)$ is finite.*

Proof. \Rightarrow : Suppose we denote $R/\mathcal{J}(R)$ by \bar{R} and the elements from \bar{R} by \bar{a} , \bar{b} etc. Let $\bar{a}\bar{b} = \bar{1}$. Then $1 - ab \in \mathcal{J}(R)$, implying that $1 - (1 - ab)1 = ab$ is a unit. That is $abc = 1 = cab$ for some c . Since R is finite it follows that $bca = 1$ and so $(\bar{b}\bar{c})\bar{a} = \bar{1}$. Hence \bar{a} has left and right inverses and thus is a unit with $(\bar{a})^{-1} = \bar{b}$.

\Leftarrow : Conversely, suppose \bar{R} is finite and that $ab = 1$. Then $\bar{a}\bar{b} = \bar{1} \Rightarrow \bar{b}\bar{a} = \bar{1} \Rightarrow 1 - ba \in \mathcal{J}(R)$. And so $1 - (1 - ba)1 = ba$ is a unit in R implying that $bac = 1 = cba$ for some $c \in R$. Hence a has left and right inverses and thus is a unit with inverse $b = a^{-1}$. This result should be contrasted with the fact that $R/\mathcal{J}(R)$ may be unit regular without R being regular (cf. [5], Lemma). Indeed, if R is any nonsemisimple Artinian ring, then R cannot be regular while $R/\mathcal{J}(R)$ is semisimple Artinian and hence unit regular.

Finiteness is closely related to the existence of group and Drazin inverses of the elements in R [2], [10]. We recall that the left (right) index $l(a)$ ($r(a)$) of $a \in R$ is the *smallest* value of $p(q)$ for which $a^{p+1}R = a^pR(Ra^{q+1} = Ra^q)$, and that if both are finite, they have to be equal, [2]. This common value is called the *index* $i(a)$ of a . A ring is called strongly π -regular if every a in R possesses a Drazin inverse. Now a has a group inverse a^\sharp in R exactly when $i(a) = 0$ or 1 that is exactly when $a^2R = aR$ and $Ra^2 = Ra$ or when $axa = a$, $xax = x$ and $ax = xa$ for some $x \in R$. The following is a generalization of the concept of finiteness.

We say that a ring R satisfies property (k, l) where k and l are nonnegative integers if, for $a \in R$, $a^{k+1}R = a^kR \Rightarrow Ra^{l+1} = Ra^l$. Clearly

all strongly π -regular rings satisfy this property [2].

PROPOSITION 2. *Let R be a ring with unity 1. Suppose R satisfies property (k, l) for some nonnegative integers k and l . Then R is finite.*

Proof. This follows immediately from the fact that if $ab = 1$ then $i(a) = r(a) = l(a) = 0$.

It is unknown if the converse is true in general, but we shall see shortly that this is the case for subrings of finite regular rings. Presently we do have the following result dealing with the ring $\begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$. This ring will be used repeatedly as a source of examples and counter examples.

PROPOSITION 3. *Let R be a ring with unity and let $R_1 = \begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$. Then*

- (i) *R is finite if and only if R_1 is finite*
- (ii) *if R has property $(1, 1)$ then so does R_1 , in which case both R and R_1 are finite.*

Proof. Since part (i) is easily established we turn to (ii). Suppose that $M = \begin{bmatrix} u & v \\ 0 & w \end{bmatrix}$, $X = \begin{bmatrix} a & c \\ 0 & d \end{bmatrix}$ and $M^2X = M$. Equating entries shows that

$$(2.3) \quad \begin{aligned} (i) \quad & u^2a = u \\ (ii) \quad & w^2d = w \\ (iii) \quad & u^2c + (uv + vw)d = v. \end{aligned}$$

Since R has property $(1, 1)$, $u^2R = uR \Rightarrow Ru^2 = Ru$ and thus $u^\#$ exists. Similarly $w^\# \in R$. Hence

$$uu^\#u^2 = u^2 \quad \text{and} \quad wd = w^\#w^2d = w^\#w.$$

Then multiplying (2.3)-iii through on the left by $uu^\#$ yields

$$(2.4) \quad u^2c + uvd + uu^\#vw^\#w = uu^\#v.$$

Subtracting this from (2.3)-iii gives

$$(2.5) \quad (1 - uu^\#)v(1 - ww^\#) = 0.$$

This, however, is exactly the consistency condition needed for M to have a group inverse: $M^\# = \begin{bmatrix} u^\# & c \\ 0 & w^\# \end{bmatrix}$, where $c = -u^\#vw^\# + (1 - uu^\#)vw^\# + u^\#v(1 - ww^\#)$, satisfying $MM^\#M = M$, $M^\#MM^\# = M^\#$ and $MM^\# = M^\#M$.

It is clear that $M^* \in R_1$. For semi-simple rings the first part of this proposition may be proven with aid of Proposition 1. It is not known if the (k, l) property is inherited by $R_{2 \times 2}$.

3. Finite regular rings. Let us first give a couple of preliminary observations for regular rings.

PROPOSITION 4. *Let R be a regular ring and let $a, b \in R$.*

$$(3.1) \quad (i) \quad Ra \cong Rb \implies aR \cong bR$$

(ii) *If a^- and $a^=$ are two inner inverses of a , then*

$$aa^- \sim aa^=, a^-a \sim a^=a, 1 - aa^- \sim 1 - aa^=, \text{ and } 1 - a^-a \sim 1 - a^=a.$$

Proof. (i) Let $Ra \cong Rb$. Then $Ra^+a \cong Rb^+b$ and so $a^+a \sim b^+b$. Hence $a^+a = p\hat{p}$, $b^+b = \hat{p}p$ for some $p \in a^+aRb^+b = a^+Rb$, and $\hat{p} \in b^+bRa^+a = b^+Ra$. If we set $q = apb^+$ and $\hat{q} = b\hat{p}a^+$, then clearly $q\hat{q} = aa^+$, $\hat{q}q = bb^+$, while $q \in aRb^+$ and $\hat{q} \in bRa^+$. Thus $aR = aa^+R \cong bb^+R = bR$. The converse follows by symmetry.

(ii) Clearly $aa^-R = aa^=R$, $Ra^-a = Ra^=a$ and $a^0 = (1 - a^-a)R = (1 - a^=a)R$, $a^0 = R(1 - aa^-) = R(1 - aa^=)$.

