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THE SCHUR SUBGROUP OF THE BRAUER GROUP

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Let K be a subfield of a cyclotomic extension L of the rational field Q . The Schur subgroup, $S(K)$, of the Brauer group of K , $B(K)$, consists of those algebra classes which contain an algebra which is isomorphic to a simple component of a group algebra QG for some finite group G .

In this paper we describe a set of generators for $S(K)$. We then use this theorem to determine the 2-primary part of $S(K)$ when L/K is cyclic and the fourth roots of unity are not in K .

NOTATION. In this paper K is a field contained in $Q(\epsilon_n)$ where ϵ_n is a primitive n th root of unity. The completion of K at a prime P is denoted K_P . If p is the integral prime dividing P , then the residue class degree of P over p is written $f(p) = f(p, K/Q)$. The ramification index of p in $Q(\epsilon_n)$ over K is $e(p) = e(p, Q(\epsilon_n)/K)$.

If A is a central simple algebra over K , then $[A]$ will denote the class of A in $B(K)$. A class $[A]$ in $B(K)$ is said to have uniformly distributed invariants of values 0 or $1/2$ if for each rational prime p , $[A]$ has the same Hasse invariant at each of the primes of K which divide p , and these invariants are either 0 or $1/2$. The common value of the invariant of $[A]$ at the primes of K dividing p is called the p -local invariant of $[A]$ and is denoted: $\text{inv}_p[A]$.

If L is an extension field of K , then the Galois group of L over K is denoted by $\text{Gal}(L/K)$, and the Frobenius automorphism of a prime p unramified in L over K is written $[L/K, p]$. Let α be a factor set $\text{Gal}(L/K) \times \text{Gal}(L/K)$ into L . Then the crossed product algebra made with L and α is denoted by $(L/K, \alpha)$. This is a central simple K algebra having L basis $\{u_\sigma\}$ for $\sigma \in \text{Gal}(L/K)$ with multiplication given by

$$\begin{aligned} u_\sigma u_\tau &= \alpha(\sigma, \tau) u_{\sigma\tau} \\ u_\sigma x &= \sigma(x) u_\sigma \quad \text{for } \sigma, \tau \in \text{Gal}(L/K), \quad x \in L. \end{aligned}$$

In case $\text{Gal}(L/K) = \langle \sigma \rangle$ is cyclic, we shall write (L, σ, a) for the crossed product in which

$$\begin{aligned} (u_\sigma)^i &= u_{\sigma^i} \quad 1 \leq i < |\sigma| \\ &= a \quad i = |\sigma|. \end{aligned}$$

If p is a rational prime which splits into an even number of primes in K over Q , then $\Omega(p)$ denotes the class of $B(K)$ with invariant $1/2$ at each of the primes of K dividing p and invariant

0 elsewhere. If p_1 and p_2 are rational primes which split into an odd number of primes in K over Q , then $\mathcal{Q}(p_1, p_2)$ denotes the class in $B(K)$ with invariant $1/2$ at each of the primes of K dividing $p_1 p_2$ and invariant 0 elsewhere.

Finally $|m|_2$ denotes the highest power of 2 which divides the integer m , and $t(q) = q^{f(q)} - 1$ for all rational primes q .

2. The generator theorem. In this section we give a set of generators for $S(K)$. This is a useful refinement of a result by Janusz [6].

LEMMA 1. *Let K be a field contained in $Q(\varepsilon_n)$ where n is odd. Suppose that $\text{Gal}(Q(\varepsilon_n)/K) = \prod_{i=1}^r \langle \phi_i \rangle$ and that $\text{Gal}(Q(\varepsilon_{i_n})/Q(\varepsilon_n)) = \langle \rho \rangle$. If $[Q(\varepsilon_n)/K, 2] = \prod \phi_i^{x_i}$, then the 2-local index of an algebra $(Q(\varepsilon_{i_n})/K, \alpha)$ is equal to 2 if and only if $\sum g_i x_i + zf(2)$ is odd where $u_\rho u_{\phi_i} = \varepsilon_i^{x_i} u_{\phi_i} u_\rho$ and $u_\rho^2 = \varepsilon_i^{2z}$.*

Proof. Set $\eta = [Q(\varepsilon_n)/K, 2]$ and suppose that η has order s . Then $u_\eta u_\rho = \varepsilon_i^z u_\rho u_\eta$ where

$$\lambda = \sum_{i=1}^r g_i x_i.$$

If λ is even we have

$$u_\rho(\varepsilon_i^{\lambda/2} u_\eta) = \varepsilon_i^{-\lambda/2} \varepsilon_i^z u_\eta u_\rho = (\varepsilon_i^{\lambda/2} u_\eta) u_\rho.$$

Let π be a prime of K dividing 2, then

$$\begin{aligned} K_{\pi \otimes K}(Q(\varepsilon_{i_n})/K, \alpha) &= \sum_{i=0}^1 \sum_{j=0}^{s-1} Q_2(\varepsilon_{i_n}) u_\rho^i u_\eta^j \\ &= \sum_{i=0}^1 \sum_{j=0}^{s-1} K_\pi(\varepsilon_i) Q_2(\varepsilon_n) u_\rho^i (\varepsilon_i^{\lambda/2} u_\eta)^j \\ &= \left(\sum_{i=0}^1 K_\pi(\varepsilon_i) u_\rho^i \right) \left(\sum_{j=0}^{s-1} Q_2(\varepsilon_n) (\varepsilon_i^{\lambda/2} u_\eta)^j \right) \\ &= (K_\pi(\varepsilon_\eta), \rho, u_\rho^2) \otimes_{K_\pi} (Q_2(\varepsilon_n), \eta, (\varepsilon_i^{\lambda/2} u_\eta)^s). \end{aligned}$$

Now $(\varepsilon_i^{\lambda/2} u_\eta)^s$ is a root of unity and $Q_2(\varepsilon_n)$ is unramified over K_π , hence by [1, Chap. V, Thm. 9.14] $(Q_2(\varepsilon_n), \eta, (\varepsilon_i^{\lambda/2} u_\eta)^s)$ has index 1. Further

$$[(K_\pi(\varepsilon_i), \rho, \varepsilon_i^{2z})] = [K_\pi \otimes_{Q_2} (Q_2(\varepsilon_i), \rho, \varepsilon_i^{2z})]$$

and $(Q_2(\varepsilon_i), \rho, \varepsilon_i^{2z})$ has index 2 if and only if z is odd, since -1 is not a norm from $Q_2(\varepsilon_i)$. Thus $K_\pi \otimes_K (Q(\varepsilon_{i_n})/K, \alpha)$ has index 2 if and only if $f(2)z$ is odd in the case that λ is even.

Now suppose that λ is odd. We have that

$$u_\rho((1 + \varepsilon_4^\lambda)u_\eta) = (1 + \varepsilon_4^{-\lambda})\varepsilon_4^\lambda u_\eta u_\rho = ((1 + \varepsilon_4^\lambda)u_\eta)u_\rho.$$

Hence

$$[K_\pi \otimes_K (Q(\varepsilon_{4n})/K, \alpha)] = [(K_\pi(\varepsilon_4), \rho, u_\rho^2) \otimes_{K_\pi} (Q(\varepsilon_n), \eta, ((1 + \varepsilon_4^\lambda)u_\eta)^s)]$$

by the same reasoning used above. We have already seen that $(K_\pi(\varepsilon_4), \rho, u_\rho^2)$ has index 2 if and only if $f(2)z$ is odd; we must look at $(Q_2(\varepsilon_n), \eta, ((1 + \varepsilon_4^\lambda)u_\eta)^s)$.

Let V_L denote the exponential valuation in the 2-adic field L . Then

$$\begin{aligned} V_{K_\pi}((1 + \varepsilon_4^\lambda)u_\eta)^s &= \frac{1}{2} V_{K_\pi(\varepsilon_4)}((1 + \varepsilon_4^\lambda)u_\eta)^s \\ &= \frac{1}{2} V_{K_\pi(\varepsilon_4)}(1 + \varepsilon_4^\lambda)^s + \frac{1}{2} V_{K_\pi(\varepsilon_4)}(u_\eta^s) \\ &= \frac{s}{2} V_{K_\pi(\varepsilon_4)}(1 + \varepsilon_4^\lambda) \end{aligned}$$

since u_η^s is a unit in $K_\pi(\varepsilon_4)$. Further, $(1 + \varepsilon_4^\lambda)$ is a prime element in $K_\pi(\varepsilon_4)$ since λ is odd. Thus $V_{K_\pi(\varepsilon_4)}(1 + \varepsilon_4^\lambda) = 1$ and

$$V_{K_\pi}((1 + \varepsilon_4^\lambda)u_\eta)^s = s/2.$$

Hence, by the definition of the Hasse invariant,

$$\begin{aligned} \text{inv}(Q_2(\varepsilon_n), \eta, ((1 + \varepsilon_4^\lambda)u_\eta)^s) &= \frac{s/2}{s} \bmod \mathbb{Z} \\ &= \frac{1}{2} \bmod \mathbb{Z}. \end{aligned}$$

Therefore, if λ is odd, we have that the index of $K_\pi \otimes_K (Q(\varepsilon_{4n})/K, \alpha)$ is 2 if and only if $f(2)z$ is even.

This completes the proof of the lemma.

We will let $S(K)_p$ denote the p -primary part of $S(K)$, and $W(K, p)$ denote the roots of unity in K with p -power order.

