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SUMMABILITY R_r FOR DOUBLE SERIES

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Let r be a positive integer. A trigonometric series T of a single variable is said to be summable R_r at θ_0 if the series obtained by r times formally integrating T has an r th symmetric derivative at θ_0 . For even values of r , summability R_r has been applied to double trigonometric series. We study here summability R_r , for odd values of r , for double trigonometric series. We obtain a connection between Bochner-Riesz summable series and series which are summable R_r .

1. Let

$$(1.1) \quad \sum_{-\infty}^{\infty} c_n e^{in\theta}$$

be a trigonometric series of a single variable. Let r be a positive integer. Suppose the series obtained by formally integrating (1.1) r times

$$(1.2) \quad c_0 \frac{\theta^r}{r!} + \sum_{n \neq 0} \frac{c_n}{(in)^r} e^{in\theta}$$

converges to a function $F(\theta)$ in a neighborhood of $\theta_0 \in (0, 2\pi)$. We will say that the series (1.1) is at θ_0 *summable by the method R_r* to sum s if $F(\theta)$ has at θ_0 an r th symmetric derivative with value s . That is, if r is even,

$$(1.3) \quad \frac{1}{2} \{F(\theta_0 + t) + F(\theta_0 - t)\} = a_0 + \frac{a_2}{2!} t^2 + \dots + \frac{s}{r!} t^r + o(t^r)$$

as $t \rightarrow 0$, and if r is odd,

$$(1.4) \quad \frac{1}{2} \{F(\theta_0 + t) - F(\theta_0 - t)\} = a_1 t + \frac{a_3}{3!} t^3 + \dots + \frac{s}{r!} t^r + o(t^r),$$

as $t \rightarrow 0$.

The following result, see [8], p. 66, establishes a connection between summability (C, α) and summability R_r for trigonometric series.

THEOREM A. *Let $\alpha > -1$ and assume the series (1.1) is summable (C, α) at θ_0 to sum s . Let r be an integer with $r > \alpha + 1$, and suppose the series (1.2) converges in a neighborhood of θ_0 . Then the series (1.1) is summable R_r to s .*

2. In two variables we will denote points $x \in E_2$ by $x = (x_1, x_2) =$

$te^{i\theta}$ and integral lattice points by $n = (n_1, n_2)$. We write

$$|x| = \sqrt{x_1^2 + x_2^2}.$$

We will say a double trigonometric series

$$(2.1) \quad T: \sum_{n \in \mathbb{Z}_2} c_n e^{i n \cdot x}$$

is *Bochner-Riesz summable* of order α at x_0 to sum s_0 if

$$\lim_{R \rightarrow \infty} \sum_{|n| < R} \left(1 - \left(\frac{|n|}{R}\right)^2\right)^\alpha c_n e^{i n \cdot x_0} = s_0.$$

Suppose r is an even number, $r = 2s$. A two dimensional analogue of summability R_r is given as follows, see [7], [4].

DEFINITION. Let $F(x)$ be defined in a neighborhood of $x_0 \in E_2$. F has at x_0 a *sth generalized Laplacian* equal to s_0 if F is integrable on each circle $|x - x_0| = t$ and

$$(2.2) \quad \frac{1}{2\pi} \int_0^{2\pi} F(x_0 + te^{i\theta}) d\theta = a_0 + \frac{a_2 t^2}{(2!)^2} + \cdots + \frac{s_0 t^{2s}}{(2^s s!)^2} + o(t^{2s})$$

as $t \rightarrow 0$.

THEOREM B. Let the series T of (2.1) be *Bochner-Riesz- m summable* at x_0 to sum s_0 , where m is a nonnegative integer, and suppose the coefficients of T satisfy

$$\sum_{n \in \mathbb{Z}_2} |n|^{-3+\varepsilon} |c_n|^2 < \infty$$

for some $\varepsilon > 0$. Let $r = 2s$ be an even integer with $r \geq m + 2$. Set

$$(2.3) \quad F(x) = \frac{c_0(x_1 + x_2)^{2s}}{2^s(2s)!} + (-1)^s \sum_{n \neq 0} \frac{c_n}{|n|^{2s}} e^{i n \cdot x}.$$

Then the *generalized sth Laplacian* of $F(x)$ exists at x_0 and is equal to s_0 .

That is, if the series (2.1) is *Bochner-Riesz- m summable* to s_0 and r is an even number with $r \geq m + 2$, then the series is also summable R_r to sum s_0 .