If $RS(A)$ and $CS(A)$ denote the row-space and column space of a matrix A respectively [1], then we have:

COROLLARY 6. *For rectangular matrixes over a regular ring R ,*

$$RS(A) \cong RS(B) \iff CS(A) \cong CS(B).$$

For future reference we add the following.

PROPOSITION 5. *In a regular ring R ,*

$$(3.2) \quad \begin{aligned} (i) \quad & abR = aR \iff {}^0b \cap Ra = (0) \iff bR + (1 - a^-a)R = R \\ (ii) \quad & Rab = Rb \iff a^0 \cap bR = (0) \iff Ra + R(1 - bb^-) = R. \end{aligned}$$

Proof. First we recall that $abR = aR \implies {}^0(ab) = {}^0a$. Now let $x \in {}^0b \cap Ra$, that is $xb = 0$ and $x = ya$ for some $y \in R$. Thus if $abR = aR$, then $0 = xb = yab \implies 0 = ya = x$. If we now write ${}^0b \cap Ra = {}^0b \cap ({}^0(1 - a^-a) + {}^0(bR + (1 - a^-a)R))$ then ${}^0b \cap Ra = (0) \implies bR + (1 - a^-a)R = R$. It is obvious that the latter implies $abR = aR$. Symmetry now yields the second result.

THEOREM 2. *Let R be a regular ring with unity 1 and let a, b, c be elements of R . Suppose that I and J are right ideals in R . The following are equivalent.*

- (i) R is finite
(ii) $aR \subseteq bR$ together with $aR \cong bR$ or $Ra \cong Rb \implies aR = bR$
(iii) $I \subseteq bR, I \cong bR \implies I = bR$
(iv) $I \subseteq J, I \cong J, J$ complemented $\implies I = J$
(v) $I \cong R \implies I = R$
(3.3) (vi) $aR \cong R \implies aR = R$
(vii) $aR \oplus I \cong cR, cR \subseteq aR \implies I = (0)$
(viii) $aR \oplus bR \cong aR \implies b = 0$
(ix) $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \sim \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \implies f = 0$
(x) $aR \dot{+} bR \cong aR \implies b = 0$
(xi) $Ra^k = Ra^{k+1} \implies a^k R = a^{k+1} R$ for some $k \geq 0$.

An analogous and equivalent result holds for left ideals.

Proof. (i) \implies (ii): Let $e = aa^+$ and $f = bb^+$. Then $e \sim f$ and $eR \subseteq fR$. By Theorem 1 $eR = fR$ and so $aR = bR$.

(ii) \implies (iii): Since $I \cong bR \implies I = aR$ for some $a \in R$, the result is clear.

(iii) \implies (iv): Any complemented one-sided ideal is principal in a ring with unity.

(iv) \implies (v): Take $J = R$ in (iv).

(v) \implies (vi): Clear with $aR = I$.

(vi) \implies (i): If $ab=1$ then $Rb=R \implies bR \cong R$. By (vi) $bR=R$, implying that b is a unit with a as inverse.

(ii) \implies (vii): Let $\phi: aR \oplus I \rightarrow cR$ be the isomorphism. Then $\phi(a, 0) = cx$ for some $x \in R$, $\phi(0, I) = I_1 \subseteq cR$ and $\phi(aR, 0) = cxR$. Hence $aR \cong cxR \subseteq aR$ and so by (iii) $aR = cxR \subseteq cR \subseteq aR$. Now $I_1 = aR \cap I_1 = cxR \cap I_1 = \phi(aR, 0) \cap \phi(0, I) = \phi[(aR, 0) \cap (0, I)] = \phi(0, 0) = (0)$. Hence $I \cong I_1 = (0)$ and so $I = (0)$ as desired.

(vii) \implies (viii): Trivial.

(viii) \Leftrightarrow (ix): By (1-3) $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \sim \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow eR \oplus fR \cong eR$ and hence setting $e = aa^+$ and $f = bb^+$ yields the desired equivalence.

(viii) \implies (x): External cancellation laws always imply the corresponding internal cancellation laws since the directness of the sum $aR \dot{+} bR$ implies that $aR \dot{+} bR \cong aR \oplus bR$.

(x) \implies (i): Let $e^2 = e$, then $eR \cong R = eR \dot{+} (1-e)R$ implies by (x) that $e = 1$, ensuring by Theorem 1 that R is finite.

(ii) \implies (xi): If $Ra^k = Ra^{k+1}$ then $a^k R \cong a^{k+1} R \subseteq a^k R$ and hence $a^k R = a^{k+1} R$ guaranteeing that $(a^k)^\#$ exists.

(xi) \implies (i): This was shown in Proposition 2.

The left analogue follow by left-right symmetry.

COROLLARY 7. *Over a finite regular ring $aR \not\cong R \oplus R$, and $aR \cong aR \oplus aR$ implies $a = 0$. In particular $R \not\cong R \oplus R$.*

REMARK. In Example 2 we shall see that the isomorphic inclusion law for right ideals:

$$(3.4) \quad I \subseteq J, I \cong J \implies I = J,$$

will not be true in general in a finite regular ring, if J is not a principal ideal.

PROPOSITION 6. *Let R be a finite regular ring. Then*

$$(i) \quad aR \subseteq baR \implies aR = baR \text{ and } Ra = Rba$$

$$(iii) \quad Ra \subseteq Rab \implies Ra = Rab \text{ and } aR = abR.$$

Proof. (i). $aR \subseteq baR \implies a = ba(ba)^+a \implies Ra = Rba(ba)^+a \subseteq Ra(ba)^+a \subseteq Ra$. Hence by Proposition 4, $aR \cong a(ba)^+aR \subseteq aR$ and thus again by (3.3) $aR = a(ba)^+aR \implies baR = ba(ba)^+aR = aR$. Lastly, $Rba \cong Ra \subseteq Rba$ and so by (3.3) $Rba = Ra$. Part (ii) follows by symmetry. We note in passing that Proposition 5 applies and that, for example, in (i) $aR = b^n aR$, $Ra = Rb^n a$ for all $n = 1, 2, \dots$.

COROLLARY 8. *In a finite regular ring R*

$$(i) \quad aR \subseteq bacR \implies aR = bacR \text{ and } Rac = Rbac.$$

$$(ii) \quad Ra \subseteq Rcab \implies Ra = Rcab \text{ and } caR = cabR.$$

PROPOSITION 7. *Let R be a subring of a finite regular ring R^* and let $a \in R$. Then*

$$aR = a^2R \implies Ra = Ra^2.$$

Proof. Since $aR = a^2R$, there exists $x \in R$, such that $a = a^2x$ and so $aR^* = a^2R^*$. Now R^* is finite regular and thus by (3.3) $R^*a = R^*a^2$, which implies that $a = ya^2$ for some $y \in R^*$. Hence $ya = y(a^2x) = (ya^2)x = ax \in R$. Now let $b = ya = ax$, then $yb = yax = bx \in R$ and also $ba = ya^2 = a$. Thus $(yb)a^2 = ya^2 = a$ or $Ra^2 = Ra$ as desired.

