LIPSCHITZ SPACES OF DISTRIBUTIONS ON THE SURFACE OF UNIT SPHERE IN EUCLIDEAN n-SPACE

Harvey Charles Greenwald

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HARVEY C. GREENWALD

In this paper Lipschitz spaces of distributions are defined and various inclusion relations are shown. Certain properties such as completeness, separability, and the density of the testing space for appropriate Lipschitz spaces are proved. The Littlewood–Paley function is defined and used to prove inclusion relationships between Lipschitz and Lebesgue spaces.

This paper is the second in a series of papers by the author of which [1] will be used extensively in this paper. As a result, a knowledge of [1] would be useful to the reader. In [1] the discussion was limited to Lipschitz spaces of functions. Here we extend the definition of a Lipschitz space to Distributions.

Conventions and notation.
$\mathbf{R}^1$ will denote the real numbers.
$\mathbf{R}^n = \{x = (x_1, \cdots, x_n): x_i \in \mathbf{R}^1, i = 1, \cdots, n\}$.
$\Sigma_{n-1} = \{x \in \mathbf{R}^n: |x| = (x_1^2 + \cdots + x_n^2)^{1/2} = 1\}$. All functions are complex valued unless otherwise stated.
$C^\infty(\Sigma_{n-1})$ is the set of indefinitely differentiable functions on Σ_{n-1}.

All statements about continuity, bounded, finiteness, etc., are made modulo sets of measure zero unless otherwise specified. By this we mean that a function that can be modified on a set of measure zero to have the property will be said to have the property.

If $f(x, r)$, where $x \in \Sigma_{n-1}$ and $0 < r < 1$, is differentiable with respect to r, we define $Tf(x, r) = d/dr(rf)(x, r)$ and $T^k f(x, r) = T(T^{k-1}f)(x, r)$ where k is an integer greater than 1. We say $f(x) = 0(g(x))$, $x \to a$, if $f(x)/g(x)$ is bounded as $x \to a$.

$f(x) = o(g(x))$, $x \to a$, if $f(x)/g(x) \to 0$ as $x \to a$.
$f(x) \simeq g(x)$, $x \to a$, if $f(x)/g(x) \to 1$ as $x \to a$.

For α real, $\bar{\alpha}$ will denote the smallest nonnegative integer larger than α. If $f(x)$ is measurable on Σ_{n-1}, we define $\|f(x)\|_p = \left[\int_{\Sigma_{n-1}} |f(x)|^p\right]^{1/p}$, $1 \leq p < \infty$, and $\|f(x)\|_\infty = \text{ess sup}_{x \in \Sigma_{n-1}} |f(x)|$ where dx is nonnormalized Lebesgue measure on Σ_{n-1}. If $f(x, r)$ is measurable in x and r where $x \in \Sigma_{n-1}$ and $0 < r < 1$, we define

$$\|f(r,x)\|_{p,dx} = \begin{cases} \left[\int_{\Sigma_{n-1}} |f(r,x)|^p dx\right]^{1/p} & \text{if } 1 \leq p < \infty \\[2em] \operatorname*{ess\,sup}_{x \in \Sigma_{n-1}} |f(r,x)| & \text{if } p = \infty \end{cases}$$

$$\|f(r,x)\|_{pq} = \begin{cases} \left[\int_0^1 \|f(r,x)\|_{p,dx} dr/1-r\right] & \text{if } 1 \leq q < \infty \\[2em] \operatorname*{ess\,sup}_{0<r<1} \|f(r,x)\|_{p,dx} & \text{if } q = \infty. \end{cases}$$

If $\alpha > 0$ and $1 \leq p, q \leq \infty$, we say $f \in \Lambda(\alpha; p, q)$ if

$$\|f(x)\|_{\alpha;p,q} = \|f(x)\|_p + \|(1-r)^{\alpha-\alpha} T^\alpha f(r,x)\|_{pq}$$

is finite.

The Poisson kernel is the function $P(rx, y) = 1/c_n(1-r^2)/|rx - y|^n$ where $|x| = |y| = 1$, $0 \leq r < 1$, and c_n is a constant so that $\int_{\Sigma_{n-1}} P(rx, y) dy = 1$ for each x. We shall also use $P(r, x \cdot y) = 1/c_n(1-r^2)/[1 - 2rx \cdot y + r^2]^{n/2}$.

