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The problem of whether there exists a composite n for which $\varphi(n) \mid n-1$ (φ is Euler's function) was first posed by D. H. Lehmer in 1932 and still remains unsolved. In this paper we prove that the number of such n not exceeding x is $O(x^{1/2}(\log x)^{3/4})$. We also prove that any such n with precisely K distinct prime factors is necessarily less than K^{2^K}. There are appropriate generalizations of these results to integers n for which $\varphi(n) \mid n-a$, a an arbitrary integer.

1. Introduction. In 1932, D. H. Lehmer [6] asked if there are any composite integers n for which $\varphi(n) \mid n-1$, φ being Euler's function. The answer to this question is still not known. Lieuwens [7] has shown that any such n is divisible by at least 11 distinct primes; Kishore [5] has recently announced the analogous result for 13 primes.

If S is any set of positive integers, denote by $N(S, x)$ the number of members of S which do not exceed x. Let L denote the set of composite n for which $\varphi(n) \mid n-1$. Although Erdös was not specifically considering the problem of estimating $N(L, x)$, as a corollary of his paper [2], we have

$$N(L, x) = O(x \exp(-c \log x \log\log\log x/\log\log x))$$

for some $c > 0$. In [11] we proved

$$N(L, x) = O(x^{2/3}(\log\log x)^{1/3}) .$$

One result of this paper is

$$N(L, x) = O(x^{1/2}(\log x)^{3/4}) . \tag{1.1}$$

There is still clearly a wide gap between the possibility $L = \varnothing$ and (1.1), for the latter does not even establish that the members of L are as scarce as squares! Note that we conjectured in [11] that for every $\varepsilon > 0$,

$$N(L, x) = O(x^{\varepsilon}) .$$

Important in proving (1.1) is the consideration for $n \in L$ of the distribution in the interval $[0, \log n]$ of the numbers $\log d$ for $d \mid n$. We show that these numbers do not leave any large gaps, in that any reasonable subinterval will contain some $\log d$.

We also prove another result of independent interest about the set L: if $n \in L$ and n is divisible by precisely K distinct primes, then

$$n < K^{2^K} . \tag{1.2}$$

This result is similar to a result of Borho [1] dealing with amicable numbers.

We establish results analogous to (1.1) and (1.2) for other sets of positive integers analogous to L. Recalling notation from [10], [11], we let

$$F(a) = \{n: n \equiv a (\bmod \varphi(n))\}$$

for each integer a. From Sierpiński [12, p. 232], we have

$$F(0) = \{1\} \cup \{2^i \cdot 3^j: i > 0,\ j \geqq 0\} . \tag{1.3}$$

We have seen in [10] that $F(0)$ plays a special role for the sets $F(a)$. Indeed, if $a \notin F(0)$, then $F(a)$ has no member of the form pa with p prime, $p \nmid a$. However, if $a \in F(0)$, then every such number pa is in $F(a)$. Hence we are naturally led to consider the subsets

$$F'(a) = \{n \in F(a): n \neq pa \text{ for } p \text{ prime, } p \nmid a\} .$$

Note that $F'(1) = L \cup \{1\}$. We shall prove

$$N(F'(a), x) = O(x^{1/2}(\log x)^{3/4}) \tag{1.4}$$

for every integer a, where the implied constant depends on a. Note that (1.3) implies $N(F(0), x) = O((\log x)^2)$, so that (1.4) is true for $a = 0$. However other results we prove will not be true for $a = 0$. Throughout the remainder of this paper, a will represent a nonzero integer.

We also prove that if $n \in F'(a)$ and n is divisible by precisely K distinct primes, then

$$n < \max\{16|a|^3, |a| \cdot K^{2^K}\} .$$

Certain results of Norton [9] (see Suryanarayana [13]) enable us to state our theorems in a sharper form than could be done otherwise. The results of Meijer [8] might yield further improvements.

We wish to thank the referee who carefully read the paper and made several helpful suggestions.