COROLLARY 9. *If R is a subring of a finite regular ring and $a \in R$, then*

$$a^{k+1}R = a^kR \implies Ra^{k+1} = Ra^k.$$

Proof. Since $a^{k+1}R = a^kR \iff a^{2k}R = a^kR$, and $R = a^{k+1} = Ra^k \iff Ra^{2k} = Ra^k$, we may apply Proposition 7 to a^k .

It is clear from Corollaries 1 and 3 and the identity $eae[e(eae)^{-1}e]eae = eae$, that if R is finite *regular*, then so is the corner ring eRe , where $e^2 = e \in R$. This should be compared with the following result by Kaplansky (unpublished), which states that eRe inherits the unit regularity from a ring R . We add the proof for completeness. [W. Desch, private communication].

PROPOSITION 8 (Kaplansky). *Let R be a unit regular ring and e be an idempotent element in R . Then eRe is unit regular.*

Proof. Let ere be an arbitrary element in eRe and $u = (ere + 1 - e)^{-1}$ be a unit. Since $(1 - e)u(1 - e) = 1 - e$, $ereu(1 - e) = 0$, $(1 - e)uere = 0$, $eu(1 - e) = u(1 - e) - (1 - e)$ and $(1 - e)ue = (1 - e)(u - 1)$, we have $ere(e(u - u(1 - e)u)e)ere = ere$ and $e(u - u(1 - e)u)e \cdot eu^{-1}e = e = eu^{-1}e \cdot e(u - u(1 - e)u)e$.

Related to the corner ring is the following well-known result which generalizes a result of [15].

PROPOSITION 9. *Let M be a right unital A -modules where A is a ring with unity 1, and let $R = \text{End}_A(M)$. If $e^2 = e \in R$ and $E = \text{End}_A(eM)$ then $eRe \cong E$.*

It should be observed here with aid of Corollary 4, that if u is a unit in a *regular* ring R and $e = e^2 \in R$, then eue is a unit in eRe exactly when $(1 - e)u^{-1}(1 - e)$ is a unit in $(1 - e)R(1 - e)$. Indeed, from (3.2) we see that

$$(3.5) \quad \begin{aligned} eueR = eR &\iff ueR + (1 - e)R = R \iff eR + u^{-1}(1 - e)R = R \\ &\iff (1 - e)u^{-1}(1 - e)R = (1 - e)R. \end{aligned}$$

4. Cancellation laws. We begin by defining four strong cancellation laws for internal direct sums of principal right ideals. Let R be a ring with unity and let $a, b, c, d \in R$. We define,

$$\begin{aligned} C_0^{\text{in}}: & \quad aR \dot{+} bR = cR \dot{+} dR, aR \cong cR \implies bR \cong dR \\ C_1^{\text{in}}: & \quad aR \dot{+} bR \cong cR \dot{+} dR, aR \cong cR \implies bR \cong dR \\ C_2^{\text{in}}: & \quad aR \dot{+} bR \cong aR \dot{+} dR \implies bR \cong dR \\ C_3^{\text{in}}: & \quad aR \dot{+} bR \cong aR \implies bR = (0). \end{aligned}$$

There are two common ways of weakening these laws. We may define for a fixed $g \in R$,

$$\begin{aligned}
C_0^{\text{in}}(g): \quad & aR + bR = gR = cR + dR, aR \cong cR \implies bR \cong dR \\
C_1^{\text{in}}(g): \quad & aR + bR \cong gR \cong cR + dR, aR \cong cR \implies bR \cong dR \\
C_2^{\text{in}}(g): \quad & aR + bR \cong gR \cong aR + dR \implies bR \cong dR \\
C_3^{\text{in}}(g): \quad & aR + bR \cong gR \cong aR \implies bR = (0),
\end{aligned}$$

and

$$\begin{aligned}
C_0^{\text{in}}(g): \quad & aR + bR = cR + dR, aR \cong gR \cong cR \implies bR \cong dR \\
C_1^{\text{in}}(g): \quad & aR + bR \cong cR + dR, aR \cong gR \cong cR \implies bR \cong dR \\
C_2^{\text{in}}(g): \quad & aR + bR \cong aR + dR, aR \cong gR \implies bR \cong dR \\
C_3^{\text{in}}(g): \quad & aR + bR \cong aR, \quad aR \cong gR \implies bR = (0).
\end{aligned}$$

These laws may be considered as *local* cancellation laws, specifying a property of element a , or element g or be considered as a *global* cancellation law, (holding for all a) specifying a property of the ring R . The case $g = 1$, of course, being of special interest. For any fixed g it is clear that locally, and hence globally, C_3^{in} and C_3^{in} coincide, while $C_{k'}^{\text{in}} \Leftarrow C_{k'}^{\text{in}} \Rightarrow C_{k'}^{\text{in}}$ for $k = 0, 1, 2, 3$. Moreover,

$$(4.1) \quad C_0^{\text{in}} \Longleftrightarrow C_1^{\text{in}} \implies C_2^{\text{in}} \implies C_3^{\text{in}}.$$

In fact, if C_0^{in} holds and $\phi: aR + bR \rightarrow cR + dR$ is an isomorphism then $\phi(a)R + \phi(b)R = cR + dR$, and so if $aR \cong cR$ then $\phi(a)R \cong cR$ ensuring that $bR \cong \phi(b)R \cong dR$. Analogous implications hold for the weaker cancellation laws.