3. The purpose of this paper is to derive a connection between Bochner-Riesz summability and summability R_r , for *odd* values of r . We use the following definition, from [5]. This definition extends the formula of (1.4) to two dimensions in a manner analogous to the extension of (1.3) to two variables by (2.2).

DEFINITION. Let $r = 2s + 1$ be an odd positive integer. Let $L(x)$ be a function defined in a neighborhood of $x_0 \in E_s$. We will say $L(x)$ has at x_0 a *generalized symmetric derivative* of order r with value s_0 if L is integrable on each circle $|x - x_0| = t$, for t small, and if

$$(3.1) \quad \frac{1}{2\pi} \int_0^{2\pi} L(x_0 + te^{i\theta})(\cos \theta + \sin \theta) d\theta \\ = a_1 t + a_3 t^3 + \cdots + \frac{s_0}{2^{2s+1} s! (s+1)!} t^{2s+1} + o(t^{2s+1})$$

as $t \rightarrow 0$.

We are able to obtain the following results which, for odd values of r , form a two dimensional version of Theorem A. We begin with the case of double trigonometric series which are Bochner-Riesz summable of integral order, since the statement and proof of our results are much simpler in this case.

THEOREM 1. Let m be a nonnegative integer. Suppose

$$(3.2) \quad T: \sum_{n \in \mathbb{Z}_2} c_n e^{in \cdot x}$$

is Bochner-Riesz- m summable at x_0 to finite sum s_0 . Let $r = 2s + 1$ be an odd integer such that $r \geq m + 1$. Suppose the coefficients of T satisfy

$$(3.3) \quad \sum_{n_1 + n_2 = 0} |n|^{-2r+3+\varepsilon} |c_n|^2 + \sum_{n_1 + n_2 \neq 0} (n_1 + n_2)^{-2} |n|^{-2r+3+\varepsilon} |c_n|^2 < \infty$$

for some $\varepsilon > 0$. Then the series

$$(3.4) \quad \frac{c_0(x_1 + x_2)^r}{(r)!(2r)!2^{s+1}} + \frac{1}{2}(x_1 + x_2) \sum'_{n_1 + n_2 = 0} \frac{c_n}{|n|^{2s}} e^{in \cdot x} \\ + \sum_{n_1 + n_2 \neq 0} \frac{-ic_n}{(n_1 + n_2)|n|^{2s}} e^{in \cdot x}$$

converges spherically to a function $L(x)$ which has at x_0 a generalized symmetric derivative of order r with value s_0 .

We are able to extend Theorem 1 to include some, but not all, fractional orders of Bochner-Riesz summability. Let β be a non-negative real number. We denote by $[\beta]$ the largest integer $\leq \beta$ and by $\langle \beta \rangle$ the fractional part of β , $\langle \beta \rangle = \beta - [\beta]$.

THEOREM 2. Let β be a nonnegative real number with $\langle \beta \rangle < 1/2$. Suppose the series (3.2) is summable Bochner-Riesz- β to finite sum s_0 . Let $r = 2s + 1$ be an odd integer with $r \geq [\beta] + 1$. Suppose the coefficients of the series (3.2) satisfy formula (3.3) for some $\varepsilon > 0$.

Then the conclusion of Theorem 1 still holds.

In particular, in the two dimensional case, Bochner-Riesz summability of order β , for $\beta < 1/2$, is enough to imply summability R_1 (which is Lebesgue summability).

4. Although Theorem 1 is a special case of Theorem 2, we give its proof separately, since its proof is much easier than that of Theorem 2. We will assume, as we may, that $c_0 = 0$, $x_0 = 0$, and $s_0 = 0$. We set

$$S_R = S_R(0) = \sum_{|n| < R} c_n ,$$

and for $\eta > 0$

$$(4.1) \quad S_R^\eta = \frac{1}{\Gamma(\eta)} \int_0^R (R-u)^{\eta-1} S_u du .$$

Note that S_R^η , as a function of R , is the fractional integral of order η of $f(R) = S_R$, see [6].

Hardy, see [2], has shown that a series $\sum c_n$ is Bochner-Riesz- η summable to 0 if and only if

$$\sum_{|n| < R} c_n \left(1 - \frac{|n|}{R}\right)^\eta \rightarrow 0$$

as $R \rightarrow \infty$. Thus, for the proof of Theorem 1 we may assume

$$(4.2) \quad S_R^m = o(R^m)$$

as $R \rightarrow \infty$.

We will need the following lemmas. The first lemma has been adapted from [7].