THEOREM 1. *Let p be a rational prime. Then $S(K)_p$ is generated by algebra classes which contain an algebra of the form $(Q(\varepsilon_{nq})/K, \alpha)$ where the values of α are in $W(Q(\varepsilon_{nq}), p)$, q is either 4 or an odd prime, and q does not divide n .*

Proof. This is a refinement of Theorem 3 of [6]. In that theorem Janusz showed the following:

1. If p is odd, or $p = 2$ and 4 divides n , then $S(K)_p$ is generated by classes which contain algebras of the following types:

(a) $(Q(\varepsilon_{nq})/K, \alpha)$, the values of α in $W(Q(\varepsilon_n), p)$ and q a prime

not dividing n .

(b) $(K(\varepsilon_{qr})/K, \beta)$, the values of β in $W(K, p)$ and q and r distinct primes not dividing n .

2. If $p = 2$ and n is odd, then $S(K)_p$ is generated by classes which contain an algebra of type (b), or of type (a') $(Q(\varepsilon_{4nq})/K, \alpha)$, the values of α in $W(Q(\varepsilon_4), 2)$ and q an odd prime not dividing n .

In order to prove Theorem 1, we must look closely at algebras of types (b) and (a').

Let $\text{Gal}(K(\varepsilon_{qr})/K) = \langle \sigma \rangle \times \langle \tau \rangle$ where $\langle \sigma \rangle = \text{Gal}(K(\varepsilon_q)/K)$ and $\langle \tau \rangle = \text{Gal}(K(\varepsilon_r)/K)$. Also let ζ be a p^d th root of unity, the highest p -power root of unity in K . Consider the algebra

$$\Delta_{qr} = (K(\varepsilon_{qr})/K, \beta) = \sum K(\varepsilon_{qr})u_\gamma \quad (\gamma \in \langle \sigma \rangle \times \langle \tau \rangle)$$

where $u_\sigma u_\tau = \zeta^x u_\sigma u_\tau$, $u_\sigma^{q-1} = \zeta^y$, and $u_\tau^{r-1} = \zeta^z$. By [8, §1], the only restrictions on x , y , and z are $(\zeta^x)^{q-1} = (\zeta^x)^{N(\tau)}$ and $(\zeta^y)^{r-1} = (\zeta^{-x})^{N(\sigma)}$ where $N(\phi) = 1 + \phi^2 + \cdots + \phi^{|\phi|-1}$. However both σ and τ fix ζ , so we get that p^d divides both $x(r-1)$ and $x(q-1)$.

Now Δ_{qr} can have nonzero invariant only at the primes of K which divide q and r . This is because these are the only primes ramified in $K(\varepsilon_{qr})/K$.

Suppose that q is odd. Let $\tau^q = [K(\varepsilon_r)/K, q]$, the Frobenius automorphism of q in $K(\varepsilon_r)/K$, and set $t = q^{f(q)} - 1$. We have that

$$\left(\frac{\beta(\sigma, \tau^q)}{\beta(\tau^q, \sigma)} \right)^{(q-1)/t} u_\sigma^{q-1} = (\varepsilon_t)^\mu$$

where $\mu = (q-1)/p^d$ and $\nu = xg + y(t/(q-1))$.

The inertia group of q in $K(\varepsilon_{qr})/K$ is $\langle \sigma \rangle$, so [7, Thm 3] implies that the q -local index of Δ_{qr} is $\max\{p^{d-s}, 1\}$ where p^s exactly divides ν .

Now suppose that p^a exactly divides $f(q)$. Then p^a divides g since $[K(\varepsilon_r)/K, q] = [K(\varepsilon_r)/Q, q]^{f(q)}$. Moreover, if $p = 2$, $f(q)$ is even, and $q \equiv 3 \pmod{4}$, then 2^{a+1} exactly divides $t/(q-1)$, otherwise p^a exactly divides $t/(q-1)$. In the case where $p = 2$, $f(q)$ is even and $q \equiv 3 \pmod{4}$, we either have $2^d > 2$ so that x is even, or $2^d = 2$ so that Δ_{qr} has q -local index 1.

Hence in all cases, $\max\{p^{d-s}, 1\}$ takes its highest possible value when p^s exactly divides $t/(q-1)$.

Now consider the algebra $(K(\varepsilon_q), \sigma, \zeta)$. Applying [7, Thm. 3] we see that the q -local index is $\max\{p^{d-e}, 1\}$ where p^e exactly divides $t/(q-1)$. Further, the local index of $(K(\varepsilon_q), \sigma, \zeta)$ at any prime unequal to q is 1. Note that $(K(\varepsilon_q), \sigma, \zeta)$ inflated to $Q(\varepsilon_{nq})/K$ has the form described in Theorem 1.

If r is even, then $K(\varepsilon_{qr}) = K(\varepsilon_q)$ so that the r -local index of Δ_{qr}

is 1. Thus, in this case, some power of $(K(\varepsilon_q), \sigma, \zeta)$ has exactly the same set of invariants as Δ_{qr} .

If r is odd, then we may replace q by r in the above argument. Hence, some power of $(K(\varepsilon_r), \tau, \zeta)$ has the same invariants at primes dividing r as Δ_{qr} does, and some power of $(K(\varepsilon_q), \sigma, \zeta)$ has the same invariants as Δ_{qr} at primes dividing q .

Thus $[\Delta_{qr}]$ is contained in the group generated by the classes described in the theorem.

Now suppose that $p = 2$ and n is odd. Let $G = \text{Gal}(Q(\varepsilon_n)/K)$ be given by the direct product

$$G = \langle \phi_1 \rangle \times \langle \phi_2 \rangle \times \cdots \times \langle \phi_k \rangle$$

where $\langle \phi_i \rangle$ has order n_i . Further, set $\langle \rho \rangle = \text{Gal}(Q(\varepsilon_{4n})/Q(\varepsilon_n))$ and $\langle \sigma \rangle = \text{Gal}(Q(\varepsilon_{nq})/Q(\varepsilon_n))$, where q is an odd prime not dividing n . Let ζ be a primitive fourth root of unity.

Consider the algebra

$$\Delta_{2q} = (Q(\varepsilon_{4nq})/K, \alpha) = \sum Q(\varepsilon_{4nq})u_\tau$$

where

$$\begin{aligned} u_\rho u_\sigma &= \zeta^{x_0} u_\sigma u_\rho, & u_\rho u_{\phi_i} &= \zeta^{x_i} u_{\phi_i} u_\rho, \\ u_\sigma u_{\phi_i} &= \zeta^{y_i} u_{\phi_i} u_\sigma, & u_{\phi_i} u_{\phi_j} &= \zeta^{y_{ij}} u_{\phi_j} u_{\phi_i}, \\ u_\rho^2 &= \zeta^z, & u_\sigma^{q-1} &= \zeta^{z_0}, & u_{\phi_i}^{n_i} &= \zeta^{z_i}, \end{aligned}$$

for $i, j = 1, 2, \dots, k$ and $i \neq j$. The conditions in [8, §1] imply that

$$\begin{aligned} z, y_i, \text{ and } y_{ij} &\text{ are even for } i, j = 1, 2, \dots, k \text{ and } i \neq j, \\ 2z_0 &\equiv x_0(q-1) \pmod{4}, \\ 2z_i &\equiv x_i n_i \pmod{4} \text{ for } i = 1, 2, \dots, k. \end{aligned}$$

We have that Δ_{2q} can have nonzero invariants only at those primes of K which divide 2, q , or some prime which ramifies in $Q(\varepsilon_n)/K$. Moreover, the invariants of Δ_{2q} can only be 0 or 1/2 since the only 2-power roots of unity in K are $\{\pm 1\}$.

Let

$$\Delta_q = (Q(\varepsilon_{nq})/K, \gamma) = \sum Q(\varepsilon_{nq})v_\tau$$

be the algebra such that

$$\begin{aligned} v_\sigma v_{\phi_i} &= \zeta^{y_i} v_{\phi_i} v_\sigma, & v_{\phi_i} v_{\phi_j} &= v_{\phi_j} v_{\phi_i}, \\ v_\sigma^{q-1} &= \zeta^{z_0^*}, & v_{\phi_i}^{n_i} &= 1, \end{aligned}$$

for $i, j = 1, 2, \dots, k$ where

$$\begin{array}{ll}
 z_0 & \text{if } q \equiv 1 \pmod{4} \\
 z_0^* = 0 & \text{if } q \equiv 3 \pmod{4} \text{ and } f(q) \text{ is even} \\
 z_0 + x_0 r & \text{if } q \equiv 3 \pmod{4} \text{ and } f(q) \text{ is odd}
 \end{array}$$

where

$$r^{-1} \equiv \frac{q^{f(q)} - 1}{q - 1} \pmod{4}.$$

Note that the y_i are all even, and that $z_0 + x_0 r$ is even when $q \equiv 3 \pmod{4}$ and $f(q)$ is odd. Thus the values of γ are all $+1$ or -1 , and Δ_q is in $S(K)$.

Further, let

$$\Delta_2 = (Q(\varepsilon_{4n})/K, \gamma') = \sum Q(\varepsilon_{4n})w_\tau$$

be the algebra such that

$$\begin{aligned}
 w_\rho w_{\phi_i} &= \zeta^{x_i} w_{\phi_i} w_\rho, & w_{\phi_i} w_{\phi_j} &= \zeta^{y_{ij}} w_{\phi_j} w_{\phi_i}, \\
 w_\rho^2 &= \zeta^{z^*}, & w_{\phi_i}^2 &= \zeta^{z_i}
 \end{aligned}$$

for $i, j = 1, 2, \dots, k$ and $i \neq j$ where

$$\begin{aligned}
 z^* &= z + x_0 & \text{if } q \equiv 3 \text{ or } 5 \pmod{8} \text{ and } f(2) \text{ is odd} \\
 &= z & \text{otherwise.}
 \end{aligned}$$

Observe that both Δ_q and Δ_2 belong to classes of the type described in the theorem.