We may similarly define the corresponding local or global cancellation laws for *external* direct sums of principal right ideals:

$$\begin{aligned}
C_1^{\text{ex}}: \quad & aR \oplus bR \cong cR \oplus dR, aR \cong cR \implies bR \cong dR \\
C_2^{\text{ex}}: \quad & aR \oplus bR \cong aR \oplus dR, \implies bR \cong dR \\
C_3^{\text{ex}}: \quad & aR \oplus bR \cong aR \implies bR = (0) \\
C_{1'}^{\text{ex}}(g): \quad & aR \oplus bR \cong gR \cong cR \oplus dR, aR \cong cR \implies bR \cong dR \\
C_{2'}^{\text{ex}}(g): \quad & aR \oplus bR \cong gR \cong aR \oplus dR, \implies bR \cong dR \\
C_{3'}^{\text{ex}}(g): \quad & aR \oplus bR \cong gR \cong aR \implies bR = (0)
\end{aligned}$$

and

$$\begin{aligned}
C_{1'}^{\text{ex}}(g): \quad & aR \oplus bR \cong cR \oplus dR, aR \cong gR \cong cR \implies bR \cong dR \\
C_{2'}^{\text{ex}}(g): \quad & aR \oplus bR \cong aR \oplus dR, aR \cong gR \implies bR \cong dR \\
C_{3'}^{\text{ex}}(g): \quad & aR \oplus bR \cong aR, \quad aR \cong gR \implies bR = (0).
\end{aligned}$$

We note that C_0^{ex} and $C_{0'}^{\text{ex}}$ are trivial, while C_0^{ex} is impossible. Indeed, if $aR \oplus bR = cR \oplus dR$ then $aR = cR$ and $bR = dR$. It is easily seen that locally, and hence globally,

$$(4.2) \quad C_1^{\text{ex}} \Longleftrightarrow C_2^{\text{ex}} \Longrightarrow C_3^{\text{ex}},$$

with an analogous result for the weak external cancellation laws.

It is always true that the *external* cancellation laws imply the corresponding *internal* cancellation laws, but the converse may not be true in general. When R is regular we may of course replace the ring elements in these laws by idempotents. By analogy to the above, we may define local and global internal and external cancellation laws for right ideals, and for right R -modules. For example,

$$\begin{aligned} \mathfrak{C}_1^{\text{ex}}: A \oplus B \cong C \oplus D, A \cong C &\Longrightarrow B \cong D, \\ \mathfrak{C}_{0'}^{\text{in}}(M): A \dot{+} B = C \dot{+} D, A \cong M \cong C &\Longrightarrow B \cong D \end{aligned}$$

and

$$\mathfrak{C}_{0''}^{\text{in}}(M): A \dot{+} B = M = C \dot{+} D, A \cong C \Longrightarrow B \cong D.$$

The latter type of cancellation law was used by Ehrlich [4] and reduces to $\mathfrak{C}_{0''}^{\text{in}}(1)$, when applied to complemented (and hence principal) ideals in R .

If we consider $\mathfrak{C}_{0''}^{\text{in}}(M)$ as a condition *on* the module M then it is implied by Fuchs' substitution property for right R -modules, which states that the right R -module M obeys the substitution property if

$$A \dot{+} B = C \dot{+} D \quad \text{and} \quad A \cong M \cong C,$$

implies that for suitable module E ,

$$A \dot{+} B = E \dot{+} B = E \dot{+} D.$$

Also, when considered as local conditions on $A = M$, it follows that

$$(4.3) \quad \mathfrak{C}_i^{\text{ex}} \Longleftrightarrow \mathfrak{C}_{i''}^{\text{ex}}(M) \quad i = 1, 2, 3.$$

Before turning to our main comparison between finite and unit regular rings, let us first examine some of the interdependence between the above cancellation laws. First, it is clear that each cancellation law for modules implies the corresponding one for ideals which in turn implies the one for principal ideals.

By analogy to the above, it is easily seen that for right R -modules

$$(4.4) \quad \mathfrak{C}_{i'} \Longleftarrow \mathfrak{C}_i \Longrightarrow \mathfrak{C}_{i''},$$

for internal as well as external cancellation laws for all possible i . Also,

$$(4.5) \quad \mathfrak{C}_0^{\text{in}} \Longleftrightarrow \mathfrak{C}_1^{\text{in}} \Longrightarrow \mathfrak{C}_2^{\text{in}} \Longrightarrow \mathfrak{C}_3^{\text{in}}$$

and

$$(4.6) \quad \mathfrak{C}_1^{\text{ex}} \iff \mathfrak{C}_2^{\text{ex}} \iff \mathfrak{C}_3^{\text{ex}},$$

with analogous implications for $\mathfrak{C}_{i'}$ and $\mathfrak{C}_{i''}$, and again $\mathfrak{C}_{3'}$ and $\mathfrak{C}_{3''}$ being identical.

As mentioned earlier the internal laws for, say (principal) ideals, may not imply the corresponding external law. We do have however that for modules,

$$(4.7) \quad \mathfrak{C}_1^{\text{in}} \iff \mathfrak{C}_1^{\text{ex}},$$

with analogous results for $\mathfrak{C}_{i'}$ and $\mathfrak{C}_{i''}$. It is exactly the equivalence of the internal and external laws which appears naturally in the study of regular rings. Lastly, if we take $M = gR$ in the cancellation laws \mathfrak{C}_i , $i = 1, 2, 3$, then because of the much stronger condition we may conclude that

$$(4.8) \quad \mathfrak{C}_i^{\text{in}}(gR) \iff \mathfrak{C}_{i'}^{\text{ex}}(gR) \iff C_{i'}^{\text{ex}}(g) \iff C_{i'}^{\text{in}}(g) \quad i = 1, 2, 3.$$

Indeed if, say, $i = 1$ all that is needed is that $C_1^{\text{in}}(g) = \mathfrak{C}_1^{\text{ex}}(gR)$.

Therefore, let $A \oplus B \cong gR \cong C \oplus D$, $A \cong C$, where A, B, C, D are right R -modules. If $\phi: A \oplus B \rightarrow gR$ and $\psi: C \oplus D \rightarrow gR$ are the isomorphism, then there is $a \in A, b \in B$, such that $\phi(a, 0) + \phi(0, b) = \phi(a, b) = g$ and hence $gR = \phi(a, 0)R + \phi(0, b)R$ with $\phi(a, 0)R \cong A$, $\phi(0, b)R \cong B$. Likewise, there are $c \in C, d \in D$ such that

$$gR = \psi(c, 0)R + \psi(0, d)R \quad \psi(c, 0)R \cong C, \psi(0, d)R \cong D.$$

Since $\phi(a, 0)R \cong \psi(c, 0)R$, it follows by $C_1^{\text{in}}(g)$ that $C \cong \phi(0, b)R \cong \psi(0, d)R \cong D$.

By analogy to the substitution property of [5] it can be shown that the cancellation law \mathfrak{C}_1 is inherited by internal as well as external direct sums of modules obeying these laws.

We have now arrived at the following relationship between finiteness, unit regularity and these cancellation laws.