If $f(x) \in L_p(\Sigma_{n-1})$, $1 \leq p \leq \infty$, then the Poisson integral of f is defined as $f(rx) = \int_{\Sigma_{n-1}} f(y)P(r, x \cdot y) dy$.

$\{Y_l^{(k)}\}$, $l = 1, \cdots, n(k)$, denotes an orthonormal basis for the spherical harmonics of degree k. $Z_y^{(k)}$ denotes the zonal harmonic of degree k with pole y.

If $F(x) \in L_1(\Sigma_{n-1})$ and $G(s) \in L_1([-1, 1], d\mu(s))$ where $d\mu(s) = (1 - s^2)^{(n-3)/2}$, the spherical convolution of F and G is the function

$$F * G(x) = \int_{\Sigma_{n-1}} F(y)G(x \cdot y) dy.$$

CHAPTER I. Lipschitz Spaces, α Real.

In this chapter the notion of a Lipschitz space for α real is defined. For this a brief discussion of distributions is necessary.

Let the testing space $S = \{\phi: \phi \in C^\infty(\Sigma_{n-1})\}$. Let $Y_l^{(k)}(x)$, $l = 1, \cdots, n(k)$, be an orthonormal basis for the spherical harmonics of degree k. $\phi \in C^\infty(\Sigma_{n-1})$ if and only if $\phi = \Sigma_{kl} a_{kl} Y_l^{(k)}$ with $a_{kl} = 0(k^{-s})$ for all reals. For a proof of see Seeley [3]. ϕ can be considered to be in $C^\infty(\mathbf{R}^n - \{0\})$ by noting that

$$\phi(x) = \sum_{k,l} a_{kl} Y_l^{(k)}(x) = \sum_{k,l} a_{kl} r^{-k} P_l^{(k)}(rx)$$

where $P_l^{(k)}$ are harmonic polynomials of degree k.

Let $D^\alpha f = \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ where $\alpha = (\alpha_1, \cdots, \alpha_n)$, α_i nonnegative integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. It is easy to see that $D^\alpha \phi = \Sigma_{k,l} a_{kl} D^\alpha Y_l^{(k)}$ and the convergence is uniform. A topology can be defined on S by letting

$$N_{\epsilon,M}(0) = \left\{ \phi \in C^\infty(\Sigma_{n-1}) \colon \sum_{|\alpha| \le M} \| D^\alpha \phi \|_\infty < \epsilon \right\}$$

be a neighborhood system at 0. A standard argument shows that with respect to this norm, S is complete.

Let the distributions S' be the set of continuous linear functionals on S. The action of $f \in S'$ on $\phi \in S$ will be denoted by $f(\phi)$. If $\phi \in S$ and $\phi = \Sigma_{k,l} a_{kl} Y_l^{(k)}$, then the Poisson integral of ϕ,

$$\phi(rx) = \sum_{k,l} a_{kl} r^k Y_l^{(k)}(x)$$

also is in S for fixed $r < 1$. It is easy to see that $\phi(rx) \to \phi(x)$ as $r \to 1^-$ in the topology of S.

If $f \in S'$ and P is the Poisson kernel, define $f * P$ by $(f * P)(\phi) = f(P * \phi)$. We call this the Poisson integral of the distribution f. In view of the above, if $f, g \in S'$ and if f and g have the same Poisson integral, they are equal as distributions.

If $f \in L_p(\Sigma_{n-1})$, $1 \le p \le \infty$, f defines a distribution by letting $f(\phi) = \int_{\Sigma_{n-1}} f(x)\phi(x)dx$. If $f \in L_p(\Sigma_{n-1})$, define $J^{-\alpha} f$ by $J^{-\alpha} f(\phi) = f(\Sigma_{k,l} a_{kl}(k+1)^\alpha Y_l^{(k)})$ where $\alpha \ge 0$ and $\phi = \Sigma_{k,l} a_{kl} Y_l^{(k)}$.

Clearly $\Sigma a_{kl}(k+1)^\alpha Y_l^{(k)} \in S$. An easy check shows that this does define a distribution. Moreover,

$$J^{-\alpha} f(rx) = \left[\sum_{k=0}^\infty (k+1)^\alpha r^k Z_{e_1}^{(k)} * f \right](x)$$

is the Poisson integral of $J^{-\alpha} f$.

In view of the above $J^\alpha f(rx) = \Sigma_{k,l} b_{kl} r^k (k+1)^{-\alpha} Y_l^{(k)}(x)$ for α real where $f \sim \Sigma_{k,l} b_{kl} Y_l^{(k)}$. This is easily seen to be harmonic on $\{Z \in \mathbf{R}^n \colon |Z| < 1\}$.