Claim. The algebra Δ_{2q} is equivalent to $\Delta_2 \otimes_K \Delta_q$ in $B(K)$.

Proof of Claim. We will show that Δ_{2q} and $\Delta_2 \otimes \Delta_q$ have the same set of invariants. This is the same as showing that the local indices of these algebras are the same at q , 2 , and the primes ramified in $Q(\varepsilon_n)/K$ because the invariants can be only 0 or $1/2$.

First consider the q -local indices of Δ_{2q} and $\Delta_2 \otimes \Delta_q$. Let the Frobenius automorphism for q in $Q(\varepsilon_{4n})/K$ be $\eta_q = \rho^r \prod \phi_i^{q^i}$, and set $t = q^{f(q)} - 1$. Then

$$\left(\frac{\alpha(\sigma, \eta_q)}{\alpha(\eta_q, \sigma)} \right)^{(q-1)/t} u_\sigma^{q-1} = (\varepsilon_i)^{(q-1)\nu_0/4}$$

where

$$\nu_0 = gx_0 + \mu \sum g_i y_i + z_0(t/(q-1))$$

where

$$\begin{aligned}\mu &= -1 & \text{if } g &= 1 \\ &= 1 & \text{if } g &= 0.\end{aligned}$$

By [6, Thm. 3], the q -local index of Δ_{2q} is given by

$$\frac{q-1}{(\nu_0(q-1), q-1)} = \begin{cases} 1 & \text{if } \nu_0 \equiv 0 \pmod{Z} \\ 2 & \text{if } \nu_0 \equiv 1/2 \pmod{Z}. \end{cases}$$

Now q does not ramify in $Q(\varepsilon_n)/K$, so the q -local index of $\Delta_2 \otimes \Delta_q$ is equal to the q -local index of Δ_q .

The restriction of η_q to $Q(\varepsilon_n)$ is the Frobenius automorphism of q in $Q(\varepsilon_n)/K$; we will denote this by η'_q .

We have that

$$\left(\frac{\gamma(\sigma, \eta'_q)}{\gamma(\eta'_q, \sigma)} \right)^{(q-1)/t} v_\sigma^{q-1} = (\varepsilon_t)^{(q-1)\nu'_0}$$

where

$$\nu'_0 = \frac{1}{4} [\sum g_i y_i + z_0^*(t/(q-1))].$$

Hence the q -local index of $\Delta_2 \otimes \Delta_q$ is given by

$$\frac{q-1}{(\nu'_0(q-1), q-1)} = \begin{cases} 1 & \text{if } \nu'_0 \equiv 0 \pmod{Z} \\ 2 & \text{if } \nu'_0 \equiv 1/2 \pmod{Z}. \end{cases}$$

Now if $q \equiv 1 \pmod{4}$, then $g = 0$ and $z_0^* = z_0$, so $\nu_0 = \nu'_0$ and Δ_{2q} has the same q -local index as $\Delta_2 \otimes \Delta_q$. If $q \equiv 3 \pmod{4}$ and $f(q)$ is even, then $g = 0$ and 4 divides $t/(q-1)$, so that $\nu' \equiv \nu'_0 \pmod{Z}$. Thus again Δ_{2q} and $\Delta_2 \otimes \Delta_q$ have the same q -local index. Finally suppose that $q \equiv 3 \pmod{4}$ and $f(q)$ is odd. In this case $g = 1$ so that

$$gx_0 + z_0(t/(q-1)) \equiv z_0^*(t/(q-1)) \pmod{4}.$$

Hence $\nu_0 \equiv \nu'_0 \pmod{Z}$ and Δ_{2q} has the same q -local index as $\Delta_2 \otimes \Delta_q$.

Now let l be a prime which ramifies in $Q(\varepsilon_n)/K$. We will compare the l -local indices of Δ_{2q} and $\Delta_2 \otimes \Delta_q$. Let $\langle \omega \rangle$ be the inertia group of l in $Q(\varepsilon_n)/K$ where $\omega = \prod \phi_i^{g_i}$, and let $\eta_l = \rho^\sigma \sigma^{g_0} \prod \phi_i^{g_i}$ be a Frobenius automorphism of l in $Q(\varepsilon_{4nq})/K$. Then $\eta'_l = \rho^\sigma \prod \phi_i^{g_i}$ and $\eta''_l = \sigma^{g_0} \prod \phi_i^{g_i}$ are Frobenius automorphisms of l in $Q(\varepsilon_n)/K$ and $Q(\varepsilon_{nq})/K$ respectively. Let e be the ramification index of l in $Q(\varepsilon_n)/K$. Then we have $v_\omega^e = 1$ and $w_\omega^e = u_\omega^e$. Moreover

$$\frac{\alpha(\omega, \eta''_l)}{\alpha(\eta''_l, \omega)} = \frac{\gamma(\omega, \eta'_l)}{\gamma(\eta'_l, \omega)} \frac{\gamma'(\omega, \eta'_l)}{\gamma'(\eta'_l, \omega)}.$$

Hence, by [7, Thm. 3], we see that Δ_{2q} and $\Delta_2 \otimes \Delta_q$ have the same

l -local index.

Finally, we must compare the 2-local indices of \mathcal{A}_{2q} and $\mathcal{A}_2 \otimes \mathcal{A}_q$. Let $\sigma^{g_0} \prod \phi_i^{g_i}$ be the Frobenius automorphism of 2 in $Q(\varepsilon_{nq})/K$, then Lemma 1 implies that the 2-local index of \mathcal{A}_{2q} is 2 if and only if $\nu = g_0 x_0 + \sum g_i x_i + (z/2)f(2)$ is odd. Further, the 2-local index of $\mathcal{A}_2 \otimes_K \mathcal{A}_q$, which is the 2-local index of \mathcal{A}_2 , is 2 if and only if $\nu' = \sum x_i g_i + (z^*/2)f(2)$ is odd.

If $f(2)$ is even, then g_0 is even since

$$[Q(\varepsilon_{nq})/K, 2] = [Q(\varepsilon_{nq})/Q, 2]^{f(2)}.$$

Thus $\nu \equiv \nu' \pmod{2}$ and \mathcal{A}_{2q} has the same 2-local index as $\mathcal{A}_2 \otimes \mathcal{A}_q$. If $f(2)$ is odd and $q \equiv 1$ or $7 \pmod{8}$, then 2 is a square modulo q , so that g must be even. Hence, once again $\nu \equiv \nu' \pmod{2}$ and \mathcal{A}_{2q} and $\mathcal{A}_2 \otimes \mathcal{A}_q$ have the same 2-local index. Finally suppose that $f(2)$ is odd and that $q \equiv 3$ or $5 \pmod{8}$. Then g is odd and $z^* = z_0 + x_0$, so $g x_0 + (z/2)f(2)$ is equivalent to $(z^*/2)f(2)$ modulo 2. Thus again $\nu \equiv \nu' \pmod{2}$.

This completes the proof of the claim and of the theorem.

3. $S(K)_2$ when $Q(\varepsilon_n)/K$ is cyclic. In this section we will completely characterize the classes in $S(K)_2$ by the behavior of of their invariants in the case where $\text{Gal}(L/K)$ is cyclic. Before beginning these calculations we need to prove the following lemma.

LEMMA 2. Suppose that $K \subset F$ are subfields of a cyclotomic field and that $[F:K]$ is not divisible by the rational prime p . If there are no p -power roots of unity in F which are not in K , then $S(F)_p = F \otimes_k S(K)_p$.

Proof. Clearly $S(F)_p \supseteq F \otimes_k S(K)_p$. We need to show containment in the other direction.

Let L be the smallest cyclotomic field containing F , and let $G = \text{Gal}(L/K)$ be given by

$$G = \prod_{i=1}^t \langle \phi_i \rangle \times \prod_{j=1}^s \langle \psi_j \rangle$$

where the order of each $\langle \phi_i \rangle$ is a power of p and the order of each $\langle \psi_j \rangle$, n_j , is relatively prime to p . It follows that $H = \text{Gal}(L/K)$ is given by

$$H = \prod_{i=1}^t \langle \phi_i \rangle \times \prod_{j=1}^{s'} \langle \psi'_j \rangle$$

where $\prod_{j=1}^s \langle \psi_j \rangle$ is a subgroup of $\prod_{j=1}^{s'} \langle \psi'_j \rangle$.

By Theorem 1, $S(F)_p$ is generated by classes containing algebras

of the form

$$(L(\varepsilon_q)/F, \alpha) = \sum_{\sigma} L(\varepsilon_q) U_{\sigma}$$

where q is either 4 or an odd prime and the values of α are p -power roots of unity.

Suppose that $U_{\psi_j}^{n_j} = \zeta^{z_j}$ where ζ is a primitive p^d th root of unity. The order of ψ_j is prime to p , so $\psi_j(\zeta) = \zeta$ unless ζ is not in F , in which case $S(F)_p = F \otimes_K S(K)_p = 1$. Set $\gamma = -z_j/n_j$ modulo p^d . Now replace U_{ψ_j} by $\zeta^{\gamma} U_{\psi_j}$ in $(L(\varepsilon_q)/F, \alpha)$. This gives an equivalent algebra, but now

$$(\zeta^{\gamma} U_{\psi_j})^{n_j} = \zeta^0 = 1.$$

Hence we might as well have started with $z_j = 0$ for $j = 1, 2, \dots, s$.