THEOREM 3. *Let R be a regular ring with unity 1, and let $e_i, f_i, i = 1, 2$ be idempotents in R .*

(α) *The following are equivalent.*

1. R is unit regular.
2. C_1^{ex}
3. $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \sim \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}, e_1 \sim f_1 \implies e_2 \sim f_2.$
4. C_1^{in}
5. $C_{1'}^{\text{in}}(1)$
6. $C_{0'}^{\text{in}}(1)$
7. $C_{1'}^{\text{ex}}(1)$

8. $\mathbb{C}_1^{\text{ex}}(R)$
9. $\mathbb{C}_1^{\text{in}}(R)$.

(β) *The following are equivalent.*

1. R is finite.
2. C_3^{ex}
3. $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \sim \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}, e_1 \sim f_1 \implies e_2 = 0$
4. C_3^{in}
5. $C_{3'}^{\text{in}}(1)$
6. $C_{3'}^{\text{ex}}(1)$
7. $\mathbb{C}_{3'}^{\text{ex}}(R)$
8. $\mathbb{C}_{3'}^{\text{in}}(R)$.

Proof. (α). (1) \Leftrightarrow (2). This has been proven by Handelman [8]. Alternatively, the unit regularity of $R_{2 \times 2}$ could be used to prove this.

(2) \Leftrightarrow (3). This follows from (1.3) on rewriting C_1^{ex} using idempotents.

(2) \Rightarrow (4) \Rightarrow (5) \Leftrightarrow (6). This always holds.

(6) \Rightarrow (1). This follows from the result of Ehrlich [4] Theorem 2, which states that if $R = \text{End}_A(M)$ is a regular ring, where M is a right A -module, and $1 \in A$, then R is unit regular exactly when

$$M = A_1 \dot{+} H = A_2 \dot{+} K, A_1 \cong A_2 \implies H \cong K.$$

Since regular rings are faithful, this includes the cancellation law $\mathbb{C}_{0'}^{\text{in}}(R)$:

$$R = I_1 \dot{+} J_1 = I_2 \dot{+} J_2, I_1 \cong I_2 \implies J_1 \cong J_2$$

for complemented right ideals, which reduces to $C_0^{\text{in}}(1)$. An alternative proof is obtained from Theorem 4 and Corollary 1 of [5]. Indeed, if $R/bR \cong R/dR$ and we pick a, c such that $aR \dot{+} bR = R = cR \dot{+} dR$, then $aR \cong cR$. Hence by $C_0^{\text{in}}(1)$, $bR \cong dR$, which by Corollary 1 of [5] ensures that R is unit regular.

The equivalence of the latter four parts was established in (4.8).

(β) The equivalence of (1)-(4) is contained in Theorem 2, while the equivalence of (5)-(8) follows from (4.8). Lastly, if C_3^{in} holds and $eR \cong R = eR \dot{+} (1 - e)R$, then $e = 1$ ensuring that R is finite.

REMARK. In his paper [8], Handelman actually showed that unit regularity is equivalent to the external cancellation law \mathbb{C}_1^{ex} for finitely generated projective modules. It is an open question whether *finite* regularity is equivalent to \mathbb{C}_3^{ex} for modules of this type.

We close this section with a counterexample showing that the isomorphism inclusion law (3.4) as well as the external cancellation

law $\mathfrak{U}_1^{\text{ex}}$ may *not* be valid for non-principal right ideals in a unit regular ring.

EXAMPLE 2. Let R be a unit regular ring and let

$$S = \left\{ M \in R_{\infty \times \infty}; M = \left[\begin{array}{c|c} A & 0 \\ \hline & d \\ 0 & d \\ & \ddots \\ & \ddots \end{array} \right], A \in R_{n \times n}, n \geq 0 \right\},$$

be the ring of all infinite matrices over R which consist of a direct sum of a finite matrix A and an infinite scalar matrix $D = \begin{bmatrix} d & 0 \\ & d \\ & \ddots \\ 0 & \ddots \end{bmatrix}$.

It is easily verified that S is again a unit regular ring. Now let $J = \left\{ M \in S; M = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, A \in R_{n \times n}, n \geq 0 \right\}$, be the ideal of all matrices

in S that have zero tail and let $a = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & \ddots & \end{bmatrix}$ and $I = HJ$, where

$H = \begin{bmatrix} 0 & & \\ 1 & 0 & \\ & 1 & \ddots \\ & & 1 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$ is the infinite row-shift matrix. Then I and J are

right ideals in S such that $I \subseteq J$, $I \cong J$ (as S -modules) $I \neq J$ and $aR + I = J$, $I \cong J$, $aR \neq (0)$.

5. **Matrices over regular rings.** In this section we shall give necessary and sufficient conditions for the matrix ring $R_{2 \times 2}$ to be finite regular or unit regular. We shall need some preliminary results dealing with 2×2 matrices over a regular ring R . If $b = paq$, with p and q invertible, then the sets of reflexive inverses $\{b^+\}$, $\{a^+\}$ are related through $\{b^+\} = q^{-1}\{a^+\}p^{-1}$. Thus each reflexive inverse b^+ is of the form $q^{-1}a^+p^{-1}$ for some $a^+ \in \{a^+\}$. Hence $bb^+ = paa^+p^{-1}$ for some a^+ , but this is not necessarily true for all aa^+ . We do have, however, that $b = paq \Rightarrow bR = paR \Rightarrow bb^+R = bR \cong aR = aa^+R \Rightarrow bb^+ \sim aa^+$, for all choices of a^+ and b^+ . It is this "decoupling" of the idempotent bb^+ from aa^+ , this allows us to select suitable b^+ and a^+ *independently* from each other.

