Pacific Journal of Mathematics

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Vol. 69, No. 2 June 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 2, 1977

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If M is a module and M is a submodule of M, then N is irreducible in M if N cannot be written as a proper intersection of two submodules of M. The purpose of this note is to study modules whose submodules can be written as a finite intersection of irreducible submodules. Such modules are characterized by the fact that their quotients all have finite Goldie dimension, so they are called q.f.d. modules.

The main result is: A module M is q.f.d. if and only if every submodule N has a finitely generated submodule T such that N/T has no maximal submodules. Because T is finitely generated this generalizes a theorem of Shock (using his ideas), who showed a q.f.d. module M having the property that every subquotient of M has a maximal submodule must be noetherian (and conversely, of course).

The q.f.d. condition also arises in the study of Krull dimension because a module with Krull dimension must be q.f.d. [1].

First, a remark, which we isolate as a lemma.

LEMMA. *A module M is* q.f.d. *if and only if each quotient of M has finite dimensional socle* (possible zero).

Proof. Every nonzero module has a quotient with nonzero socle. (Take $A \subset M$ to be maximal with respect to not containing m, $0 \neq m \in M$.) So an infinite direct sum of modules has a quotient with infinite dimensional socle.

THEOREM. *A module M is* q.f.d. *if and only if every submodule N contains a finitely generated submodule T, such that N/T has no maximal submodules.*

Proof. Suppose that X is a module such that every finitely generated submodule of X is contained in a maximal submodule of X. Having chosen maximal submodules $M_1, \cdots, M_n$ and elements $x_1, \cdots, x_n$ such that $x_i \notin M_i$, but $x_i \in M_j$ for $j > i$, choose a maximal submodule M_{n+1} containing $x_1R + \cdots + x_nR$ and an element x_{n+1} not in M_{n+1}.

Let $\bar{X} = X/\bigcap_{i=1}^{\infty} M_i$. Then, $\bar{X} = \bigcap_{i=1}^{n} M_i \oplus \bigcap_{i=n+1}^{\infty} M_i$, because, if we denote the right hand summand by M, we have a strict descending chain $M \supset M \cap M_n \supset M \cap M_n \cap M_{n-1} \cdots$ so that M has the same composition length as $X/\bigcap_{i=1}^{n} M_i$.

Thus, $\bar{X}$ has direct sums of arbitrary size, and so is infinite dimensional.

Conversely, we wish to show that if for every $N \subset M$, we can always find such a T, then M has no quotient which contains an infinite direct sum. If M/K does, then by the lemma we can find a K' with M/K' having an infinite direct snm of simple submodules. Let $K' \subset S \subset M$ be such that S/K' is this infinite direct sum. Choose T finitely generated such that S/T has no maximal submodules. Then $S/T + K'$ has no maximal submodules. But $S/T + K'$ is a homomorphic image of the semisimple module S/K' so $S/T + K'$ is semisimple, and always has maximal submodules if it is not zero. So, it must be zero and $S = T + K'$. Since T is finitely generated S/K' must be, a clear impossibility.

REMARK. The above naturally raises the question, when are finitely generated modules finite dimensional? We observe the following:

PROPOSITION. *If cyclic modules are finite dimensional then finitely generated modules are.*

Proof. Let $E(M)$ denote the injective hull of the module M. Let M be generated by $\{m_1, \cdots, m_r\}$. Let $E(M) = E(m_1R) \oplus K_1$, and write $m_2 = a_1 + k_1$. Then $K_1 = E(k_1R) \oplus K_2$. So $E(M) = E(m_1R) \oplus E(k_1R) \oplus K_2$, and $m_1R + m_2R \subset E(m_1R) \oplus E(k_1R)$. Continue in this fashion to get $E(M)$ as a finite direct sum of injective hulls of cyclic modules. These are finite dimensional, so M is.

Since there are non-noetherian valuation rings (ideals are linearly ordered), there are non-noetherian rings whose finitely generated modules are finite dimensional. The result cited in the first paragraph is 4.10 of [4].

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Received July 22, 1976.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72 00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 69, No. 2 June, 1977