Now suppose that $U_{\psi_j} U_{\tau} = \zeta^{x_j} U_{\tau} U_{\psi_j}$ for some τ in $\text{Gal}(L(\varepsilon_q)/F)$, τ not in $\langle \psi_j \rangle$. Then

$$\begin{aligned} 1 = U_{\psi_j}^{n_j} &= (U_{\tau}^{-1} U_{\psi_j} U_{\tau})^{n_j} = \prod_{i=0}^{n_j-1} \psi_j^i(\zeta^{x_j}) \\ &= \zeta^{n_j x_j}. \end{aligned}$$

However n_j is prime to p , so x_j must be 0. Thus $U_{\psi_j} U_{\tau} = U_{\tau} U_{\psi_j}$ for all $\tau \in \text{Gal}(L(\varepsilon_q)/F)$. This is true for all ψ_j , $j = 1, 2, \dots, s$.

Therefore

$$[(L(\varepsilon_q)/F, \alpha)] = [(E_1/F, \alpha_1) \otimes_F (E_2/F, \alpha_2)]$$

where E_1 is the field fixed by $\prod_{i=1}^t \langle \phi_i \rangle$ and E_2 is the field fixed by $\prod_{j=1}^s \langle \psi_j \rangle$. Moreover α_1 is the trivial factor set, so $[(E_1/F, \alpha_1)] = [F]$.

Further, $[(E_2/F, \alpha_2)] = [F \otimes_K (E_2/K, \alpha'_2)]$ where α'_2 restricted to $\prod_{i=1}^t \langle \phi_i \rangle$ equals α_2 and α'_2 is trivial on $\text{Gal}(F/K)$. This makes α'_2 a factor set by the same reasoning we used to ascertain that α is equivalent to a factor set with nontrivial values only on $\prod_{i=1}^t \langle \phi_i \rangle$.

This completes the proof of the lemma.

Notice that this lemma implies that an algebra class $[A]$ in $S(F)_p$ has q_i -local index p^{a_i} for some sets of primes q_1, \dots, q_t if and only if there is an algebra class $[D]$ in $S(K)_p$ with exactly the same local indices. Hence, if we can find the possible local indices for classes in $S(F)_p$, then we have found them for classes in $S(K)_p$.

In the following theorems we assume that $[K:Q]$ is even. We may do this because $S(K)$ consists of all classes in $B(K)$ with uniformly distributed invariants of value 0 or 1/2 if $[K:Q]$ is odd. This follows from [2].

A. $S(K)_2$ when n is odd.

THEOREM 2. *Let K be a field contained in $L = Q(\varepsilon_n)$ where n*

is odd such that $\text{Gal}(L/K)$ is cyclic and $[K:Q]$ is even. Then the 2-primary part of $S(K)$ consists of those classes $[A]$ in $B(K)$ with uniformly distributed invariants of value 0 or $1/2$ which satisfy the following conditions.

(I) For a prime p which divides n , $\text{inv}_p[A] = 0$ if $e(P)$ is odd or if $[L:K]/e(P)$ is even.

(II) For any prime q , $\text{inv}_q[A] = 0$ if $f(q)$ is even and a Frobenius automorphism of q is a square in $\text{Gal}(L/K)$.

(III) Let p be a prime which divides n to which (I) does not apply. Suppose that $f(p)$ is odd and that $|(p-1)/e(p)|_2 \geq |p'-1|_2$ for every prime p' which divides n and is unequal to p . Then the invariant of $[A]$ is $1/2$ at an even number of primes in the set

$$\{p\} \cup \{\text{primes } q: (q/p) = -1 \text{ and } (q, n) = 1\}$$

where (q/p) is the Legendre symbol.

Proof. Let $G = \text{Gal}(L/K)$ be $\langle \phi \rangle$ and have order $m = 2^c c'$, $(2, c') = 1$.

Step 1. We need to determine the invariants of the generators of $S(K)_2$ given in Theorem 1.

(a) Let $\Delta_q = \Delta_q(x, y, z)$ be an algebra

$$\Delta_q = (L(\varepsilon_q)/K, \alpha) = \sum_{\tau} L(\varepsilon_q) U_{\tau}$$

where q is an odd prime not dividing n and the values of α are in $\{\pm 1\}$. Let $\langle \gamma \rangle = \text{Gal}(L(\varepsilon_q)/L)$. Then the factor set α is determined by the integers x , y , and z where

$$U_{\tau} U_{\phi} = (-1)^x U_{\phi} U_{\tau},$$

$$(U_{\tau})^{q-1} = (-1)^y,$$

$$(U_{\phi})^m = (-1)^z.$$

The restrictions given in [8, §1] reduce to:

$$x = 0 \quad \text{if } m \text{ is odd.}$$

Suppose that the Frobenius automorphism of q in L/K is ϕ^g . Set $t(q) = q^{f(q)} - 1$. Then

$$\left(\frac{\alpha(\gamma, \phi^g)}{\alpha(\phi^g, \gamma)} \right)^{(q-1)/t(q)} U_{\tau}^{q-1} = (\varepsilon_{t(q)})^{((q-1)/2)\nu}$$

where $\nu = xg + y(t(q)/(q-1))$. The inertia group of q in $L(\varepsilon_q)/K$ is $\langle \gamma \rangle$, so [8, Thm. 3] implies that the q -local index of $[\Delta_q]$ is given by

$$\begin{aligned} \frac{q-1}{(\nu(q-1)/2, q-1)} &= 1 && \text{if } \nu \text{ is even} \\ &= 2 && \text{if } \nu \text{ is odd.} \end{aligned}$$

Now $t(q)/(q-1)$ is odd if and only if $f(q)$ is odd, so we get

$$(3.1) \quad \text{inv}_q [A_q] = 1/2 \iff xg + yf(q) \quad \text{is odd.}$$

Now suppose that p divides n . Let $\gamma^h \phi^{h'}$ be a Frobenius automorphism for p in $L(\varepsilon_q)/K$, and let $\langle \phi^a \rangle$ be the inertia group of p in $L(\varepsilon_q)/K$. Then

$$\left(\frac{\alpha(\phi^a, \gamma^h \phi^{h'})}{\alpha(\gamma^h \phi^{h'}, \phi^a)} \right)^{e(p)/t(p)} (U_\phi^a)^{e(p)} = (\varepsilon_{t(p)})^{(e(p)/2)\nu'}$$

where $\nu' = xah + \mu z(t(p)/e(p))$,

$$\begin{aligned} \text{where} \quad \mu &= 0 && \text{if } a = 0 \\ &= 1 && \text{if } a \neq 0. \end{aligned}$$

Thus the p -local index of $[A_q]$ is given by

$$\begin{aligned} \frac{e(p)}{(\nu'e(p)/2, e(p))} &= 1 && \text{if } \nu' \text{ is even} \\ &= 2 && \text{if } \nu' \text{ is odd.} \end{aligned}$$

Hence

$$(3.2) \quad \text{inv}_p [A_q] = 1/2 \iff xah + \mu z \left(\frac{t(p)}{e(p)} \right) \quad \text{is odd.}$$

(b) Let $A_2 = A_2(x, y, z)$ be the algebra

$$A_2 = (L(\varepsilon_4)/K, \alpha) = \sum_{\tau} L(\varepsilon_4) U_\tau$$

where the values of α are in $\{\pm 1, \pm \varepsilon_4\}$. If $\langle \rho \rangle = \text{Gal}(L(\varepsilon_4)/L)$, then the factor set α is determined by the integers x , y , and z where

$$U_\rho U_\phi = (\varepsilon_4)^x U_\phi U_\rho,$$

$$(U_\rho)^2 = (\varepsilon_4)^y,$$

$$(U_\phi)^m = (\varepsilon_4)^z.$$

The restrictions on x , y , and z are

$$(3.3) \quad \begin{aligned} &y \text{ is even} \\ &xm + 2z \equiv 0 \pmod{4}. \end{aligned}$$

Let $[L/K, 2] = \phi^g$. Then by Lemma 1,

$$(3.4) \quad \text{inv}_2 [\Delta_2] = 1/2 \Leftrightarrow xy + (y/2)f(2) \quad \text{is odd}.$$

Now let p be a prime dividing n . Let $\rho^k \phi^{k'}$ be a Frobenius automorphism of p in $L(\varepsilon_4)/K$, and let $\langle \phi^a \rangle$ be the inertia group of p in L/K . Then

$$\left(\frac{\alpha(\phi^a, \rho^k \phi^{k'})}{\alpha(\rho^k \phi^{k'}, \phi^a)} \right)^{e(p)/t(p)} (U_\phi^a)^{e(p)} = (\varepsilon_{t(p)})^{(e(p)/4)\nu''}$$

where
$$\nu'' = xak + \mu z \left(\frac{t(p)}{e(p)} \right)$$

where
$$\begin{aligned} \mu &= 0 & \text{if } a &= 0 \\ &= 1 & \text{if } a \neq 0. \end{aligned}$$

Thus

$$(3.5) \quad \text{inv}_p [\Delta_2] = 1/2 \Leftrightarrow \frac{xak}{2} + \frac{\mu z}{2} \left(\frac{t(p)}{e(p)} \right) \quad \text{is odd}.$$

Finally observe that if l is a finite prime which does not divide nq , then l does not ramify in $L(\varepsilon_q)/K$ and so $\text{inv}_l [\Delta_q] = 0$.