2. Preliminary results. If n is an integer at least 2, denote by $\omega(n)$ the number of distinct prime factors of n, $P(n)$ the largest prime factor of n, and $p(n)$ the least prime factor of n.

In our work with the sets $F'(a)$ it will be convenient to isolate the square free members. Note that every member of $F'(1)$ is square free. Let

$$F''(a) = \{n \in F'(a) \colon n \text{ is square free}\} .$$

LEMMA 1. $N(F'(a), x) \leqq 4a^2 + \sum_{d \mid a} N(F''(a/d), x/d)$.

Proof. Let $n \in F'(a)$, $4a^2 < n \leqq x$. If $n = pa$ for some prime p, then $p \mid a$, so $n \leqq a^2$. Hence $n \neq pa$ for every prime p. Let m be the maximal square free divisor of n and let $d = n/m$. Then every prime factor of d also divides m. Hence $\varphi(m) = \varphi(n)/d$, so that $d \mid a$ and $m \in F(a/d)$. Since $m \neq pa/d$ for every prime p, we have $m \in F''(a/d)$.

Hence all we need verify is that if n_1, $n_2 \in F'(a)$ with maximal square free divisors m_1, m_2, and if n_1, $n_2 > 4a^2$, then $m_1 = m_2$ implies $n_1 = n_2$. Now for any n we have

$$\varphi(n) > \sqrt{n}/2 \tag{2.1}$$

(Sierpiński [12, p. 230]). Suppose $m_1 = m_2$. Then n_1 and n_2 have the same set of prime factors. This implies $n_1/\varphi(n_1) = n_2/\varphi(n_2)$. Let $k_i = (n_i - a)/\varphi(n_i)$ for $i = 1, 2$. Then

$$k_1 + a/\varphi(n_1) = k_2 + a/\varphi(n_2) .$$

From (2.1) and the assumption n_1, $n_2 > 4a^2$, we have $0 < |a/\varphi(n_i)| < 1$ for $i = 1, 2$. But k_1, k_2 are integers and $a/\varphi(n_1)$, $a/\varphi(n_2)$ have the same sign, so

$$a/\varphi(n_1) = a/\varphi(n_2) .$$

But $n_1/\varphi(n_1) = n_2/\varphi(n_2)$, so $n_1 = n_2$, which was to be proved.

LEMMA 2. *If* $n \geqq 16a^2$, $n \in F''(a)$, *then*
(i) $k \doteq (n-a)/\varphi(n)$ *is a positive integer at least* 2;
(ii) *if* $m \mid n$, $m \neq n$, *then* $m/\varphi(m) < k$;
(iii) *there is a prime* $q > P(n)$ *with* $nq/\varphi(nq) > k$;
(iv) $\omega(n) \geqq 3$.

Proof. (i) First we note that n is composite. Indeed if $n = p$, a prime, then the condition $p \in F''(a)$ implies $p - 1 \mid a - 1$ and $a \neq 1$. Then $p \leqq |a| + 2 < 16a^2$, a contradiction.

Now $n = k\varphi(n) + a$, so if $k \leqq 0$, then $n \leqq a$. Suppose $k = 1$. Since n is composite, n has a divisor d with $\sqrt{n} \leqq d < n$. Then $\varphi(n) \leqq n - d \leqq n - \sqrt{n}$. Then

$$a = n - \varphi(n) \geqq \sqrt{n} \geqq 4|a| ,$$

a contradiction.

(ii) It is sufficient to prove (ii) for $m = n/p$ where $p = P(n)$. From (2.1) and the assumption $n \geqq 16a^2$, we have $|a/\varphi(n)| < 1/2$. Hence from the equation $n/\varphi(n) = k + a/\varphi(n)$ and (i) we have

$$(2.2) \qquad (3/4)k \leqq k - 1/2 < n/\varphi(n) < k + 1/2 .$$

Then $m/\varphi(m) < k + 1/2 < 2k$, so

$$(2.3) \qquad k\varphi(m) > m/2 .$$

Now

$$(2.4) \qquad a = n - k\varphi(n) = mp - k\varphi(mp) = p(m - k\varphi(m)) + k\varphi(m) .$$

If $m = k\varphi(m)$, then (2.4) implies $a = k\varphi(m)$, so that $a = m$ and $n \notin F''(\mathrm{a})$. Hence $m \neq k\varphi(m)$. If $m > k\varphi(m)$, then (2.3), (2.4) imply

$$a \geqq p + k\varphi(m) > p + m/2 \geqq (2pm)^{1/2} > n^{1/2} \geqq 4|a| ,$$

a contradiction. Hence $m < k\varphi(m)$.