LEMMA 1. Suppose $\sum_{n \in \mathbb{Z}} c_n e^{in \cdot x}$ is Bochner-Riesz- $(m+1)$ summable to 0 at $x = 0$, and suppose the coefficients c_n satisfy condition (3.3) of Theorem 1, with $r \geq m+1$. Then

$$(4.3) \quad S_R^k = o(R^{r+1/2}) ,$$

as $R \rightarrow \infty$, for $k = 0, 1, \dots, m+1$.

Proof. We first note that for $n_1 + n_2 \neq 0$,

$$\begin{aligned} & \sum_{n_1 + n_2 \neq 0} (n_1 + n_2)^{-2} |n|^{-2r+3+\varepsilon} |c_n|^2 \\ & \geq \frac{1}{4} \sum_{n_1 + n_2 \neq 0} |n|^{-2} |n|^{-2r+3+\varepsilon} |c_n|^2 \\ & = \frac{1}{4} \sum_{n_1 + n_2 \neq 0} |n|^{-2r+1+\varepsilon} |c_n|^2 . \end{aligned}$$

Thus, from (3.3),

$$\sum_{n_1+n_2 \neq 0} |n|^{-2r+1+\varepsilon} |c_n|^2 < \infty,$$

and therefore

$$\sum_{n \in \mathbb{Z}_2} |n|^{-2r+1+\varepsilon} |c_n|^2 < \infty.$$

Using Schwartz's inequality,

$$\begin{aligned} \sum_{|n| < R} |c_n| &= \sum_{|n| < R} (|n|^{1/2(-2r+1+\varepsilon)} |c_n|)(|n|^{-1/2(-2r+1+\varepsilon)}) \\ (4.4) \quad &\leq \left(\sum_{n \in \mathbb{Z}_2} |n|^{-2r+1+\varepsilon} |c_n|^2 \right)^{1/2} \left(\sum_{|n| < R} |n|^{2r-1-\varepsilon} \right)^{1/2} \\ &= C \cdot (R^{2r+1-\varepsilon})^{1/2} \\ &= o(R^{r+1/2}) \end{aligned}$$

as $R \rightarrow \infty$.

Now fix an integer j .

$$\begin{aligned} \sum_{|i| < R} c_i(R - |i| + j)^{m+1} &= \sum_{|i| < R+j} c_i(R - |i| + j)^{m+1} \\ &\quad - \sum_{R \leq |i| < R+j} c_i(R - |i| + j)^{m+1}. \end{aligned}$$

Since $\sum c_n e^{i n \cdot x}$ is Bochner-Riesz- $(m+1)$ summable to 0 at 0,

$$\sum_{|i| < R+j} c_i(R - |i| + j)^{m+1} = o(R^{m+1})$$

as $R \rightarrow \infty$.

$$\sum_{R \leq |i| < R+j} c_i(R - |i| + j)^{m+1} = o(R^{r+1/2}),$$

because of (4.4). Thus,

$$\begin{aligned} \sum_{|i| < R} c_i(R - |i| + j)^{m+1} &= o(R^{m+1}) + o(R^{r+1/2}) \\ (4.5) \quad &= o(R^{r+1/2}), \end{aligned}$$

as $R \rightarrow \infty$.

We next use the fact, see [7], that there are number C_{jk} , for $j = 1, \dots, m+2$, $k = 0, \dots, m+1$ such that for all complex numbers z ,

$$\sum_{j=1}^{m+2} C_{jk}(z + j)^{m+1} = z^k.$$

Thus, for $0 \leq k \leq m+1$,

$$\begin{aligned} S_R^k &= \frac{1}{k!} \sum_{|i| < R} c_i(R - |i|)^k \\ &= \frac{1}{k!} \sum_{|i| < R} c_i \sum_{j=1}^{m+2} C_{jk}(R - |i| + j)^{m+1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{m+2} \frac{1}{k!} C_{jk} \sum_{|i| < R} c_i (R - |i| + j)^{m+1} \\
&= \sum_{j=1}^{m+2} \frac{1}{k!} C_{jk} o(R^{r+1/2}) \\
&= o(R^{r+1/2}),
\end{aligned}$$

by (4.5). This proves Lemma 1.

LEMMA 2. Let $x = (x_1, x_2) = te^{i\theta} \in E_2$ and $n = (n_1, n_2) \in \mathbb{Z}_2$, with $|n| \neq 0$. Define

$$(4.6) \quad g_n(x) = \begin{cases} \frac{1}{2}(x_1 + x_2)e^{in \cdot x} & \text{if } n_1 + n_2 = 0 \\ -ie^{in \cdot x} / (n_1 + n_2) & \text{if } n_1 + n_2 \neq 0. \end{cases}$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta})(\cos \theta + \sin \theta) d\theta = \frac{J_1(|n|t)}{|n|},$$

where $J_1(z)$ is the Bessel function of the first kind of order 1.