If $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is an arbitrary matrix in the regular ring $R_{2 \times 2}$, then it can be shown that [10] there exists a reflexive inverse M^+ of M such that *simultaneously*

$$(5.1) \quad MM^+ = \begin{bmatrix} e_1 & 0 \\ \alpha & e_2 \end{bmatrix}, \quad M^+M = \begin{bmatrix} f_1 & \beta \\ 0 & f_2 \end{bmatrix},$$

where

$$(5.2) \quad \begin{aligned} e_1 &= aa^+ + uu^+(1 - aa^+), \quad e_2 = ss^+(1 - vv^+) + vv^+ \\ f_1 &= a^+a + (1 - a^+a)v^+v, \quad f_2 = u^+u + (1 - u^+u)s^+s \\ \alpha &= (1 - ss^+)(1 - vv^+)[ba^+ + zu^+(1 - aa^+)] \\ \beta &= [a^+c + (1 - a^+a)v^+z](1 - u^+u)(1 - s^+s) \end{aligned}$$

and

$$u = (1 - aa^+)c, \quad v = b(1 - a^+a), \quad z = d - ba^+c$$

with $s = (1 - vv^+)z(1 - u^+u)$. In these expressions a^+ is arbitrary but the same choice of a^+ must be used throughout. Moreover (5.1)–(5.2) remain valid if $(\cdot)^+$ is replaced by $(\cdot)^-$ throughout. Since MM^+ and M^+M are triangular idempotents they are similar to diagonal matrices. Indeed, for example

$$(5.3) \quad \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} = \begin{bmatrix} 1 & f_1\beta - \beta f_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 & \beta \\ 0 & f_2 \end{bmatrix} \begin{bmatrix} 1 & \beta f_2 - f_1\beta \\ 0 & 1 \end{bmatrix}.$$

We thus may state

PROPOSITION 10. *If $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R_{2 \times 2}$, where R is a regular ring, then*

$$(5.4) \quad MR_{2 \times 2} \cong \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R_{2 \times 2}$$

for suitable idempotents e_1, e_2 given by (5.2).

COROLLARY 10. *If $E^2 = E \in R_{2 \times 2}$, where R is a regular ring, then $E \sim \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$ for suitable idempotents e_1 and e_2 in R .*

Because (5.1)–(5.3) also hold for matrices of the form $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \in R_{n \times n}$, where $A \in R_{p \times p}$ and $D \in R_{q \times q}$, with $p + q = n$, we may extend these results by induction to $R_{n \times n}$. In particular, if $M = \begin{bmatrix} a & c^r \\ b & D \end{bmatrix}$, where $a \in R$, b and c are columns in R^{n-1} and $D \in R_{(n-1) \times (n-1)}$, then there exists an M^+ such that $MM^+ \approx \begin{bmatrix} e_1 & 0 \\ 0 & E_1 \end{bmatrix}$ for suitable idempotents $e_1 \in R$ and $E_1 \in R_{(n-1) \times (n-1)}$. Hence by Lemma 1, $MR^n \cong$

$\begin{bmatrix} e_1 R \\ E_1 R^{n-1} \end{bmatrix} \cong e_1 R \oplus E_1 R^{n-1}$, and so we may now repeat the reduction with E_1 . By induction we obtain $MR^n \cong e_1 R \oplus \cdots \oplus e_n R$ for suitable idempotents $e_i \in R, i = 1, \dots, n$, and thus we have

COROLLARY 11. *If $M \in R_{n \times n}$, where R is a regular ring, then*

$$MR_{n \times n} \cong \begin{bmatrix} e_1 & & 0 \\ & e_2 & \\ & & \ddots \\ 0 & & & e_n \end{bmatrix} R_{n \times n},$$

for suitable idempotents $e_i \in R, i = 1, \dots, n$. In particular if $E^2 = E \in R_{n \times n}$ then $E \sim \begin{bmatrix} e_1 & 0 \\ & \ddots \\ 0 & e_n \end{bmatrix}$ for some $e_i^2 = e_i \in R, i = 1, \dots, n$.

We now recall from Theorem 2, that if R is regular then $R_{2 \times 2}$ is finite regular if and only if $MR_{2 \times 2} \cong R_{2 \times 2} \Rightarrow MR_{2 \times 2} = R_{2 \times 2}$. So if we let $E = MM^+$ be given by (5.1), then $ER_{2 \times 2} = MR_{2 \times 2} \cong I_2 R_{2 \times 2}$ where I_n is the identity matrix in $R_{n \times n}$. From (1.2) we see that this holds exactly when $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R^2 \cong R^2$, or

$$(5.5) \quad e_1 R \oplus e_2 R \cong R \oplus R.$$

Since $R_{2 \times 2}$ is regular exactly when R is, we have proven the following.

THEOREM 4. *If R is a regular ring with unity 1, then $R_{2 \times 2}$ is finite if and only if for any two idempotents $e_1, e_2 \in R$,*

$$(5.6) \quad e_1 R \oplus e_2 R \cong R \oplus R \implies e_1 = e_2 = 1.$$

It is well-known that finiteness for $R_{2 \times 2}$ implies that of R . Let us now examine the converse question for regular rings. We begin by noting that

$$(5.7) \quad \begin{aligned} MX = I \text{ for some } X \in R_{2 \times 2} &\iff MM^- = I \text{ for all } M^- \\ &\iff MM^+ = I \text{ for some } M^+, \end{aligned}$$

and hence using (5.1)–(5.2) we see that

$$(5.8) \quad \begin{aligned} MX = I \text{ for some } X \in R_{2 \times 2} &\iff (1 - uu^+)(1 - aa^+) = 0, \\ &\quad (1 - ss^+)(1 - vv^+) = 0 \end{aligned}$$

while

$$(5.9) \quad \begin{aligned} YM = I \text{ for some } Y \in R_{2 \times 2} &\iff (1 - a^+a)(1 - v^+v) = 0, \\ &\quad (1 - u^+u)(1 - s^+s) = 0. \end{aligned}$$

Thus $R_{2 \times 2}$ is finite if and only if these are equivalent. We shall show now that if R is regular, then $R_{2 \times 2}$ will not be finite if there exist elements a and x in R , with a not unit regular, such that

$$(5.10) \quad R = aR + x(a^\circ).$$

We begin by noting that (5.8) and (5.9) are, with aid of regularity, equivalent to

$$(5.11) \quad \begin{aligned} (i) \quad & uR = (1 - aa^+)R \\ (ii) \quad & sR = (1 - vv^+)R \\ (iii) \quad & Rv = R(1 - a^+a) \\ (iv) \quad & Rs = R(1 - u^+u). \end{aligned}$$

Suppose we select $c = 1, d = 1 + ba^+c$ and $b = (1 - aa^+)x$. Then $z = 1$ and $u = 1 - aa^+$ is idempotent so that (i) holds. Now if (5.10) holds, then by (3.2)

$$(5.12) \quad uR = (1 - aa^+)R = (1 - aa^+)x(1 - a^+a)R = vR.$$

Hence, by regularity $(1 - vv^+)u = 0$ so that $s = (1 - vv^+)(1 - u) = 1 - vv^+$, if we pick $u^+ = u$. It now follows that (ii) and (iv) also hold. Indeed $Rs = R(1 - vv^+) = {}^0v = {}^0u = R(1 - u)$. Now consider the last identity $Rv = R(1 - a^+a)$. This implies that

$$(1 - aa^+)R = vR \cong (1 - a^+a)R,$$

and so $aa^+ \approx a^+a$ since always $aa^+ \sim a^+a$. But the latter is locally equivalent to a being unit regular [12], which is excluded by assumption. The structure of (5.10) should be compared with the representation $R = aR + u(a^\circ)$, u a unit which is equivalent to a being unit regular [12]. We note in passing that none of the "obvious" choices for x seem to work. For example, $x = 1$ implies that $a^2R = aR$, which if R is finite, ensures that a^* exists so that a is unit regular. Let us now turn to unit regular rings.

