

Pacific Journal of Mathematics

**PRODUCTS OF COMPACT SPACES WITH BI- k AND RELATED
SPACES**

ANDREW J. BERNER

PRODUCTS OF COMPACT SPACES WITH $bi - k$ AND RELATED SPACES

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The main theorem of this paper characterizes $bi - k$ spaces as those spaces whose product with every compact space is sequentially k .

1. **Introduction.** The classes of $bi - k$ spaces, countably $bi - k$ spaces and singly $bi - k$ spaces were studied in [5], and the class of sequentially k spaces was introduced in [3]. The following implications hold among these spaces, without the assumption of any separation axioms: $bi - k \rightarrow$ countably $bi - k \rightarrow$ singly $bi - k \rightarrow$ sequentially k . Also, all k spaces are sequentially k , and all Hausdorff sequentially k spaces are k spaces. (These classes will be defined at the end of this introduction.)

THEOREM 1.1. *The following are equivalent:*

- (a) X is a $bi - k$ space.
- (b) $X \times Y$ is a singly $bi - k$ space for every compact Hausdorff space Y .
- (c) $X \times Y$ is sequentially k for every compact space Y .

This theorem is proved in § 2.

REMARK 1.2. Cohen [4] proved that the product of a k space with a (locally) compact Hausdorff space is a k space. Noble [6] showed this is false without the Hausdorff assumption, but in Noble's example, the product was a $bi - k$ space. Theorem 1.1 ($c \leftrightarrow a$) shows that the product of a k space with a compact space need not even be sequentially k .

REMARK 1.3. Michael [5] has asked whether the product of two countably $bi - k$ spaces must be countably $bi - k$. Examples have been given showing it is consistent with Zermelo-Fraenkel set theory that this is false. Theorem 1.1 can be used to give an absolute counterexample. All we need is a countably $bi - k$ space that is not $bi - k$. Let X be the subspace of the product of uncountably many copies of $\{0, 1\}$ consisting of points that are 1 on only countably many coordinates (i.e., a Σ -product centered at the point all of whose coordinates are 0). Arhangel'skii proved that this space is countably $bi - k$ (in fact, countably bi -sequential) but not $bi - k$ [2]. Thus, by Theorem 1.1, there is a compact Hausdorff space Y_1 and a

compact T_1 space Y_2 such that $X \times Y_1$ is not singly $bi - k$, thus not countably $bi - k$, and $X \times Y_2$ is not even sequentially k .

DEFINITION 1.4. [3] If S is a subset of a topological space X and (S_i) is a nested sequence of subsets of X , then (S_i) is an S -sequence if whenever (s_i) is a sequence of points with $s_i \in S_i$ for each i , then (s_i) has an accumulation point in S .

DEFINITION 1.5. A space X is a $bi - k$ space if whenever \mathcal{F} is a filter base containing the open sets around a point $p \in X$, there is a compact set $S \subset X$ and a nested sequence of sets (S_i) such that (S_i) meshes with \mathcal{F} and (S_i) is an S -sequence.

DEFINITION 1.6. A space X is a countably $bi - k$ space if whenever (F_i) is a nested sequence of sets accumulating at a point p (i.e., $p \in \text{cl}(F_i)$ for each i) there is a nested sequence of sets (S_i) accumulating at p and a compact set S such that $S_i \subset F_i$ for each i and (S_i) is an S -sequence.

DEFINITION 1.7. A space X is a singly $bi - k$ space if whenever $p \in \text{cl}(F)$, there is a compact set S and a nested sequence of sets (S_i) accumulating at p such that $S_i \subset F$ for each i and (S_i) is an S -sequence.

DEFINITION 1.8. A space X is sequentially k if whenever a set F is not closed there is a point $p \in \text{cl}(F) - F$, a compact set S and a nested sequence of sets (S_i) accumulating at p such that $S_i \subset F$ for each i and (S_i) is an S -sequence.

2. **Proof of Theorem 1.1.** In [2], Michael proved that a space X is countably $bi - k$ if and only if $X \times I$ is singly $bi - k$ (where I is the unit interval). The heart of the proof (in one direction) involves coding a bad nested sequence (S_i) of subsets of X (i.e., a witnessing sequence to the statement ' X is not countably $bi - k$ ') as a single bad subset of $X \times I$. This idea of coding is hinted at in the following proof of Theorem 1.1.

$$a \longrightarrow b \quad \text{and} \quad a \longrightarrow c:$$

The product of two (or even countably many) $bi - k$ spaces is $bi - k$ [5]. Since compact spaces are $bi - k$, the product of a $bi - k$ space and a compact space is $bi - k$, and thus singly $bi - k$ and sequentially k .

$$\text{not } a \longrightarrow \text{not } b \quad \text{and} \quad \text{not } a \longrightarrow \text{not } c:$$

Both implications make use of the following construction.