Proof. This is the lemma from [5].

5. Proof of Theorem 1. Let

$$T_R(x) = \sum_{\substack{|n| < R \\ n_1 + n_2 = 0}} \frac{1}{2} (x_1 + x_2) \frac{c_n}{|n|^{2s}} e^{in \cdot x} + \sum_{\substack{|n| < R \\ n_1 + n_2 \neq 0}} \frac{-ic_n}{(n_1 + n_2)|n|^{2s}} e^{in \cdot x}.$$

The hypothesis (3.3) insures that

$$L(x) = \lim_{R \rightarrow \infty} T_R(x)$$

exists a.e. on each circle $|x| = t$, see [3], Theorem 1. Also, by Theorem 2 of [3],

$$\int_0^{2\pi} \sup_R |T_R(te^{i\theta})| d\theta < \infty,$$

so, using Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta \\
&= \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} T_R(te^{i\theta})(\cos \theta + \sin \theta) d\theta \\
&= \lim_{R \rightarrow \infty} \sum_{|n| < R} \frac{c_n}{|n|^{2s}} \frac{1}{2\pi} \int_0^{2\pi} g_n(te^{i\theta})(\cos \theta + \sin \theta) d\theta
\end{aligned}$$

where $g_n(x)$ is defined by (4.6). Using Lemma 2 we get

$$\begin{aligned}
 (5.1) \quad & \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta \\
 &= \lim_{R \rightarrow \infty} \sum_{|n| < R} \frac{c_n}{|n|^{2s}} \frac{J_1(|n|t)}{|n|} \\
 &= \lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \frac{J_1(|n|t)}{|n|^r} \\
 &= t^r \lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \gamma(|n|t),
 \end{aligned}$$

where $\gamma(t) = z^{-r} J_1(z)$.

We express the last sum as an integral and integrate by parts $m + 1$ times.

$$\begin{aligned}
 (5.2) \quad & \sum_{|n| < R} c_n \gamma(|n|t) = S_R \gamma(Rt) - \int_0^R S_u \frac{d}{du} \gamma(ut) du \\
 &= S_R \gamma(Rt) - S_R^1 \frac{d}{dR} \gamma(Rt) + \int_0^R S_u^1 \frac{d^2}{du^2} \gamma(ut) du \\
 &\quad \vdots \\
 &= S_R \gamma(Rt) - S_R^1 \frac{d}{dR} \gamma(Rt) + \dots + (-1)^m S_R^m \frac{d^m}{dR^m} \gamma(Rt) \\
 &\quad + (-1)^{m+1} \int_0^R S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du.
 \end{aligned}$$

From Lemma 1,

$$S_R^k = o(R^{r+1/2}) \quad \text{for } k = 0, \dots, m.$$

Repeatedly using the relations from [1],

$$(5.3) \quad \frac{d}{dz} (z^{-n} J_n(z)) = z^{-n} J_{n+1}(z)$$

and

$$J_n(z) = o(z^{-1/2}),$$

as $z \rightarrow \infty$, we get

$$(5.4) \quad \frac{d^k}{dz^k} \gamma(z) = o(z^{-r-1/2})$$

as $z \rightarrow \infty$. So, for $k = 0, \dots, m$

$$\begin{aligned}
 (5.5) \quad & S_R^k \frac{d^k}{dR^k} \gamma(Rt) = o(R^{r+1/2}) o(R^{-r-1/2}) \\
 &= o(1),
 \end{aligned}$$

as $R \rightarrow \infty$. Thus, returning to (5.2),

$$\lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \gamma(|n|t) = (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du ,$$

and (5.1) becomes,

$$\begin{aligned} (5.6) \quad & \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta \\ &= t^r \lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \gamma(|n|t) \\ &= t^r (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du . \end{aligned}$$

Now we make use of the series expansion for $J_1(z)$, [1], p. 4.

$$\begin{aligned} (5.7) \quad J_1(z) &= \sum_{k=0}^\infty \frac{(-1)^k (\frac{1}{2}z)^{1+2k}}{k!(k+1)!} \\ &= a_1 z + a_3 z^3 + \dots . \end{aligned}$$

Then,

$$\begin{aligned} \gamma(z) &= z^{-r} J_1(z) \\ &= z^{-r} (a_1 z + a_3 z^3 + \dots + a_{r-2} z^{r-2} + a_r z^r + \dots) . \end{aligned}$$