(iii) If $a > 0$, clearly any prime $q > P(n)$ will do. Hence assume $a < 0$. We first prove

$$(2.5) \qquad P(n) < n/2|a| .$$

Indeed from (2.2) we have (with $m = n/P(n)$)

$$\frac{3}{4}k < \frac{n}{\varphi(n)} = \frac{m}{\varphi(m)} \cdot \frac{P(n)}{P(n) - 1} \leqq \frac{2m}{\varphi(m)} .$$

Then from (ii) and (2.4) we have

$$P(n) = (a - k\varphi(m))/(m - k\varphi(m)) \leqq |a| + k\varphi(m) < |a| + (8/3)m .$$

If (2.5) fails, we have $m = n/P(n) \leqq 2|a|$, and it follows that $P(n) < (19/3)|a|$ and $n = mP(n) < 16a^2$, a contradiction.

By Chebyshev's theorem there is a prime q with $n/2|a| < q < n/|a|$, and by (2.5), $q > P(n)$. Also

$$\frac{nq}{\varphi(nq)} > \frac{n}{\varphi(n)} \cdot \frac{n/|a|}{n/|a| - 1} = \frac{n^2}{\varphi(n)(n + a)}$$
$$= \frac{kn^2}{(n - a)(n + a)} > k ,$$

since $n^2 > n^2 - a^2 > 0$.

(iv) We noted in the proof of (i) that $\omega(n) \geqq 2$. Suppose $\omega(n) = 2$. Let $n = pq$ with $p < q$. Let r be a prime with $r > q$ and $pqr/\varphi(pqr) > k \geqq 2$ (using (i) and (iii)). Since $(2/1)(3/2)(5/4) < 4$, we have $k = 2$ or 3.

If $k = 3$, then since $(2/1)(5/4)(7/6) < 3$, we have $n = pq = 6 < 16a^2$.

Suppose $k = 2$. Since $(5/4)(7/6)(11/10) < 2$, we have $p = 2$ or 3. By (ii), $p/\varphi(p) < 2$, so $p = 3$. Since $(3/2)(7/6)(11/10) < 2$, we have $q = 5$. That is, $n = pq = 15 < 16a^2$.

LEMMA 3. *Suppose k, n are natural numbers with n square free and $n/\varphi(n) > k$. If $m \mid n$ and $m/\varphi(m) < k$, then*

$$p(n/m) < \omega(n/m)\cdot(m+1) .$$

Proof. Let $r = \omega(n/m)$, $p = p(n/m)$. Then

$$k < \frac{n}{\varphi(n)} \leqq \frac{m}{\varphi(m)} \cdot \left(\frac{p}{p-1}\right)^r ,$$

so that

$$m/k\varphi(m) > (1 - 1/p)^r \geqq 1 - r/p .$$

Hence

$$p < \frac{rk\varphi(m)}{k\varphi(m) - m} = r\left(1 + \frac{m}{k\varphi(m) - m}\right) \leqq r(m+1) .$$

3. Members of $F''(a)$ with K prime factors.

THEOREM 1. *Suppose $n \geqq 16a^2$, $n \in F''(a)$, $K = \omega(n)$. Let the prime factorization of n be $p_1 p_2 \cdots p_K$ where $p_1 > p_2 > \cdots > p_K$. Then for $1 \leqq i \leqq K$, we have*

$$p_i < (i+1)(1 + \prod_{j=i+1}^{K} p_j) .$$

Proof. Let $m = \prod_{j=i+1}^{K} p_j$. By (iii) of Lemma 2 there is a prime $q > p_1$ with $nq/\varphi(nq) > k$. By (ii) of Lemma 2, $m/\varphi(m) < k$. Since $p_i = p(nq/m)$ and $i + 1 = \omega(nq/m)$, Lemma 3 completes the proof.

