INNER-OUTER FACTORIZATION OF FUNCTIONS WHOSE FOURIER SERIES VANISH OFF A SEMIGROUP

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Let G be a compact, connected, Abelian group. Its dual, Γ, is discrete and can be ordered. Let Γ_1 be a semigroup which is a subset of the positive elements for some ordering, but which contains the origin of Γ. Let $H^p(\Gamma_1)$ be the subspace of $L^p(G)$ consisting of functions which have vanishing off Γ_1. The question that this paper is concerned with is what conditions on a function in $H^p(\Gamma_1)$ assure an inner-outer factorization.

An inner function is a function $f \in H^\infty(\Gamma_1)$ such that $|f|=1$ a.e. (dx) on G. A function $f \in H^p(\Gamma_1)$ is said to be outer if

$$\int_G \log |f(x)| = \log \left| \int_G f(x)dx \right| > -\infty .$$

A function $f \in H^1(\Gamma_1)$ is said to be in the class $LRP(\Gamma_1)$ if $\log |f| \in \Gamma_1(G)$ and $\log |f|$ has Fourier coefficients equal to zero off $\Gamma_1 \cup -\Gamma_1$. The main result of the paper is that if Γ_1 is the intersection of half planes and $f \in H^1(\Gamma_1)$ with $\int_G \log |f(x)| dx > -\infty$ then f has an inner-outer factorization if and only if $\log |f|$ is in $LRP(\Gamma_1)$.

A semigroup, P, in Γ_1 is called a half plane if $P \cup -P = \Gamma$ and $P \cap -P = \{0\}$. Helson and Lowdenslager [2] proved that if Γ_1 is a half plane then every function $f \in H^p(\Gamma_1)$ with $\int \log |f| dx > -\infty$ has a factorization as a product of an outer function, $h \in H^p(\Gamma_1)$ and an inner function, g, and this factorization is unique up to multiplication by constants of magnitude 1. From now on we shall assume $\int \log |f| dx > -\infty$.

Helson and Lowdenslager also showed [3] that if u is a real function such that u and e^u are summable, and v is the conjugate function of u with respect to the half plane, Γ_1, then e^{u+iv} is an outer function in $H^1(\Gamma_1)$. Conversely, if a summable outer function has the representation e^{u+iv} with u and v real then u is summable and v is equal to its conjugate modulo 2π except for an additive constant.

Let P be a half plane which contains Γ_1. Then, for $u \in L_R^1(G)$ there exists a conjugate function, v, which is unique if we assume $v(0) = 0$, such that $u + iv$ has its Fourier series supported on P. The function, v, is in L^p, $p < 1$. If u has its Fourier coefficients supported only on $\Gamma_1 \cup -\Gamma_1$ then $u + iv$ has its Fourier coefficients supported only on Γ_1 [4, Chap. 8, §7]. Therefore, $f \in LRP(\Gamma_1)$ if

and only if $\log|f| \in L^1$ and $\log|f|$ is the real part of a function whose Fourier coefficients vanish off Γ_1.

THEOREM. *Assuming that $f \in H^1(\Gamma_1)$ and $\int \log|f|\,dx > -\infty$, and that $\Gamma_i = \bigcap_{i \in I} P_i$, then f has an inner-outer factorization if and only if $\log|f|$ has its Fourier coefficients vanish off $\Gamma_1 \cup -\Gamma_1$.*

Proof. Assume $f \in LRP(\Gamma_1)$. Let $u = \log|f| \in L^1(G)$. Take any $i \in I$ and consider P_i. $\Gamma_1 \subset P_i$ and f has an inner-outer factorization with respect to P_i. The outer factor is given by e^{u+iv_i} where v_i is the conjugate function to u with respect to P_i. Since u has its Fourier coefficients supported on $\Gamma_1 \cup -\Gamma_1$, it follow that v_i also has its Fourier coefficients supported there. Therefore, v_i is the same as the conjugate function of u with respect to any of the other half planes P_j, $j \in I$. Therefore, the outer factor of f in $H^1(P_i)$ is given by e^{u+iv_i}. Also, if P_j is any of the other half planes whose intersection gives Γ_1, then the outer factor f in $H^1(P_j)$ is e^{u+iv_j}. Therefore, $e^{u+iv_i} \in \bigcap_{i \in I} H^1(P_i)$, which is just equal to $H^1(\Gamma_1)$. For each half plane P_i, $i \in I$, we have that the inner factor is given by $fe^{-(u+iv_i)}$. Therefore, the inner factor is also in $H^1(\Gamma_1)$.