We define a polynomial $P(z)$ as follows. If $r = 1$, let $P(z) \equiv 0$. Otherwise, let

$$P(z) = a_1 z + a_3 z^3 + \dots + a_{r-2} z^{r-2}$$

where the a_i 's are given by (5.7). Now we let

$$(5.8) \quad \lambda(z) = \gamma(z) - z^{-r} P(z) .$$

Then $\lambda(z)$ is an entire function in the plane and

$$\gamma(z) = z^{-r} P(z) + \lambda(z) .$$

Returning to (5.6),

$$\begin{aligned} (5.9) \quad & \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta \\ &= t^r (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du \\ &= t^r (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \{(ut)^{-r} P(ut) + \lambda(ut)\} du \\ &= t^r (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \{(ut)^{-r} P(ut)\} du \\ &\quad + t^r (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut) du \\ &= A + t^r B(t) . \end{aligned}$$

Since $c_0 = 0$, therefore $S_u^m = 0$ for $0 \leq u < 1$. Thus we may replace the interval of integration of the integral involving A by the interval $(1/2, \infty)$.

$$\begin{aligned}
 A &= t^r (-1)^{m+1} \int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}} \{(ut)^{-r} P(ut)\} du \\
 &= t^r (-1)^{m+1} \int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}} \left(\sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} a_k (ut)^{k-r} \right) du \\
 &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} t^{r+k-r} a_k (-1)^{m+1} \int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}} u^{k-r} du \\
 &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} t^k a_k (-1)^{m+1} \int_{1/2}^{\infty} o(u^m) O(u^{k-r-m-1}) du \\
 &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} t^k a_k (-1)^{m+1} \int_{1/2}^{\infty} o(u^{k-r-1}) du \\
 &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} b_k t^k .
 \end{aligned}$$

Returning to (5.9),

$$\begin{aligned}
 &\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta}) (\cos \theta + \sin \theta) d\theta \\
 &= A + t^r B(t) \\
 &= b_1 t + b_3 t^3 + \dots + b_{r-2} t^{r-2} + 0 \cdot t^r + t^r B(t) .
 \end{aligned}$$

The proof of Theorem 1 will be complete when we establish $B(t) \rightarrow 0$ as $t \rightarrow 0$.

$$\begin{aligned}
 (5.10) \quad B(t) &= (-1)^{m+1} \int_0^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut) du \\
 &= \int_0^{1/t} + \int_{1/t}^{\infty} \\
 &= B_1(t) + B_2(t) .
 \end{aligned}$$

To estimate $B_1(t)$ we use the fact that $\lambda(z)$ is entire, so for $|z| \leq 1$,

$$\left| \frac{d^k}{dz^k} \lambda(z) \right| < K .$$

Since $|ut| \leq 1$ in the interval of integration involving $B_1(t)$,

$$\left| \frac{d^{m+1}}{du^{m+1}} \lambda(ut) \right| \leq t^{m+1} K$$

in this interval.

$$\begin{aligned}
B_1(t) &= (-1)^{m+1} \int_0^{1/t} o(u^m) t^{m+1} K du \\
&= o(t^{m+1}) \int_0^{1/t} u^m du \\
&= o(t^{m+1}) \left(\frac{1}{t} \right)^{m+1} \\
&= o(1)
\end{aligned}$$

as $t \rightarrow 0$.

For the estimate of $B_2(t)$ we use the decomposition

$$\lambda(z) = \gamma(z) - z^{-r} P(z).$$

Clearly, as $z \rightarrow \infty$

$$\frac{d^{m+1}}{dz^{m+1}} z^{-r} P(z) = O(z^{-m-3}),$$

and by (5.4),

$$\frac{d^{m+1}}{dz^{m+1}} \gamma(z) = O(z^{-r-1/2}).$$

Thus, for $z \rightarrow \infty$

$$(5.11) \quad \frac{d^{m+1}}{dz^{m+1}} \lambda(z) = O(z^{-r-1/2}),$$

and

$$\begin{aligned}
B_2(t) &= (-1)^{m+1} \int_{1/t}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut) du \\
&= (-1)^{m+1} \int_{1/t}^{\infty} o(u^m) t^{m+1} O(ut)^{-r-1/2} du \\
&= o(t^{m+1-r-1/2}) \int_{1/t}^{\infty} o(u)^{m-r-1/2} du \\
&= o(t^{m-r+1/2}) o\left(\frac{1}{t}\right)^{m-r+1/2} \\
&= o(1).
\end{aligned}$$

(Note we needed $m - r - 1/2 < -1$ to perform the last integration.) Thus $B_2(t) \rightarrow 0$ as $t \rightarrow 0$, and returning to (5.10), the proof of Theorem 1 is complete.