Now assume that $[L:K]$ is odd. Then $S(K)_2 = K \otimes_q S(Q)$ by [5, Cor. 2]. This means that there is an algebra class $[A]$ in $S(K)_2$ with $\text{inv}_q [A] = 1/2$ if and only if the order of the decomposition group of q in K/Q , $f(q)e(q, K/Q)$, is odd.

For each prime p which divides n , we must have that $e(p, K/Q)$ is even and $e(p)$ is odd. Thus condition (I) of the theorem applies, and is satisfied. Further, every element in $\text{Gal}(L/K)$ is a square, so condition (II) reduces to: For any prime q , $\text{inv}_q [A] = 0$ if $f(q)$ is even. Hence this condition is satisfied. Condition (III) is trivially satisfied since condition (I) applies to each prime p which divides n .

Suppose now that q is a prime not dividing n such that $f(q)$ is odd. Then the decomposition group of q in K/Q has odd order. Thus the algebra $K \otimes_q (Q(\varepsilon_q), \gamma, -1)$ has invariant $1/2$ at q and invariant 0 elsewhere, where $\langle \gamma \rangle = \text{Gal}(Q(\varepsilon_q)/Q)$ and $q' = q$ unless q is even, in which case $q' = 4$. Note that K cannot be a real field in this case, so that the invariants of any algebra in $B(K)$ are 0 at the infinite primes of K .

We have now shown that the theorem holds if $[L:K]$ is odd. For the rest of the proof we shall assume that $[L:K]$ is even. By Lemma 2, we may assume that $[L:K] = 2^c$ for $c \geq 1$.

Suppose that K is a real field. Pick a prime p such that $f(p)e(p, K/Q)$ is even. This can always be done since $[K:Q]$ is assumed to be even. Consider the algebra $K \otimes_q D_p$ where $[D_p] \in S(Q)$ has invariant $1/2$ only at p and the infinite prime p_∞ . Then $[K \otimes D_p]$

has invariant $1/2$ just at the infinite primes of K . Hence $\Omega(p_\infty)$ is in K . This settles the case with respect to the infinite primes since $B(C) = \{1\}$ where C is the complex numbers. For the remainder of the proof, "prime" will mean "finite prime."

Step 2. Condition (I) is satisfied.

Suppose that p is a prime which divides n , and that $e(p) \neq 2^t$. Then a is even where $\langle \phi^a \rangle$ is the inertia group of p in L/K . Hence $(p-1)/e(p)$ is even because it is divisible by a if $e(p) \neq 1$. Thus (3.2) implies that $\text{inv}_p [A_q] = 0$ for all odd primes q which do not divide n . Now consider A_2 . If $a = 0$, then (3.5) implies that $\text{inv}_p [A_2] = 0$ since $\mu = 0$. If $a \neq 0$, then $2^t \geq 4$ so that $p \equiv 1 \pmod{4}$. Hence $[Q(\varepsilon_4)/Q, p] = 1$, so in (3.5) we have that $k = 0$. Moreover, (3.3) implies that z is even, so $\text{inv}_p [A_2] = 0$.

We have shown that each of the generators of $S(K)_2$ has 0 invariant at p . Hence $\text{inv}_p [A] = 0$ for all $[A]$ in $S(K)_2$ and condition (I) is satisfied.

Step 3. Condition (II) is satisfied.

Suppose that p is a prime dividing n such that $f(p)$ is even and condition (I) does not apply to p . Note that the identity element in $\text{Gal}(L/K)$ is a Frobenius automorphism for p in L/K in this case, so condition (II) does apply to p .

Observe that $t(p)/e(p)$ is even, and in the case where $e(p) = 2$, $t(p)/e(p)$ is divisible by 4. This is so because $f(p)$ is even and $e(p) = 2^t$ must divide $p-1$.

Let l be either 4 or an odd prime not dividing n , and suppose that γ^h is a Frobenius automorphism for p in $L(\varepsilon_l)/K$ where $\langle \gamma \rangle = \text{Gal}(L(\varepsilon_l)/L)$. If l is an odd prime then h must be even since $f(p)$ is even. If $l = 4$, then $h = 0$. Further, by (3.3), z is even when $e(p) \geq 4$. Thus (3.2) and (3.5) imply that $\text{inv}_p [A_{l'}] = 0$ where $l' = l$ if l is odd or $l' = 2$ if $l = 4$.

Hence, for p , condition (II) is satisfied on the generators of $S(K)_2$. Therefore condition (II) is satisfied for all primes which divide n .

Now suppose that q is a prime which does not divide n such that $f(q)$ is even and $[L/K, q] = \phi^g$ is a square in $\text{Gal}(L/K)$. Then g is even so that $gx + f(q)y$, or $gx + f(q)y/2$ in the case of $q = 2$, is even for all permissible values of x and y . Thus, by (3.1) and (3.4), $\text{inv}_q [A_q] = 0$.

Classes of the type $[A_q]$ are the only classes amongst the generating classes given by Theorem 1 which might possibly have

nonzero invariant at primes of K dividing q . Hence $\text{inv}_q[A] = 0$ for all $[A]$ in $S(K)_2$, and condition (II) is satisfied for primes which do not divide n .

Step 4. For each prime l to which conditions (I) and (II) do not apply, there is a class $[A]$ in $S(K)_2$ such that $\text{inv}_l[A] = 1/2$.

First suppose that q is a prime which does not divide n . If $f(q)$ is odd, then the algebra

$$\begin{aligned} \mathcal{A}_q^0 &= \mathcal{A}_q(0, 2, 0) & \text{if } q = 2 \\ &= \mathcal{A}_q(0, 1, 0) & \text{if } q \neq 2 \end{aligned}$$

has invariant $1/2$ at q and invariant 0 elsewhere. Hence $\Omega(q) = [\mathcal{A}_q^0]$ if $f(q)$ is odd.

Suppose that $f(q)$ is even and that $[L/K, q] = \phi^g$ where g is odd. By (3.1) and (3.4), the algebra

$$\begin{aligned} \mathcal{A}_q^1 &= \mathcal{A}_q(1, 0, 1) & \text{if } q = 2 \text{ and } 2^e = 2 \\ &= \mathcal{A}_q(1, 0, 0) & \text{otherwise} \end{aligned}$$

has invariant $1/2$ at q .

Now let p be a prime which divides n such that neither condition (I) nor condition (II) applies to p . Hence, $f(p)$ is odd. Pick an odd prime q not dividing n such that $[Q(\varepsilon_p)/Q, q] = \psi$ where $\langle \psi \rangle = \text{Gal}(Q(\varepsilon_p)/Q)$. There exist infinitely many such q by the Tchebotarev density theorem. This choice of q insures that $q \equiv 1 \pmod{4}$ and that $(q/p) = -1$. Hence, by quadratic reciprocity, $(p/q) = -1$. Thus h must be odd where γ^h is a Frobenius automorphism of p in $L(\varepsilon_q)/K$. Then by (3.2) $\text{inv}_p[\mathcal{A}_q^1] = 1/2$ where \mathcal{A}_q^1 is the algebra described above. This is because α is odd if condition (I) does not apply.

Step 5. If condition (III) does not apply, then $\Omega(l)$ is in $S(K)_2$ for every prime l to which conditions (I) and (II) do not apply.

Let p be a prime which divides n such that condition (I) does not apply to p . This means that p is totally ramified in L/K . Hence p is the only prime which is ramified in L/K , and so p is the only prime dividing n to which condition (I) does not apply.

Now suppose that condition (II) does not apply to p . We saw in Step 3 that this means that $f(p)$ is odd. Further suppose that $|(p-1)/e(p)|_2 < |p'-1|_2$ for some prime $p' \neq p$ which divides n . Pick an odd prime q_0 which does not divide n such that $[L(\varepsilon_p)/Q, q_0] = \psi\psi'$ where ψ generates $\text{Gal}(Q(\varepsilon_p)/Q)$ and ψ' generates $\text{Gal}(Q(\varepsilon_{p'})/Q)$. Now $f(q_0)$ is divisible by the same power of 2 as $p'-1$ is, hence $[L/K, q_0] = \phi^g$ where g is even. Thus $\text{inv}_{q_0}[\mathcal{A}_{q_0}^1] = 0$. However our

choice of q_0 insures that $q_0 \equiv 1 \pmod{4}$ and that $(q_0/p) = -1$. Thus the argument at the end of Step 3 gives $\text{inv}_p [\mathcal{A}_{q_0}^1] = 1/2$. Since p is the only prime dividing n at which $\mathcal{A}_{q_0}^1$ can have nonzero invariants, we have that $\Omega(p) = [\mathcal{A}_{q_0}^1]$.

Now let q be a prime which does not divide n such that condition (II) does not apply to q . We saw in Step 3 that $\Omega(q)$ is in $S(K)_2$ if $f(q)$ is odd. Further, if $f(q)$ is even, we have that $\text{inv}_q [\mathcal{A}_q^1] = 1/2$. Thus, if $\text{inv}_p [\mathcal{A}_q^1] = 0$, we have $\Omega(q) = [\mathcal{A}_q^1]$. If $\text{inv}_p [\mathcal{A}_q^1] = 1/2$, then $\Omega(q) = [\mathcal{A}_q^1] \otimes_k \Omega(p)$.

Step 6. Condition (III) is satisfied.