Conversely, assume that f has an inner-outer factorization, gh, in $H^1(\Gamma_1)$. Choose P_j, $j \in I$, then the outer factor, h, of f in $H^1(\Gamma_1)$, and hence in $H^1(P_j)$, is given by e^{u+v_j}, where v_j is the conjugate function of $u = \log|f|$ with respect to P_j. Since this is true for all P_j, $j \in I$, it follows that e^{iv_j} is the same regardless of which half plane, P_j is used. Now assume P_k is another of the half planes whose intersection is Γ_1. Then $e^{iv_k} = e^{iv_j}$ where v_k is the conjugate function of u with respect to P_j. It follows that $v_k(x) = v_j(x) + 2n\pi$ where n might change from point to point. We will now show that $n = 0$. Consider the function $h^{1/2}$ which is outer in $H^2(\Gamma_1) \subset H^1(\Gamma_1)$. It follows that $\log|h^{1/2}| = u/2$. The conjugate function of $u/2$ with respect to p_j is $v_j/2$ and its conjugate function with respect to P_k is $v_k/2$. By the Helson and Lowdenslager theorem $h^{1/2} = e^{(u+iv_j)/2}$ and also $h^{1/2} = e^{(u+iv_k)/2}$. Therefore,

$$h = h^{1/2}h^{1/2} = e^{u+i(v_j+v_k)/2}.$$

Hence

$$v_k(x) = (v_j(x) + v_k(x))/2 + 2n\pi.$$

So,

$$v_k(x) = v_j(x) + 4n\pi.$$

Now consider $h^{1/4} = e^{(u+iv_j)/4} = e^{(u+iv_k)/4}$. Therefore,

$$h = h^{3/4}h^{1/4} = e^{3(u+iv_k)/4}e^{(u+iv_j)/4} = e^{u+i(3v_k+v_j)/4} .$$

Hence,

$$v_k(x) = v_j(x) + 8n\pi .$$

By considering the 2^mth roots of h we can show that the difference between v_k and v_j must be $2^{m+1}n\pi$. This must hold for all values of m. The only integer for which this is true is 0. Therefore u has the same conjugate function with respect to each of the half planes.

We will show that u has its Fourier coefficients supported of $\Gamma_1 \cup -\Gamma_1$. Suppose that $\hat{u}(\gamma) \neq 0$, where $\gamma \notin \Gamma_1 \cup -\Gamma_1$. Then there exists P_j, $j \in I$ such that $\gamma \notin P_j$. There also exists P_k, $k \in I$, such that $\gamma \notin -P_k$. Let v_j be the conjugate functions of u with respect to the half plane, P_j and let v_k be the conjugate function of u with respect to P_k. Since $\gamma \notin P_j$, we have

$$\hat{v}_j(\gamma) = i\hat{u}(\gamma)$$

[4, Chap. 8, §7]. Likewise, since $\gamma \notin -P_k$ it follows that $\gamma \in P_k \backslash 0$ and that

$$\hat{v}_k(\gamma) = -i\hat{u}(\gamma) .$$

But since $\hat{u}(\gamma) \neq 0$, we have $\hat{v}_j(\gamma) \neq \hat{v}_k(\gamma)$, and hence v_j and v_k are different functions. But we have just shown that u has the same conjugate function with respect to each half plane. Therefore $\hat{u}(\gamma)=0$ and u has its Fourier series supported on $\Gamma_1 \cup -\Gamma_1$. Therefore $f \in LRP(\Gamma_1)$.

COROLLARY. *If $f \in H^1(\Gamma_1)$ where Γ_1 is the intersection of half planes and $f \in LRP(\Gamma_1)$, then $f = p_1p_2$ where p_1, $p_2 \in H^2(\Gamma_1)$ and $|p_1|^2 \equiv |p_2|^2 \equiv |f|$*

EXAMPLE. In [1] Ebenstein discusses the H^p functions on a semigroup which is the intersection of a countable collection of half planes. This semigroup fulfills the hypothesis of the theorem. Let T^ω be the compact group which is the Cartesian product of countably many circles. The dual $\sum_{i=I}^{\infty} Z$, is the direct sum of countably many copies of the integers. Define $A \subset \sum_{i=1}^{\infty} Z$ by

$$A = \{x: x_i \geqq 0 \quad \text{for all } i\} .$$

We may define $H^p(T^\omega)$, $p \geqq 1$ as the subset of $L^p(T^\omega)$ consisting of these functions whose Fourier coefficients vanish off A. The semigroup, A, is the intersection of half planes P_i defined as follows:

$$P_i = \{x: x_i \geqq 0 \ , \quad \text{if } x_i = 0 \text{ then } x_1 \geqq 0 \ ,$$
$$\text{if } x_1, x_2, \cdots, x_j = 0 \text{ then } x_{j+1} = 0\} \ .$$

Therefore the theorem applies to $H^p(T^\omega)$.

REMARK. One might hope that certain theorems which hold for the H^p spaces of the disk would remain true, at least for the class $LRP(\Gamma_1)$. One such theorem is Szego's theorem which states if $w \in L^1(dx)$ and $w \geqq 0$, then

$$\inf_{g \in A_0} \int |1 - g|^2 \, w dx = \exp \int \log{(w)} dx$$

where A_0 consists of those polynomials supported on Γ_1, with zero-th coefficient equal to zero. This theorem is true if Γ_1 is a half plane [4, Chap. 8, § 3]. Rudin has an example [5, Theorem 4.4.8] of a function, f, which is outer, but does not span. This same function can be used to show that Szego's theorem fails even for the class LRP.

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