6. Proof of Theorem 2. We may assume that the fractional part of β is not zero. Otherwise Theorem 2 reduces to Theorem 1. Write $\beta = m + \alpha$, where m is an integer and $0 < \alpha < 1/2$.

We again assume $c_o = 0$, $x_o = 0$, $s_o = 0$. We proceed as in the beginning of the proof of Theorem 1.

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta \\ & = t^r \lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \gamma(|n|t), \end{aligned}$$

with $\gamma(z) = z^{-r} J_1(z)$.

As in the proof of Theorem 1 we integrate the last sum by parts. We now integrate by parts $m + 2$ times.

$$\begin{aligned} \sum_{|n| < R} c_n \gamma(|n|t) &= S_R \gamma(Rt) - \int_0^R S_u \frac{d}{du} \gamma(ut) du \\ &\vdots \\ (6.1) \quad &= S_R \gamma(Rt) - S_R^1 \frac{d}{dR} \gamma(Rt) + \cdots + (-1)^{m+1} S_R^{m+1} \frac{d^{m+1}}{dR^{m+1}} \gamma(Rt) \\ &\quad + (-1)^{m+2} \int_0^R S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du. \end{aligned}$$

We are now assuming the series (3.1) is summable Bochner-Riesz- β to 0 at $x_0 = 0$, so it is also summable Bochner-Riesz- $(m + 1)$ to 0 at $x_0 = 0$. Therefore we may again apply Lemma 1. For $0 \leq k \leq m + 1$,

$$\begin{aligned} S_R^k \frac{d^k}{dR^k} \gamma(Rt) &= o(R^{r+1/2}) O(R^{-r-1/2}) \\ &= o(1), \end{aligned}$$

as $R \rightarrow \infty$, so

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta \\ (6.2) \quad & = t^r \lim_{R \rightarrow \infty} \sum_{|n| < R} c_n \gamma(|n|t) \\ & = t^r (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du. \end{aligned}$$

We define $P(z)$ and $\lambda(z)$ as in the proof of Theorem 1:

$$P(z) = \begin{cases} 0 & \text{if } r = 1 \\ a_1 z + a_3 z^3 + \cdots + a_{r-2} z^{r-2} & \text{if } r \neq 1 \end{cases}$$

and

$$\lambda(z) = \gamma(z) - z^{-r} P(z).$$

Then (6.2) becomes,

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta \\ & = t^r (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} [(ut)^{-r} P(ut) + \lambda(ut)] du \end{aligned}$$

$$\begin{aligned}
&= t^r(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} [(ut)^{-r} P(ut)] du \\
&\quad + t^r(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\
&= A(t) + t^r B(t) .
\end{aligned}$$

$$\begin{aligned}
A &= t^r(-1)^m \int_{1/2}^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \left(\sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} a_k (ut)^{k-r} \right) du \\
&= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} t^{r+k-r} a_k (-1)^m \int_{1/2}^\infty o(u)^{m+1} \frac{d^{m+2}}{du^{m+2}} u^{k-r} du \\
&= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} b_k t^k .
\end{aligned}$$

Hence,

$$(6.3) \quad \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta = \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} b_k t^k + t^r B(t)$$

where

$$(6.4) \quad B(t) = (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du .$$

To complete the proof of Theorem 2 we must show $B(t) \rightarrow 0$ as $t \rightarrow 0$.

If $f(u)$ is a function defined for $u > 0$ and η is a positive real number, denote by

$$I^\eta f(z) = \frac{1}{\Gamma(\eta)} \int_0^z (z-u)^{\eta-1} f(u) du ,$$

the fractional integral of order η , see [6]. Now if we set

$$f(u) = S_u = \sum_{|n| < u} c_n ,$$

then by (4.1),

$$S_u^\eta = I^\eta S_u ,$$

so

$$\begin{aligned}
S_u^{m+1} &= I^{m+1} S_u \\
&= I^{1-\alpha} I^{m+\alpha} S_u \\
&= I^{1-\alpha} S_u^{m+\alpha} .
\end{aligned}$$

Thus,

$$\begin{aligned}
S_u^{m+1} &= \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{1-\alpha-1} S_z^{m+\alpha} dz \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{-\alpha} S_z^{m+\alpha} dz .
\end{aligned}$$