Let p be a prime dividing n to which condition (I) does not apply. Further suppose that $f(p)$ is odd and that $|(p-1)/e(p)|_2 \geq |p'-1|_2$ for every prime $p' \neq p$ which divides n . This hypothesis, and the assumption that $[K:Q]$ is even, forces $p \equiv 1 \pmod{4}$. We also have that $\langle \phi \rangle$ is the inertia group of p in L/K .

Let q be a prime not dividing n such that $\text{inv}_p [\mathcal{A}_q^1] = 1/2$ where \mathcal{A}_q^1 is one of the generators of $S(K)_2$ given in Theorem 1. Let $[L/K, q] = \phi^g$ and let γ^h be a Frobenius automorphism of p in $L(\varepsilon_q)/K$ where $\langle \gamma \rangle = \text{Gal}(L(\varepsilon_q)/K)$, $q' = q$ if q is odd, and $q' = 4$ if $q = 2$.

(a) Suppose that q is odd. Then by (3.2), hx must be odd. However, h is odd if and only if $(p/q) = -1$ since $f(p)$ is odd. So, by the law of quadratic reciprocity, $(q/p) = -1$ and so $f(q)$ is divisible by the same power of 2 as $(p-1)/e(p)$ is. This implies that g is odd. Hence $\text{inv}_q [\mathcal{A}_q^1] = 1/2$.

(b) Suppose that $q = 2$. Then $h = 0$ since $[Q(\varepsilon_2)/Q, p] = 1$. Thus $z/2(t(p)/e(p))$ must be odd. This means that $t(p)/e(p) \equiv 2 \pmod{4}$ and z is odd. By (3.3), this can only occur when x is odd and $e(p) = 2$. Thus $p \equiv 5 \pmod{8}$, so that $(2/p) = -1$. This implies that $f(2)$ is even and that q is odd. Hence, by (3.4) $\text{inv}_2 [\mathcal{A}_2^1] = 1/2$.

Now let q be a prime not dividing n such that $(q/p) = -1$ and $\text{inv}_q [\mathcal{A}_q''] = 1/2$ where \mathcal{A}_q'' is one of the algebras described in Theorem 1. Let $[L/K, q] = \phi^g$ and let γ^h be a Frobenius automorphism of p in $L(\varepsilon_q)/K$ where $\langle \gamma \rangle = \text{Gal}(L(\varepsilon_q)/K)$ and $q' = q$ if q is odd or $q' = 4$ if $q = 2$.

By (3.1) and (3.4), xg is odd. If q is odd, then h is odd so that $\text{inv}_p [\mathcal{A}_q''] = 1/2$. So suppose that $q = 2$. Then we must have $p \equiv 5 \pmod{8}$. This implies that $t(p)/e(p) \equiv 2 \pmod{4}$, and, by (3.3), that z is odd. Hence (3.5) implies that $\text{inv}_p [\mathcal{A}_2''] = 1/2$.

We have now shown that

$$\text{inv}_p [\mathcal{A}_q] = 1/2 \Leftrightarrow \text{inv}_q [\mathcal{A}_q] = 1/2 \quad \text{and} \quad (q/p) = -1.$$

Since every algebra class $[A]$ in $S(K)_2$ is generated by classes of

this form, we have shown that condition (III) is satisfied.

Further, this proves that $\Omega(q)$ is in $S(K)_2$ if $(q/p) = 1$ and condition (II) does not apply to q . This is because $[A_q]$ can have nonzero invariants only at p and q ; we saw in Step 3 that we could arrange for nonzero invariants at q and we have just seen that we cannot get nonzero invariants at p .

This completes the proof of the theorem.

B. $S(K)_2$ when n is even.

Now suppose that $L = Q(\varepsilon_n)$ is a cyclotomic field containing ζ , a primitive 2^s th root of unity for $s \geq 2$. Further suppose that $K \subset L$ does not contain a fourth root of unity, and that $\text{Gal}(L/K) = \langle \phi \rangle$ has order $2^s c'$, $(c', 2) = 1$.

Let $\text{Gal}(Q(\zeta)/Q) = \langle \rho \rangle \times \langle \psi \rangle$ where $\rho(\zeta) = \zeta^{-1}$ and $\psi(\zeta) = \zeta^5$. Then we may assume that $\phi = \rho\psi^{2^{r-2}}\tau$ where the order of $\langle \psi^{2^{r-2}} \rangle = 2^{s-r}$ divides the order of $\langle \tau \rangle$. Thus $\phi(\zeta) = \zeta^{-h}$ where $h = 5^{2^{r-2}}$. We will keep this notation for the rest of this section.

We must determine the invariants of the generators of $S(K)_2$ given in Theorem 1.

Let $A_q = A_q(x, y, z)$ be the algebra

$$A_q = (L(\varepsilon_q)/K, \alpha) = \sum_{\tau} L(\varepsilon_q) U_{\tau}$$

where q is a prime not dividing n and the values of α are in $\langle \zeta \rangle$. Let $\langle \gamma \rangle = \text{Gal}(L(\varepsilon_q)/L)$. The factor set α is determined by the integers x, y , and z where

$$\begin{aligned} U_{\tau} U_{\phi} &= \zeta^x U_{\phi} U_{\tau}, \\ U_{\tau}^{q-1} &= \zeta^y, \\ U_{\phi}^{2^s c'} &= \zeta^z. \end{aligned}$$

The conditions in [8, §1] require that

$$\begin{aligned} \text{(i)} \quad & \zeta^z = (\zeta^z)^{\phi} = \zeta^{-hz} \\ \text{(ii)} \quad & (\zeta^y)^{-h-1} = (\zeta^y)^{\phi-1} \\ & = (\zeta^{-x})^{N(\tau)} \\ & = \zeta^{-x(q-1)} \\ \text{(iii)} \quad & 1 = (\zeta^z)^{\tau-1} = (\zeta^z)^{N(\phi)} \end{aligned}$$

where $N(\tau) = 1 + \tau^2 + \dots + \tau^{|\tau|-1}$ for a group element τ . Hence

$$\begin{aligned} \text{(3.6) (a)} \quad & 2^{s-1} \text{ divides } z, \\ \text{(b)} \quad & y(h+1) - x(q-1) \equiv 0 \pmod{2^s}, \\ \text{(c)} \quad & 2 \text{ divides } x \text{ if } c = s - r. \end{aligned}$$

Now suppose that $[L/K, q] = \phi^g$. Then

$$\left(\frac{\alpha(\gamma, \phi^g)}{\alpha(\phi^g, \gamma)} \right)^{(q-1)/t(q)} U_{\gamma}^{q-1} = (\varepsilon_{t(q)})^{(q-1)\nu}$$

where

$$\nu = \frac{1}{2^s} \left[x \left(\frac{1 - (-h)^g}{1 + h} \right) + y \left(\frac{t(q)}{q-1} \right) \right].$$

Thus the q -local index of Δ_q is given by

$$\begin{aligned} \frac{q-1}{((q-1)\nu, q-1)} &= 1 & \text{if } \nu \equiv 0 \pmod{Z} \\ &= 2 & \text{if } \nu \equiv 1/2 \pmod{Z}. \end{aligned}$$

Hence

$$(3.7) \quad \text{inv}_q [\Delta_q] = 1/2 \Leftrightarrow \nu \equiv 1/2 \pmod{Z}.$$

Now suppose that p is an odd prime which divides n . Let $\gamma^b \phi^{b'}$ be a Frobenius automorphism of p in $L(\varepsilon_q)/K$, and let $\langle \phi^a \rangle$ be the inertia group of p in $L(\varepsilon_q)/K$.

Then

$$\left(\frac{\alpha(\phi^a, \gamma^b \phi^{b'})}{\alpha(\gamma^b \phi^{b'}, \phi^a)} \right)^{e(p)/t(p)} (U_{\phi^a})^{e(p)} = \varepsilon_{t(p)}^{e(p)\nu_p}$$

where

$$\nu_p = \frac{1}{2^s} \left[xb \left(\frac{1 - h^a}{1 + h} \right) + \mu z \left(\frac{p^{f(p)} - 1}{e(p)} \right) \right]$$

where

$$\begin{aligned} \mu &= 0 & \text{if } a &= 0 \\ &= 1 & \text{if } a &\neq 0. \end{aligned}$$

Hence

$$(3.8) \quad \text{inv}_p [\Delta_q] = 1/2 \iff \nu_p \equiv 1/2 \pmod{Z}.$$

Finally suppose that 2 is ramified in L/K . Our assumption that the order of $\langle \psi^{2^{r-2}} \rangle$ divides the order of $\langle \tau \rangle$ implies that in this case $\text{Gal}(L/K) = \langle \rho \rangle$.