THEOREM 5. *If R is a regular ring with unity then $R_{2 \times 2}$ is unit regular exactly when for any idempotents $e_1, e_2, f_1, f_2 \in R$,*

$$(5.13) \quad \begin{aligned} e_1R \oplus e_2R \cong f_1R \oplus f_2R &\implies (1 - e_1)R \oplus (1 - e_2)R \\ &\cong (1 - f_1)R \oplus (1 - f_2)R. \end{aligned}$$

Proof. That (5.13) is necessary is an immediate consequence of a result of Handelman (see [8, Theorem 2]).

Now suppose (5.13) holds, and that $E^2 = E \sim F = F^2$. By (5.1) there exist 1 - 2 inverses E^+, F^+ such that

$$EE^+ \approx \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}, FF^+ \approx \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}.$$

Hence by (1.1)-ii,

$$I - EE^+ \sim \begin{bmatrix} 1 - e_1 & 0 \\ 0 & 1 - e_2 \end{bmatrix} \quad \text{and} \quad I - FF^+ \sim \begin{bmatrix} 1 - f_1 & 0 \\ 0 & 1 - f_2 \end{bmatrix}.$$

But now we may apply (3.1)-ii with $E^- = E$ and $E^+ = E^+$ to show that

$$I - E \sim I - EE^+ \sim \begin{bmatrix} 1 - e_1 & 0 \\ 0 & 1 - e_2 \end{bmatrix}, \quad I - F \sim \begin{bmatrix} 1 - f_1 & 0 \\ 0 & 1 - f_2 \end{bmatrix}.$$

Consequently if $E \sim F$, then

$$ER_{2 \times 2} = EE^+R_{2 \times 2} \cong \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}R_{2 \times 2} \cong \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}R_{2 \times 2} \cong FR_{2 \times 2},$$

and so by (1.3) $e_1R \oplus e_2R \cong f_1R \oplus f_2R$. Using (5.13) and (1.3) once more yields $\begin{bmatrix} 1 - e_1 & 0 \\ 0 & 1 - e_2 \end{bmatrix} \sim \begin{bmatrix} 1 - f_1 & 0 \\ 0 & 1 - f_2 \end{bmatrix}$ as desired. This result is easily seen to generalize to $R_{n \times n}$, with aid of Corollary 8.

COROLLARY 12. *If R is unit regular then (5.13) holds.*

We may also use Theorem 3 α to characterize the unit regularity of $R_{2 \times 2}$.

THEOREM 6. *If R is a regular ring then $R_{2 \times 2}$ is unit regular if and only if for any idempotents $e_i, f_i, g_i, h_i \in R, i = 1, 2$,*

$$(5.14) \quad \left. \begin{aligned} e_1R \oplus e_2R \oplus f_1R \oplus f_2R &\cong g_1R \oplus g_2R \oplus h_1R \oplus h_2R \\ e_1R \oplus e_2R &\cong g_1R \oplus g_2R \end{aligned} \right\} \\ \implies f_1R \oplus f_2R \cong h_1R \oplus h_2R.$$

Proof. By Theorem 3 α , $R_{2 \times 2}$ is unit regular if and only if for all $A, B, C, D \in R_{2 \times 2}$ such that

$$(5.15) \quad AR_{2 \times 2} \oplus BR_{2 \times 2} \cong CR_{2 \times 2} \oplus DR_{2 \times 2} \quad \text{and} \quad AR_{2 \times 2} \cong CR_{2 \times 2},$$

it follows that $BR_{2 \times 2} \cong DR_{2 \times 2}$. Suppose now that (5.14) and (5.15) hold. Then by (5.4) there exist e_i, f_i, g_i and $h_i, i = 1, 2$ such that

$$\begin{aligned} AR_{2 \times 2} &\cong \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}R_{2 \times 2} \cong \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}R_{2 \times 2} \cong CR_{2 \times 2}, \quad \text{and} \\ &\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}R_{2 \times 2} \oplus \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}R_{2 \times 2} \cong \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}R_{2 \times 2} \oplus \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}R_{2 \times 2}. \end{aligned}$$

With aid of Lemma 1 this implies that $e_1R \oplus e_2R \cong g_1R \oplus g_2R$ and $e_1R \oplus e_2R \oplus f_1R \oplus f_2R \cong g_1R \oplus g_2R \oplus h_1R \oplus h_2R$. Hence by (5.14) $\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} R^2 \cong \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} R^2$ which by Lemma 1 ensures that

$$BR_{2 \times 2} \cong \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} R_{2 \times 2} \cong \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} R_{2 \times 2} \cong DR_{2 \times 2}.$$

Conversely, if $R_{2 \times 2}$ is unit regular, it suffices to take

$$A = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}, B = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}, C = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \text{ and } D = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$$

and apply Theorem 3 α and Lemma 1.

REMARK. We may replace the idempotents in Theorem 6 by arbitrary regular ring elements.

Our last result will be the converse of a theorem by Henriksen [13].