Returning to (6.4)

$$\begin{aligned}
 B(t) &= (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\
 &= \lim_{R \rightarrow \infty} (-1)^m \int_0^R \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{-\alpha} S_z^{m+\alpha} dz \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\
 &= \lim_{R \rightarrow \infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du dz \\
 &= \lim_{R \rightarrow \infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} H(z, t, R) dz,
 \end{aligned}$$

where

$$\begin{aligned}
 H(z, t, R) &= \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du. \\
 B(t) &= \lim_{R \rightarrow \infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^{1/t} S_z^{m+\alpha} H(z, t, R) dz \\
 &\quad + \lim_{R \rightarrow \infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^R S_z^{m+\alpha} H(z, t, R) dz \\
 &= B_1(t) + B_2(t).
 \end{aligned}$$

We will make separate estimates of $H(z, t, R)$ for $B_1(t)$ and for $B_2(t)$.

First, in the interval of integration involving $B_1(t)$, $0 \leq z \leq 1/t$.

$$\begin{aligned}
 H(z, t, R) &= \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\
 (6.5) \quad &= \int_z^{1/t} + \int_{1/t}^R \\
 &= H_1 + H_2.
 \end{aligned}$$

Using the fact that λ is entire,

$$\begin{aligned}
 |H_1| &\leq \int_z^{1/t} (z-u)^{-\alpha} t^{m+2} \cdot K du \\
 &\leq K t^{m+2} \int_z^{1/t} (z-u)^{-\alpha} du \\
 &= O(t^{m+2}) \left(\frac{1}{t} - z \right)^{1-\alpha}.
 \end{aligned}$$

We estimate H_2 by employing (5.11)

$$\begin{aligned}
 H_2 &= \int_{1/t}^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\
 &= \int_{1/t}^\infty (u-z)^{-\alpha} t^{m+2} O(ut)^{-r-1/2} du
 \end{aligned}$$

$$\begin{aligned}
&= O(t^{m-r+3/2})\left(\frac{1}{t} - z\right)^{-\alpha} \int_{1/t}^{\infty} u^{-r-1/2} du \\
&= O(t^{m-r+3/2})\left(\frac{1}{t} - z\right)^{-\alpha} \left(\frac{1}{t}\right)^{-r+1/2} \\
&= O(t^{m+1})\left(\frac{1}{t} - z\right)^{-\alpha}.
\end{aligned}$$

Returning to (6.5),

$$H(z, t, R) = O(t^{m+2})\left(\frac{1}{t} - z\right)^{1-\alpha} + O(t^{m+1})\left(\frac{1}{t} - z\right)^{-\alpha}.$$

and

$$\begin{aligned}
B_1(t) &= \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^{1/t} S_z^{m+\alpha} H(z, t, R) dz \\
&= \int_0^{1/t} o(z^{m+\alpha}) \left\{ O(t^{m+2})\left(\frac{1}{t} - z\right)^{1-\alpha} + O(t^{m+1})\left(\frac{1}{t} - z\right)^{-\alpha} \right\} dz \\
&= o\left(\frac{1}{t}\right)^{m+\alpha} \left\{ O(t^{m+2}) \int_0^{1/t} \left(\frac{1}{t} - z\right)^{1-\alpha} dz + O(t^{m+1}) \int_0^{1/t} \left(\frac{1}{t} - z\right)^{-\alpha} dz \right\} \\
&= o\left(\frac{1}{t}\right)^{m+\alpha} \left\{ O(t^{m+2})\left(\frac{1}{t}\right)^{2-\alpha} + O(t^{m+1})\left(\frac{1}{t}\right)^{1-\alpha} \right\} \\
&= o(1),
\end{aligned}$$

as $t \rightarrow 0$.

It remains to be shown that $B_2(t) \rightarrow 0$. In the interval of integration for B_2 , $1/t \leq z \leq R$, and

$$\begin{aligned}
H(z, t, R) &= \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\
&= \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \left(\frac{-P(ut)}{(ut)^r} \right) du \\
&\quad + \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du \\
&= H_a + H_b.
\end{aligned}$$

$$\begin{aligned}
H_a &= - \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \left(\sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} a_k(ut)^{k-r} \right) du \\
&= \int_z^R (u - z)^{-\alpha} t^{m+2} O(ut)^{-m-4} du \\
&= t^{-2} \left\{ \int_z^{2z} (u - z)^{-\alpha} O(u)^{-m-4} du + \int_{2z}^{\infty} (u - z)^{-\alpha} O(u)^{-m-4} du \right\} \\
&= t^{-2} \{ O(z)^{1-\alpha} z^{-m-4} + O(z^{-\alpha}) z^{-m-3} \} \\
&= t^{-2} O(z^{-m-\alpha-3}).
\end{aligned}$$