THEOREM 2. *Suppose $n \geqq 16a^2$, $n \in F''(a)$, $K = \omega(n)$. Then there is a positive constant β independent of the choice of a, n such that*

$$p(n) < \beta K^{1/2}(\log K)^{1/2} . \tag{3.1}$$

In addition, if $K \geqq 4$, then $p(n) \leqq K - 1$.

Proof. Let $p = p(n)$. Since there is a prime $q > P(n)$ with $nq/\varphi(nq) > k \geqq 2$ ((i) and (iii) of Lemma 2), it follows from Norton [9, Theorem 4] that there is an absolute constant $\beta_1 > 0$ with

$$K + 1 = \omega(nq) > \beta_1 p^2/\log p .$$

By Theorem 1, $\log p < \log(2(K+1)) < \beta_2 \log K$ for some $\beta_2 > 0$ ((iv) of Lemma 2). Hence there is an absolute constant $\beta > 0$ such that $p^2 < \beta^2 K \log K$, which proves (3.1).

Now assume $K \geqq 4$. Then $p \leqq K-1$ if $p = 2$ or 3. From $nq/\varphi(nq) > 2$, we have $K+1 \geqq 7$ if $p = 5$, so $p \leqq K-1$ in this case too. If $p \geqq 7$ we similarly get $K+1 \geqq 15$, so that using a result of Grün [3], we have

$$p < (2/3)(K+1) + 2 < K - 1 .$$

THEOREM 3. *If* $n \in F''(a)$, $K = \omega(n)$, *then*

$$n < \max\{16a^2, K^{2^K}\} .$$

Proof. Assume $n \geqq 16a^2$. By (iv) of Lemma 2 we have $K \geqq 3$. If $K = 3$, we can show as follows that $n \leqq 435 < 3^{2^3}$. Write $n = pqr$ where $p < q < r$ are primes. By Lemma 2 there is a prime $s > r$ such that

$$pqrs/\varphi(pqrs) > k \geqq 2 , \tag{3.2}$$

$$pq/\varphi(pq) < k . \tag{3.3}$$

We proceed as with the proof of (iv) of Lemma 2. Say $k \geqq 3$. Then (3.2) implies $k = 3$, $p = 2$, $q \leqq 5$ or $k = 4$, $n = pqr = 30$. In the former case, (3.3) implies $q = 5$, so (3.2) implies $n = pqr = 70$. Now say $k = 2$. Then (3.2), (3.3) imply $p = 3$. Then (3.2) implies $q = 5$, $r \leqq 29$ (so $n \leqq 3 \cdot 5 \cdot 29 = 435$) or $q = 7$, $r \leqq 13$ (so $n \leqq 3 \cdot 7 \cdot 13 = 273$).

Assume $K \geqq 4$. Let the prime factorization of n be $p_1 p_2 \cdots p_K$ where $p_1 > p_2 > \cdots > p_K$. By Theorem 2,

$$p_K + 1 \leqq K .$$

By Theorem 1, $p_{K-1} < K(p_K + 1) \leqq K^2$. Hence

$$p_{K-1}p_K + 1 < K^3 .$$

Again by Theorem 1, $p_{K-2} < (K-1)(p_{K-1}p_K + 1)$, so that

$$p_{K-2}p_{K-1}p_K + 1 < p_{K-2}(p_{K-1}p_K + 1) < (K-1)K^6 < K^7 .$$

Continuing in this fashion we get

$$n = p_1 p_2 \cdots p_K < K^{2^K - 1} < K^{2^K} .$$

THEOREM 4. *If* $n \in F'(a)$, $K = \omega(n)$, *then*

$$n < \max\{16|a|^3, |a| \cdot K^{2^K}\} .$$

Proof. Assume $n \geqq 16|a|^3$. Following the proof of Lemma 1, we find a positive integer d with $d \mid (n, a)$ and $n/d \in F''(a/d)$. Then $n/d \geqq 16a^2$, so Theorem 3 implies $n/d < K^{2^K}$. Hence