THEOREM 7. *If $R_{n \times n}$ is unit-regular for some $n \geq 1$, then R is unit regular.*

Proof. It is well-known that if $R_{n \times n}$ is regular with identity for some $n \geq 1$, then so is R . So let $e \sim f$ with $e = p\hat{p}$, $f = p\hat{p}$ and $p \in eRf$, $\hat{p} \in fRe$. Then $E = \begin{bmatrix} e & 0 \\ 0 & I_{n-1} \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} \hat{p} & 0 \\ 0 & I_{n-1} \end{bmatrix} = P\hat{P}$ and $F = \begin{bmatrix} f & 0 \\ 0 & I_{n-1} \end{bmatrix} = \begin{bmatrix} \hat{p} & 0 \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & I_{n-1} \end{bmatrix} = \hat{P}P$ where $P \in eR_{n \times n}F = \begin{bmatrix} eRf & ? \\ ? & ? \end{bmatrix}$ and $\hat{P} \in fR_{n \times n}E = \begin{bmatrix} fRe & ? \\ ? & ? \end{bmatrix}$. Hence $E \sim F$ and so by the unit regularity of $R_{n \times n}$, $I - E \sim I - F$. That is,

$$\begin{bmatrix} 1 - e & 0 \\ 0 & 0 \end{bmatrix} R_{n \times n} \cong \begin{bmatrix} 1 - f & 0 \\ 0 & 0 \end{bmatrix} R_{n \times n}$$

as $R_{n \times n}$ -modules, which implies by (1.2) that

$$\begin{bmatrix} (1 - e)R \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - e & 0 \\ 0 & 0 \end{bmatrix} R^n \cong \begin{bmatrix} 1 - f & 0 \\ 0 & 0 \end{bmatrix} R^n = \begin{bmatrix} (1 - f)R \\ 0 \end{bmatrix}$$

as R -modules.

Hence $(1 - e) \sim (1 - f)$ as desired.

We remark that this result may also be obtained from Theorem 5 extended to $R_{n \times n}$, which is far less transparent however.

6. Conclusions. We have compared some of the properties of finite regular and unit regular rings, and have shown that both are

closely related to the study of generalised inverses on the one hand, and a study of cancellation laws on the other. We have seen that the finiteness property may often replace the concept of rank (cf. Prop. 6) in matrix calculations, and that the finiteness of $R_{2 \times 2}$ depends, for a regular ring on the existence of certain nonunit regular elements.

It will be of interest to study the semigroup and subgroup structure of finite regular and unit regular rings.

We conclude with several open problems.

1. If R is finite does $a^2R = aR \Rightarrow Ra^2 = Ra$?
2. Does C_1^{in} imply regularity?
3. Does $C_1^{\text{in}} \Rightarrow C_1^{\text{ex}}$ in arbitrary rings? If the answer to 2 is affirmative, then the answer is yes.
4. Is a finite *regular* subring of a unit regular ring also unit regular?
5. Is finite regularity equivalent to $\mathfrak{C}_3^{\text{ex}}$ for finitely *generated projective* modules?
6. When does $I \subseteq J, I \cong J \Rightarrow I = J$ for right ideals?
7. If R is finite regular can $R \oplus R \cong R \oplus R \oplus R$ as R -modules hold?
8. Is the (k, l) property inherited by $R_{2 \times 2}$?
9. What sort of finite regular ring satisfy C_2^{in} or C_2^{ex} ?

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REFERENCES

1. A Ben Israel and T. N. E. Greville, *Generalized Inverses Theory and Applications*, Wiley, New York, 1974.
2. M. P. Drazin, *Pseudo-inverses in associated rings and semigroups*, Amer. Math. Monthly, **65** (1958), 506-514.
3. G. Ehrlich, *Unit regular rings*, Portugal Math., **27** (1969), 209-212.
4. ———, *Units and one-sided units in regular rings*, Trans. Amer. Math. Soc., **216** (1976), 81-90.
5. L. Fuchs, *On the substitution property for modules*, Monatsh. Math., (1971), 198-204.
6. K. R. Goodearl, *Prime ideals in regular self-injective rings*, Canad. J. Math., **25** (1973), 829-839.
7. K. R. Goodearl and D. Handelman, *Simple self-injective rings*, Communications in Algebra, **3** (1975), 797-834.
8. D. Handelman, *Perspectivity and cancellation in regular rings*, J. of Algebra, to appear.
9. R. E. Hartwig, *Generalized Inverses, EP Elements and Associates*, to appear.
10. ———, *Block generalized inverses*, Arch. Ratl. Mech. Anal., **61** (1976), 197-251.
11. R. E. Hartwig and J. Shoaif, *Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices*, submitted for publication.
12. R. E. Hartwig and J. Luh, *A note on the group structure of unit regular ring elements*, submitted for publication.

13. M. Henriksen, *On a class of regular rings that are elementary divisor rings*, Archiv der Mathematik, **24** (1973), 133-141.
14. I. Kaplansky, *Rings of Operators*, W. A. Benjamin Inc., New York, 1968.
15. I. Vidav, *Modules over regular rings*, Mathematica Balkanica, **1** (1971), 287-292.

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V. V. Anh and P. D. Tuan, <i>On starlikeness and convexity of certain analytic functions</i>	1
Willard Ellis Baxter and L. A. Casciotti, <i>Rings with involution and the prime radical</i>	11
Manuel Phillip Berriozabal, Hon-Fei Lai and Dix Hayes Pettey, <i>Noncompact, minimal regular spaces</i>	19
Sun Man Chang, <i>Measures with continuous image law</i>	25
John Benjamin Friedlander, <i>Certain hypotheses concerning L-functions</i>	37
Moshe Goldberg and Ernst Gabor Straus, <i>On characterizations and integrals of generalized numerical ranges</i>	45
Pierre A. Grillet, <i>On subdirectly irreducible commutative semigroups</i>	55
Robert E. Hartwig and Jiang Luh, <i>On finite regular rings</i>	73
Roger Hugh Hunter, Fred Richman and Elbert A. Walker, <i>Finite direct sums of cyclic valuated p-groups</i>	97
Atsushi Inoue, <i>On a class of unbounded operator algebras. III</i>	105
Wells Johnson and Kevin J. Mitchell, <i>Symmetries for sums of the Legendre symbol</i>	117
Jimmie Don Lawson, John Robie Liukkonen and Michael William Mislove, <i>Measure algebras of semilattices with finite breadth</i>	125
Glenn Richard Luecke, <i>A note on spectral continuity and on spectral properties of essentially G_1 operators</i>	141
Takahiko Nakazi, <i>Invariant subspaces of weak-* Dirichlet algebras</i>	151
James William Pendergrass, <i>Calculations of the Schur group</i>	169
Carl Pomerance, <i>On composite n for which $\varphi(n) \mid n - 1$. II</i>	177
Marc Aristide Rieffel and Alfons Van Daele, <i>A bounded operator approach to Tomita-Takesaki theory</i>	187
Daniel Byron Shapiro, <i>Spaces of similarities. IV. (s, t)-families</i>	223
Leon M. Simon, <i>Equations of mean curvature type in 2 independent variables</i>	245
Joseph Nicholas Simone, <i>Metric components of continuous images of ordered compacta</i>	269
William Charles Waterhouse, <i>Pairs of symmetric bilinear forms in characteristic 2</i>	275