Suppose X is not $bi - k$. Then there is a point $p \in X$ and a filter base \mathcal{F} of subsets of X such that \mathcal{F} contains the open sets around p , but there is no compact $S \subset X$ and nested sequence of sets (S_i) such that (S_i) meshes with \mathcal{F} and is an S -sequence. Thus, in particular, there is an $F \in \mathcal{F}$ such that $p \notin F$ and therefore if $F_1 \in \mathcal{F}$ and $F_2 \in \mathcal{F}$, then $F_1 \cap F_2 - \{p\} \neq \emptyset$.

Define a base for a new topology on X as follows. If $x \in X - \{p\}$, then $\{x\}$ is open, and if $F \in \mathcal{F}$, then $F \cup \{p\}$ is a neighborhood of p . This refines the original topology on X . Let X' be X with this new topology. Note that X' is completely regular.

Let $Y_1 = \beta(X')$, the Stone-Ćech compactification of X' (actually, any Hausdorff compactification will do), and let Y_2 be the one point compactification of X' . Note that Y_2 is a T_1 space, but is definitely not Hausdorff.

Claim 1. $X \times Y_1$ is not singly $bi - k$.

Proof. Let $C = \{(x, x) : x \in X - \{p\}\}$. C is a subset of $X \times X' \subset X \times Y_1$. If $U \times V$ is a basic open set around (p, p) , then U and $V \cap X'$ are both in \mathcal{F} , so $U \cap V \cap X' - \{p\} \neq \emptyset$. Thus $(U \times V) \cap C \neq \emptyset$, i.e., $(p, p) \in \text{cl}(C) - C$. Suppose $X \times Y_1$ is singly $bi - k$. Then there is a compact set $K \subset X \times Y_1$, and a nested sequence (K_i) of subsets of C such that $(p, p) \in \text{cl}(K_i)$ for each i , and (K_i) is a K -sequence. Then, if $\pi_x : X \times Y_1 \rightarrow X$ is the projection map, the sequence $(\pi_x(K_i))$ is a $\pi_x(K)$ -sequence and $\pi_x(K)$ is compact in X . Suppose $F \in \mathcal{F}$. Then there is an open set $V \subset Y_1$ such that $V \cap X' = F \cup \{p\}$. But then for each i , there is an x_i such that $(x_i, x_i) \in K_i \cap (X \times V)$. Then $x_i \in F$, so $\pi_x(K_i) \cap F \neq \emptyset$. Thus $(\pi_x(K_i))$ is a $\pi_x(K)$ -sequence meshing with \mathcal{F} . This violates the choice of \mathcal{F} , so $X \times Y_1$ is not singly $bi - k$.

Claim 2. $X \times Y_2$ is not sequentially k .

Proof. Let $Y_2 = X' \cup \{\alpha\}$.

Let $C = (X \times \{\alpha\}) \cup \{(y, x) : y \in \text{cl}_x(\{x\}) \text{ and } x \in X' - \{p\}\}$. (Nobody said X was a T_1 space!) As in the proof of Claim 1, $(p, p) \in \text{cl}(C) - C$. Suppose $(x, y) \in \text{cl}(C) - C$. Then $y \neq \alpha$. Suppose $y \neq p$. Then since $(x, y) \notin C$, it follows that $x \notin \text{cl}_x(\{y\})$ so there is an open (in X) set U such that $x \in U$ but $y \notin U$. But then $U \times \{y\}$ is open (in $X \times Y_2$) and $(U \times \{y\}) \cap C = \emptyset$. Thus $y = p$.

Could $X \times Y_2$ be sequentially k ? If so, then there is a point $(z, p) \in \text{cl}(C) - C$, a compact set $K \subset X \times Y_2$ and a sequence (K_i)

such that $(z, p) \in \text{cl}(K_i)$ and $K_i \subset C$ for each i , and (K_i) is a K -sequence. Again, let $\pi_X: X \times Y_2 \rightarrow X$ be the projection map. Let $D_i = \{x: \text{there is a } y \in \pi_X(K_i) \text{ such that } y \in \text{cl}_X(\{x\})\}$. Suppose $x_i \in D_i$ for each i . There is, for each i , a point $y_i \in \pi_X(K_i) \cap \text{cl}_X(\{x_i\})$. Since $(\pi_X(K_i))$ is a $\pi_X(K)$ -sequence, the sequence (y_i) has an accumulation point $k \in \pi_X(K)$. But, since any open set containing y_i also contains x_i , k is an accumulation point of (x_i) . Thus (D_i) is a $\pi_X(K)$ -sequence.

Suppose $F \in \mathcal{S}$. Since $F \cup \{p\}$ is open in Y_2 , for each i $(X \times (F \cup \{p\})) \cap K_i \neq \emptyset$. But $\alpha \notin F \cup \{p\}$ so there is a point $(y, x) \in K_i$ with $x \in F$ and $y \in \text{cl}_X(\{x\})$. Thus $x \in D_i \cap F$. Therefore D_i is a $\pi_X(K)$ -sequence meshing with \mathcal{S} , and $\pi_X(K)$ is compact. As in the case of Claim 1, this contradicts the choice of \mathcal{S} , so $X \times Y_2$ is not sequentially- k .

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