We change variables in the interval for H_b by letting $x = ut$.

$$\begin{aligned}
 H_b(z, t, R) &= \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du \\
 &= \int_{tz}^{tR} \left(\frac{x}{t} - z \right)^{-\alpha} t^{m+2} \frac{d^{m+2}}{dx^{m+2}} \gamma(x) \frac{dx}{t} \\
 &= t^{m+1+\alpha} \int_{tz}^{tR} (x - tz)^{-\alpha \gamma^{(m+2)}}(x) dx \\
 &= t^{m+1+\alpha} \left\{ \int_{tz}^{tz+1} + \int_{tz+1}^{tR} \right\} \\
 &= H'_b + H''_b .
 \end{aligned}$$

Recall that $1/t \leq z$, so in the interval of integration for H_b , $x > tz \geq 1$. Thus, by (5.11)

$$|\gamma^{(m+2)}(x)| \leq Cx^{-r-1/2} ,$$

and

$$\begin{aligned}
 H'_b &= t^{m+1+\alpha} \int_{tz}^{tz+1} (x - tz)^{-\alpha \gamma^{(m+2)}}(x) dx \\
 &= t^{m+1+\alpha} O(tz)^{-r-1/2} \int_{tz}^{tz+1} (x - tz)^{-\alpha} dx \\
 &= t^{m+1+\alpha} O(tz)^{-r-1/2} .
 \end{aligned}$$

We estimate H''_b by integrating by parts.

$$\begin{aligned}
 H''_b &= t^{m+1+\alpha} \int_{tz+1}^{tR} (x - tz)^{-\alpha \gamma^{(m+2)}}(x) dx \\
 &= t^{m+1+\alpha} (x - tz)^{-\alpha \gamma^{(m+1)}}(x) \Big|_{tz+1}^{tR} \\
 &\quad + t^{m+1+\alpha} \alpha \int_{tz+1}^{tR} (x - tz)^{-\alpha-1 \gamma^{(m+1)}}(x) dx \\
 &= t^{m+1+\alpha} (x - tz)^{-\alpha \gamma^{(m+1)}}(x) \Big|_{tz+1}^{tR} \\
 &\quad + t^{m+1+\alpha} O(tz)^{-r-1/2} \int_{tz+1}^{tR} (x - tz)^{-\alpha-1} dx \\
 &= t^{m+1+\alpha} (tR - tz)^{-\alpha \gamma^{(m+1)}}(tR) - t^{m+1+\alpha \gamma^{(m+1)}}(tz + 1) \\
 &\quad + t^{m+1+\alpha} O(tz)^{-r-1/2} \left(\frac{1}{-\alpha} \right) \{ (tR - tz)^{-\alpha} - 1 \} \\
 &= t^{m+1+\alpha} (tR - tz)^{-\alpha} O(tz)^{-r-1/2} + t^{m+1+\alpha} O(tz)^{-r-1/2} \\
 &= t^{m+1+\alpha} O(tz)^{-r-1/2} .
 \end{aligned}$$

Hence, in the interval of integration for B_2 ,

$$\begin{aligned}
 H_b(z, t, R) &= H'_b + H''_b \\
 &= t^{m+1+\alpha} O(tz)^{-r-1/2} ,
 \end{aligned}$$

and

$$\begin{aligned} H(z, t, R) &= H_a + H_b \\ &= t^{-2}O(z^{-m-\alpha-3}) + t^{m+1+\alpha}O(tz)^{-r-1/2}. \end{aligned}$$

So,

$$\begin{aligned} B_2(t) &= \lim_{R \rightarrow \infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^R S_z^{m+\alpha} H(z, t, R) dz \\ &= \lim_{R \rightarrow \infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^R o(z)^{m+\alpha} \{t^{-2}O(z^{-m-\alpha-3}) + t^{m+1+\alpha}O(tz)^{-r-1/2}\} dz \\ &= t^{-2} \int_{1/t}^{\infty} o(z^{m+\alpha-m-\alpha-3}) dz + t^{m+1+\alpha-r-1/2} \int_{1/t}^{\infty} o(z^{m+\alpha-r-1/2}) dz \\ &= t^{-2} o(z^{-2}) \Big|_{1/t}^{\infty} + t^{m+1/2+\alpha-r} o(z^{m+\alpha-r+1/2}) \Big|_{1/t}^{\infty} \\ &= o(1). \end{aligned}$$

(Note that the hypothesis $\alpha < 1/2$ is necessary here to insure that the last integral converge.) This completes the proof of Theorem 2.

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