Let $\eta = \gamma^b$ be a Frobenius automorphism of 2 in $L(\varepsilon_q)/K$. Let f be the order of $\langle \eta \rangle$. We have

$$\begin{aligned} U_{\rho}((1 + \zeta^{ab})U_{\eta}) &= (1 + \zeta^{-ab})U_{\rho}U_{\eta} \\ &= (1 + \zeta^{-ab})\zeta^{ab}U_{\eta}U_{\rho} \\ &= [(1 + \zeta^{ab})U_{\eta}]U_{\rho}. \end{aligned}$$

Let π be a prime of K which divides 2. Then

$$\begin{aligned} K_\pi \otimes \Delta_q &= \sum_{i=0}^1 \sum_{j=0}^{f-1} K_\pi(\varepsilon_4) K_\pi(\varepsilon_q) U_\rho^i U_\eta^j \\ &= \sum_{i=0}^1 \sum_{j=0}^{f-1} K_\pi(\varepsilon_4) K_\pi(\varepsilon_q) U_\rho^i [(1 + \zeta^{xb}) U_\eta]^j \\ &\cong \sum_{i=0}^1 K_\pi(\varepsilon_4) U_\rho^i \otimes_{K_\pi} \sum_{j=0}^{f-1} K_\pi(\varepsilon_q) [(1 + \zeta^{xb}) U_\eta]^j \\ &\cong (K_\pi(\varepsilon_4), \rho, U_\rho^2) \otimes_{K_\pi} (K_\pi(\varepsilon_q), \eta, [(1 + \zeta^{xb}) U_\eta]^f). \end{aligned}$$

Now $[(K_\pi(\varepsilon_4), \rho, U_\rho^2)] = K_\pi \otimes_{Q_2} (Q_2(\varepsilon_4), \rho, \zeta^z)$. Hence $\text{inv}(K_\pi(\varepsilon_4), \rho, U_\rho^2)$ may be assumed to be 0, since otherwise $e(2, K/Q)$ would be odd which would mean that $K = Q(\varepsilon_{n/4})$. The Schur subgroup of a cyclotomic field is given in [5].

Now let V' and V be the exponential valuations of $K_\pi(\varepsilon_4)$ and K_π respectively. Since $e(K_\pi(\varepsilon_4)/K) = 2$, we have

$$\begin{aligned} V[(1 + \zeta^{xb}) U_\eta]^f &= \frac{1}{2} V'[(1 + \zeta^{xb}) U_\eta]^f \\ &= \frac{1}{2} [V'(1 + \zeta^{xb})^f + V'(U_\eta^f)] \\ &= \frac{1}{2} f V'(1 + \zeta^{xb}). \end{aligned}$$

Now $V'(1 + \zeta^{xb})$ is odd if and only if xb is odd since $1 + \zeta^{xb}$ is a prime element of $K_\pi(\varepsilon_4)$ when xb is odd. Thus from the definition of the Hasse invariant we get

$$\begin{aligned} \text{inv}(K_\pi \otimes \Delta_q) &= 0 && \text{if } xb \text{ is even} \\ &= 1/2 && \text{if } xb \text{ is odd.} \end{aligned}$$

Thus

$$(3.9) \quad \text{inv}_2[\Delta_q] = 1/2 \iff \mu_0 xb \quad \text{is odd}$$

where

$$\begin{aligned} \mu_0 &= 0 && \text{if } 2 \text{ is unramified in } L/K \\ &= 1 && \text{if } 2 \text{ is ramified in } L/K. \end{aligned}$$

Observe that q and the primes which divide n are the only primes which might ramify in $L(\varepsilon_q)/K$. Hence, these are the only primes at which Δ_q can have nonzero invariants.

THEOREM 3. *The 2-primary part of $S(K)$ consists of all classes $[A]$ in $B(K)$ with uniformly distributed invariants of value 0 or $1/2$ which satisfy the following conditions.*

(I) For a prime p which divides n , $\text{inv}_p[A] = 0$ if any of the following hold:

(a) $e(p)$ is odd;

(b) $f(p)$ is even;

(c) $[L:K(\zeta)]/e(p)$ is an even integer.

(II) For q a prime which does not divide n , $\text{inv}_q[A] = 0$ if either

(a) $t = s - r$ and $f(q)$ is even, or

(b) $t \neq s - r$, $f(q)$ is even, and $q^{f(q)} \equiv (-h)^s \pmod{2^{s+1}}$ where $[L/K, q] = \phi^s$.

(III) Let p be a prime which divides n such that condition (I) does not apply to p . If $|e(p, K/Q)|_2 \geq |e(p', K/Q)|_2$ for every prime $p' \neq p$, then the invariant of $[A]$ is $1/2$ at an even number of primes in the set

$$\{p\} \cup \{\text{primes } q: (p/q) = -1, (q, n) = 1\}$$

where (p/q) is the Legendre symbol.

Proof. We have assumed that $\langle \phi \rangle$ has even order. Hence, by Lemma 2, we may assume that $[L:K] = 2^s$.

First suppose that K is a real field. Pick an odd prime q such that $f(q)e(q, K/Q)$ is even. There will always be such a prime since $[K:Q]$ must be even. Then the algebra $K \otimes_Q (Q(\varepsilon_q), \tau, -1)$ where $\langle \tau \rangle = \text{Gal}(Q(\varepsilon_q)/Q)$ has invariant $1/2$ only at the infinite primes of K . Thus $\Omega(p_\infty)$ is in $S(K)_2$ when K is real.

For the rest of the proof, "prime" will mean "finite prime."

Step 1. Condition (I) is satisfied.

Let p be a prime which divides n . If $e(p) = 1$, then p is unramified in $L(\varepsilon_q)/K$ for any prime q not dividing n . Hence $\text{inv}_p[A] = 0$ for all $[A]$ in $S(K)_2$. Now suppose that $e(p)$ is even.

If $p \neq 2$ and $\langle \phi^a \rangle$ is the inertia group of p in L/K , then 2^{s-r} divides a , or if $s = r$, 2 divides a . Since the power of 2 dividing a must divide $(p-1)/e(p)$, we have that $t(p)/e(p)$ is even. Further $h = 5^{2r-2}$ so $(h^a - 1)/(h + 1)$ is not divisible by 2^s if and only if 2^{s-r+1} does not divide a , or if $s = r$, if and only if 4 does not divide a . However this happens if and only if $[L:K] = 2^{s-r}e(p)$, or if $s = r$, if and only if $[L:K] = 2e(p)$. Thus we have

$$\frac{h^a - 1}{h + 1} \not\equiv 0 \pmod{2^s} \iff [L:K(\zeta)]/e(p) \text{ is odd.}$$

Let q be a prime which does not divide n and let $\gamma^b \phi^{b'}$ be a

Frobenius automorphism of p in $L(\varepsilon_q)/K$ where $\langle \gamma \rangle = \text{Gal}(L(\varepsilon_q)/L)$. Then we may rewrite (3.8) to read

$$(3.10) \quad \text{inv}_p [\Delta_q] = 1/2 \iff ([L: K(\zeta)]/e(p))xb \quad \text{is odd}$$

since 2^{s-1} divides z . Since b is even if $f(p)$ is even, (3.10) implies condition (I) for $p \neq 2$.

If γ^b is a Frobenius automorphism for 2 in $L(\varepsilon_q)/K$, then b is even if $f(2)$ is even. Thus (3.9) gives condition (I)(b). Since $\text{Gal}(L/K) = \langle \rho \rangle$ when 2 is ramified in L/K , we see that condition (I) (c) never applies to 2.

Step 2. Condition (II) holds.

Let q be a prime not dividing n and let $[L/K, q] = \phi^g$. We consider the invariants of algebras of the form $\Delta_q = \Delta_q(x, y, z)$. We have

$$\phi^g(\zeta) = \zeta^{(-h)^g} = \zeta^{q^{f(q)}}.$$

Hence $q^{f(q)} = (-h)^g + V2^s$ for some integer V .

Further, by (3.6) (b), we have

$$y = \frac{x(q-1) + W2^s}{1+h}$$

for some integer W . Thus we may rewrite (3.7) to read

$$(3.11) \quad \text{inv}_q [\Delta_q] = 1/2 \iff \left(\frac{W}{1+h} \right) \left(\frac{q^{f(q)} - 1}{q-1} \right) + \frac{xV}{h+1} \equiv 1/2 \pmod{Z}.$$

Now $t(q)/(q-1)$ is even if $f(q)$ is even. Moreover x is even if $t = s - r$ and V is even if $q^{f(q)} \equiv (-h)^g \pmod{2^{s+1}}$. Hence condition (II) is obtained directly from (3.11).

Step 3. For each prime l to which conditions (I) and (II) do not apply, there is a class $[A]$ in $S(K)_2$ such that $\text{inv}_l [A] = 1/2$.

Suppose that q is a prime which does not divide n such that condition (II) does not apply to q . If $f(q)$ is odd, then the algebra

$$\Delta_q^0 = \Delta_q(0, 2^{s-1}, 0)$$

has invariant $1/2$ at q since $W = (h+1)/2$ is odd.

If $f(q)$ is even, $t \neq s - r$, and $q^{f(q)} \not\equiv (-h)^g \pmod{2^{s+1}}$, then consider the algebra

$$\Delta'_q = \Delta_q\left(\frac{h+1}{2}, \frac{q-1}{2}, 0\right).$$

We have that $t(q)/(q-1)$ is even and that V is odd, thus (3.11) implies that $\text{inv}_q [A'_q] = 1/2$.

Now let p be a prime which divides n such that condition (I) does not apply to p . Pick a prime q which does not divide n such that $[Q(\varepsilon_{4p})/Q, q] = \psi_p$, where ψ_p generates $\text{Gal}(Q(\varepsilon_{4p})/Q(\varepsilon_4))$. This choice of q insures that $q \equiv 1 \pmod{4}$ and that $(q/p) = -1$. Hence, by quadratic reciprocity, $(p/q) = -1$ so that b is odd where $\gamma^b \phi^{b'}$ is a Frobenius automorphism of p in $L(\varepsilon_q)/K$ and $\langle \gamma \rangle = \text{Gal}(L(\varepsilon_q)/L)$. Hence, by (3.10) and (3.9) $\text{inv}_p [A'_q] = 1/2$.