$$n < d \cdot K^{2^K} \leqq |a| \cdot K^{2^K} .$$

4. A combinatorial lemma.

LEMMA 4. *Suppose* $\delta \geqq 0$, $a_1 \geqq a_2 \geqq \cdots \geqq a_t > 0$, $B_i = \sum_{j=i}^{t} a_j$ *for* $1 \leqq i \leqq t$, *and*

$$a_i \leqq \delta + B_{i+1} \tag{4.1}$$

for $1 \leqq i \leqq t-1$. *Then given any* y *with* $0 \leqq y < B_1$, *there is a subset* S *of* $\{1, 2, \cdots, t\}$ *with*

$$y - \delta - a_t < \sum_{i \in S} a_i \leqq y .$$

Proof. We may assume $y \geqq \delta + a_t$ for otherwise take $S = \varnothing$. We have

$$B_1 > y , \quad B_t \leqq y . \tag{4.2a,b}$$

Let $s(0) = 0$. Say we have either constructed a set S as called for or we have inductively found an integer sequence $s(0) < s(1) < \cdots < s(i-1) < t$ where $i \geqq 1$ and

$$\sum_{j=1}^{i-1} a_{s(j)} + B_{s(i-1)+1} > y , \tag{4.3a}$$

$$\sum_{j=1}^{i-1} a_{s(j)} + B_t \leqq y . \tag{4.3b}$$

Let $s(i)$ be maximal with

$$\sum_{j=1}^{i-1} a_{s(j)} + B_{s(i)} > y .$$

By (4.3a), (4.3b), $s(i)$ exists and $s(i-1) < s(i) < t$. Then since $a_{s(i)} + B_{s(i)+1} = B_{s(i)}$, we have

$$\sum_{j=1}^{i} a_{s(j)} + B_{s(i)+1} > y . \tag{4.4a}$$

Note that $\sum_{j=1}^{i-1} a_{s(j)} + B_{s(i)+1} \leqq y$. Then we may assume

$$\sum_{j=1}^{i-1} a_{s(j)} + B_{s(i)+1} \leqq y - \delta - a_t , \tag{4.5}$$

for otherwise we may take

$$S = \{s(1), s(2), \cdots, s(i-1), s(i)+1, s(i)+2, \cdots, t\} .$$

Then from (4.5) and from (4.1) applied to $a_{s(i)}$, we have

$$\sum_{j=1}^{i} a_{s(j)} + B_t = \sum_{j=1}^{i} a_{s(j)} + a_t$$
$$\leqq \sum_{j=1}^{i-1} a_{s(j)} + \delta + B_{s(i)+1} + a_t \leqq y \ ;$$

that is,

$$\sum_{j=1}^{i} a_{s(j)} + B_t \leqq y \ . \tag{4.4b}$$

Since there is not an infinite increasing sequence of positive integers all less than t, this process must terminate with the construction of a suitable set S.

5. Estimates for $N(F'(a), x)$.

THEOREM 5. *For every a, $N(F'(a), x) = O(x^{1/2}(\log x)^{3/4})$, where the implied constant depends on a.*

Proof. In view of Lemma 1, it will be sufficient to prove for every a that $N(F''(a), x) = O(x^{1/2}(\log x)^{3/4})$, where the implied constant depends on a. We record for future reference: there are positive constants α, γ with

$$n/\varphi(n) < \alpha \log\log n \ , \quad n \geqq 3 \tag{5.1}$$

$$\omega(n) < \gamma \log n/\log\log n \ , \quad n \geqq 3 \ . \tag{5.2}$$

(Hardy and Wright [4, pp. 353-355].)