Step 4. If condition (III) does not apply, then $\Omega(l)$ is in $S(K)_2$ for every prime l to which conditions (I) and (II) do not apply.

Let p be a prime dividing n to which condition (I) does not apply. Then p is totally ramified in $L/K(\zeta)$. Further, since the inertia group of a prime in $Q(\varepsilon_n)/K$ must be a subgroup of its inertia group in $Q(\varepsilon_n)/Q$, we have that p is the only prime which is ramified in L/K . Thus p is the only prime dividing n to which condition (I) does not apply.

Suppose that $|e(p, K/Q)|_2 < |e(p', K/Q)|_2$ for some prime $p' \neq p$ which divides n . Let $2^\lambda = |e(p, K/Q)|_2$.

(a) Assume that p' is odd.

Pick a prime q_0 not dividing n such that $[L/Q, q_0] = \psi_p \psi_{p'}$, where $\langle \psi_{p'} \rangle = \text{Gal}(Q(\varepsilon_{p'})/Q)$ and $\psi_p = \psi$ if $p = 2$ or $\langle \psi_p \rangle = \text{Gal}(Q(\varepsilon_p)/Q)$ if $p \neq 2$. There are infinitely many such q_0 by the Tchebotarev density theorem. Our choice of q_0 insures that $q_0 \equiv 5 \pmod{8}$ if $p = 2$ or $(q_0/p) = -1$ if $p \neq 2$. Thus $(p/q_0) = -1$ since $q_0 \equiv 1 \pmod{4}$ by choice. Let γ generate $\text{Gal}(L(\varepsilon_{q_0})/L)$ and let $\gamma^b \phi^{b'}$ be a Frobenius automorphism for p in $L(\varepsilon_{q_0})/K$. Then b must be odd. Thus $\text{inv}_p [A'_{q_0}] = 1/2$ by (3.9) and (3.10). On the other hand, $f(q_0)$ is divisible by $|p' - 1|_2$ since $[L/K, q_0] \in \text{Gal}(L/K(\zeta))$ if $p \neq 2$ and $[L/K, q_0] = 1$ if $p = 2$. Hence $q_0^{f(q_0)}$ and h^g , where $[L/K, q_0] = \phi^g$, are both equivalent to 1 modulo 2^{s+1} . This is clear if $p = 2$; if $p \neq 2$, then $q_0 \equiv 1 \pmod{2^s}$ and ϕ^g must be a square in $\text{Gal}(L/K(\zeta))$ by our choice of q_0 . Thus condition (II) applies to q_0 , so $\text{inv}_{q_0} [A_{q_0}] = 0$. Hence $\Omega(p) = [A'_{q_0}]$.

(b) Assume that $p' = 2$, that is that $2^{s-2} > 2^\lambda$.

Pick a prime q_1 not dividing n such that $[L(\varepsilon_{2^{s+1}})/Q, q'] = \psi_p \psi_{2'}^{2^{s-2}-2}$, where ψ_p is the generator of the Sylow-2 subgroup of $\text{Gal}(Q(\varepsilon_p)/Q)$ such that $\psi_p^{2^{\lambda+r-s}}(\varepsilon_p) = \phi(\varepsilon_p)$, and $\psi_{2'}$ is the automorphism sending $\varepsilon_{2^{s+1}}$ to $\varepsilon_{2^{s+1}}^5$. Now

$$[L(\varepsilon_{2^{s+1}})/K, q_1] = (\psi_p^{2^{\lambda+r-s}} \psi_{2'}^{2^{s-2}-2})^g$$

for some g , $2 \leq g \leq 2^{s-r}$. Hence $[L/K, q_1] = \phi^g$. Further,

$$\psi_{2'}^{2^r-2g}(\varepsilon_{2^s+1}) = (\varepsilon_{2^s+1})^{h^g} = (\varepsilon_{2^s+1})^{q_1^{f(q_1)}},$$

so $h^g \equiv q_1^{f(q_1)} \pmod{2^{s+1}}$. This implies that $\text{inv}_{q_1}[\mathcal{A}'_{q_1}] = 0$ since we arranged for $f(q_1)$ to be even.

On the other hand, we picked q_1 so that $q_1 \equiv 1 \pmod{4}$ and $(q_1/p) = -1$. Hence $(p/q_1) = -1$. Thus, by (3.10), $\text{inv}_p[\mathcal{A}'_{q_1}] = 1/2$. Therefore $\Omega(p) = [\mathcal{A}'_{q_1}]$.

Now let q be a prime which does not divide n such that condition (II) does not apply to q . By Step 3, there is an algebra \mathcal{A}_q^* such that $\text{inv}_q[\mathcal{A}_q^*] = 1/2$. If $\text{inv}_p[\mathcal{A}_q^*] = 0$, then $\Omega(q) = [\mathcal{A}_q^*]$. If $\text{inv}_p[\mathcal{A}_q^*] = 1/2$, then $\Omega(q) = [\mathcal{A}_q^*] \otimes_K \Omega(p)$.

Step 5. Condition (III) holds.

Suppose that p is a prime dividing n to which condition (I) does not apply. Further suppose that $|e(p, K/Q)|_2 \geq |e(p', K/Q)|_2$ for every prime $p' \neq p$ which divides n .

Let q be a prime not dividing n . Let $\langle \gamma \rangle = \text{Gal}(L(\varepsilon_q)/L)$ and $\gamma^b \phi^{b'}$ be a Frobenius automorphism for p in $L(\varepsilon_q)/K$.

First suppose that $\text{inv}_p[\mathcal{A}_q^*] = 1/2$ where \mathcal{A}_q^* is an algebra of the form \mathcal{A}_q . From (3.9) and (3.10) we see that this implies that xb is odd. Thus b is odd, which means that $(p/q) = -1$. Further, if $p \neq 2$, then our hypotheses insure that $p \equiv 1 \pmod{4}$. Thus $(q/p) = -1$ if $p \neq 2$, or $q \equiv 3$ or $5 \pmod{8}$ if $p = 2$. Suppose $p \neq 2$, then $|e(p, K/Q)/2^{s-r}|_2 > 2^{r-2}$, so the full 2-part of $e(p, K/Q)$ is equal to $|f(q)|_2$. Hence $q^{f(q)} \equiv 1 \pmod{2^{s+1}}$ and $[L/K, q] = \phi^{2^{s-r}}$. Since $h^{2^{s-r}} \not\equiv 1 \pmod{2^{s+1}}$, we have by (3.11) that $\text{inv}_q[\mathcal{A}_q^*] = 1/2$. In the case where $p = 2$, $|f(q)|_2 = 2^{s-2}$ so $q^{f(q)} \not\equiv 1 \pmod{2^{s+1}}$. However $[L/K, q] = 1$. Thus, by (3.11), $\text{inv}_q[\mathcal{A}_q^*] = 1/2$.

Now suppose that $(p/q) = -1$ and $\text{inv}_q[\mathcal{A}_q^*] = 1/2$. Since $(p/q) = -1$ we have that b is odd. Further, $(q/p) = -1$ if $p \neq 2$ or $q \equiv 3$ or $5 \pmod{8}$ if $p = 2$. Hence $f(q)$ is divisible by $|e(p, K/Q)/2^{s-r}|_2$ if $p \neq 2$ or by 2^{s-r} if $p = 2$. This means that $f(q)$ is even so that xv is odd. Thus xb is odd. Hence (3.9) and (3.10) imply that $\text{inv}_p[\mathcal{A}_q^*] = 1/2$.

We have just shown that

$$\text{inv}_p[\mathcal{A}_q] = 1/2 \iff \text{inv}_q[\mathcal{A}_q] = 1/2 \quad \text{and} \quad (q/p) = -1.$$

Since every algebra class $[A]$ in $S(K)_2$ is a product of classes of the form $[\mathcal{A}_q]$, this gives condition (III).

In addition, this shows that $\Omega(q)$ is in $S(K)_2$ where q is a prime not dividing n such that $(q/p) = 1$ and condition (II) does not apply to q . This is because there is an algebra $[\mathcal{A}_q^*]$ with $\text{inv}_q[\mathcal{A}_q^*] = 1/2$ by Step 3, and we have just seen that $\text{inv}_p[\mathcal{A}_q^*] = 0$.

This completes the proof of the theorem.

We have now determined the Schur subgroup of all fields K , not containing a fourth root of unity, which have a cyclic extension of the form $Q(\varepsilon_n)$. Observe that subfields of $Q(\varepsilon_{p^d})$ are included as special cases. The Schur group of these fields was first found by Yamada [8].

REFERENCES

1. A. A. Albert, *Structure of Algebras*, Amer. Math. Soc. Colloq. Publications XXIV, Providence, 1961.
2. M. Bernard, *The Schur subgroup, I*, J. of Algebra, **22**, No. 2 (1972), 374-377.
3. M. Bernard and M. Schacher, *The Schur subgroup, II*, J. of Algebra, **22**, No. 2 (1972), 378-385.
4. K. L. Fields, *On the Brauer-Speiser theorem*, Bull. Amer. Math. Soc., **77**, No. 2 (1971), 477-478.
5. G. J. Janusz, *The Schur group of cyclotomic field*, J. of Number Theory, (to appear).
6. ———, *Generators for the Schur group of local and global number fields*, Pacific J. Math., (to appear).
7. T. Yamada, *Characterization of the simple components of the group algebra over the p -adic number field*, J. Math. Soc. Japan, **23** (1971), 295-310.
8. ———, *The Schur subgroup of the Brauer group, I*, J. of Algebra, **27** (1973), 579-589.

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