Let $n \in F''(a)$, $16a^2 \leqq n \leqq x$, $K = \omega(n)$. Let the prime factorization of n be $p_1 p_2 \cdots p_K$ where $p_1 > p_2 > \cdots > p_K$. We may assume $n > x^{1/2}(\log x)^{3/4}$. Theorem 1 implies

$$\log p_i < \log(2K) + \sum_{j=i+1}^{K} \log p_j \ , \quad 1 \leqq i \leqq K-1 \ .$$

We apply Lemma 4 with

$$\delta = \log(2K) \ , \quad t = K \ , \quad a_i = \log p_i \ , \quad y = \frac{1}{2}\log x + \frac{3}{4}\log\log x \ .$$

Hence there is an integer m with $m \mid n$ and

$$y - \delta - \log p_K < \log m \leqq y \ .$$

Then

$$x^{1/2}(\log x)^{3/4}/2Kp_K < m \leqq x^{1/2}(\log x)^{3/4} \ .$$

By (3.1), (5.2), we have

$$\begin{aligned} 2Kp_K &< 2\beta K^{3/2}(\log K)^{1/2} \\ &< 2\beta(\gamma \log x/\log\log x)^{3/2}(\log(\gamma \log x/\log\log x))^{1/2} \\ &< \gamma'(\log x)^{3/2}(\log\log x)^{-1} \end{aligned}$$

for some $\gamma' > 0$. Hence

$$\begin{aligned} f(x) \doteq (1/\gamma')x^{1/2}(\log x)^{-3/4}\log\log x &< m \\ \leqq x^{1/2}(\log x)^{3/4} &\doteq g(x) . \end{aligned}$$

For each integer m in the above interval we now count the number of choices for $n \in F''(a)$ with $n \leqq x$ and $m \mid n$. Since $\varphi(m) \mid \varphi(n)$ for such n, we have

$$n \equiv 0 (\operatorname{mod} m) , \quad n \equiv a (\operatorname{mod} \varphi(m)) ,$$

so by the generalized Chinese remainder theorem, there are at most $1 + x/[m, \varphi(m)]$ choices for such n (here $[\ ,\ ]$ denotes least common multiple). Now $(m, \varphi(m)) \mid (n, \varphi(n))$ and $(n, \varphi(n)) \mid a$. Hence for each m, there are at most (using (5.1))

$$\begin{aligned} 1 + x/[m, \varphi(m)] &= 1 + x(m, \varphi(m))/m\varphi(m) \\ &\leqq 1 + |a|x/m\varphi(m) < 1 + |a|\alpha x \log\log x/m^2 \end{aligned}$$

choices for $n \in F''(a)$ with $n \leqq x$ and $m \mid n$.

Hence we have

$$\begin{aligned} N(F''(a), x) &\leqq 16a^2 + x^{1/2}(\log x)^{3/4} + \sum_{f(x) < m \leqq g(x)} (1 + |a|\alpha x \log\log x/m^2) \\ &= O(x^{1/2}(\log x)^{3/4}) + O(x \log\log x \sum_{f(x) < m} 1/m^2) \\ &= O(x^{1/2}(\log x)^{3/4}) + O(x \log\log x/f(x)) \\ &= O(x^{1/2}(\log x)^{3/4}) . \end{aligned}$$

REMARK. Both the referee and D. Suryanarayana kindly suggest the use of a fact due to Landau,

$$\sum_{m > y} 1/m\varphi(m) = O(1/y) ,$$

in the proof of Theorem 5, rather than (5.1). This enables us to get the slightly stronger estimate

$$N(F'(a), x) = O(x^{1/2}(\log x)^{3/4}(\log\log x)^{-1/2}) \tag{5.3}$$

where the implied constant depends on a. In addition we note that if those $n \leqq x$ for which $p(n) \leqq (\log x)^{1/4}$ are treated separately from the remaining choices for n, then an extra factor of $1/\log\log x$ may be introduced on the right of (5.3). It is conceivable that further improvements are possible, even in the exponent on $\log x$ (perhaps by considering a sharper version of Lemma 4 where the constant δ is replaced by a variable δ_i which is usually small). It would seem to take a completely new idea however to lower the exponent on x.

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