The above definition of $J^{-\alpha} f$ agrees with the $J^{-\alpha} f$ defined in Proposition 5.10 of Greenwald [1] for $f \in \Lambda(\alpha + \beta; p, q)$. Hence,

PROPOSITION 1.1. J^β maps $\Lambda(\alpha; p, q)$ isomorphically onto $\Lambda(\alpha + \beta; p, q)$ provided only $\alpha, \alpha + \beta > 0, 1 \le p, q \le \infty$.

DEFINITION. The Lebesgue space $L_{p,\alpha}(\Sigma_{n-1}) = \{f \in S': f = J^\alpha\psi, \psi \in L_p(\Sigma_{n-1})\}$ for α real, $1 \leq p \leq \infty$. Define $\|f\|_{p,\alpha} = \|\psi\|_p$. Thus $L_{p,\alpha}$ is a Banach space that is an isometric image of $L_p(\Sigma_{n-1})$.

DEFINITION. For $\alpha \leq 0$, $1 \leq p$, $q \leq \infty$, let $\Lambda(\alpha; p, q)$ be the set of distributions $f \in L_{p,\alpha-\frac{1}{2}}$ for which $\|(1-r)^{\tilde\alpha-\alpha}T^{\tilde\alpha}f(rx)\|_{pq}$ is finite. For $\alpha \leq 0$, define $\|f\|_{\alpha;p,q} = \|f\|_{p,\alpha-\frac{1}{2}} + \|(1-r)^{\tilde\alpha-\alpha}T^{\tilde\alpha}f(rx)\|_{pq}$.

Note. We are essentially defining $\Lambda(-\alpha; p, q)$ to be $J^{-\alpha-\frac{1}{2}}[\Lambda(\frac{1}{2}; p, q)]$. The choice of $\frac{1}{2}$ is arbitrary. Any $\beta > 0$, would work as well.

REMARK. Let α be real and $1 \leq p$, $q \leq \infty$. Let $f \in L_{p,\alpha-\frac{1}{2}}$. Then if k is any nonnegative integer greater than α, the following norms are equivalent:
  (i) $\|(1-r)^{k-\alpha}T^k f(rx)\|_{pq} + \|f\|_{p,\alpha-\frac{1}{2}}$
  (ii) $\|(1-r)^{\tilde\alpha-\alpha}T^{\tilde\alpha}f(rx)\|_{pq} + \|f\|_{p,\alpha-\frac{1}{2}}$.

Proof. This is an immediate consequence of (2.3) of Greenwald [1].

PROPOSITION 1.2. *Let α be real and $1 \leq p$, $q \leq \infty$. Then the set of distributions $f \in L_{p,\alpha-\frac{1}{2}}$ for which $\|(1-r)^{\tilde\alpha-\alpha}T^{\tilde\alpha}f(rx)\|_{pq} < \infty$ normed with $(*) = \|(1-r)^{\tilde\alpha-\alpha}T^{\tilde\alpha}f(rx)\|_{pq} + \|f\|_{p,\alpha-\frac{1}{2}}$ is topologically and algebraically equal to $\Lambda(\alpha; p, q)$.*

Proof. For $\alpha \leq 0$ this is the definition. Hence it suffices to consider the case $\alpha > 0$.
  (a) Assume that $f \in L_{p,\alpha-\frac{1}{2}}$ and that $(*) < \infty$. We want to show that $f \in \Lambda(\alpha; p, q)$. There is a $g \in L_p(\Sigma_{n-1})$ such that

$$f(rx) = [g * G^{\alpha-\frac{1}{2}}(r, \cdot)](x).$$

Hence $\|f(rx)\|_{p,dx} \leq M_\alpha \|g\|_p = M_\alpha \|f\|_{p,\alpha-\frac{1}{2}}$ if $r \leq \frac{1}{2}$.
  By Proposition 5.1 of Greenwald [1], $f(rx)$ is the Poisson integral of a function $h \in L_p(\Sigma_{n-1})$ and $\|h\|_p \leq M_\alpha(*)$. Now h and f have the same Poisson integral and thus are equal as distributions. Hence f is a function and is in $L_p(\Sigma_{n-1})$. Therefore $f \in \Lambda(\alpha; p, q)$ and $\|f\|_{\alpha;p,q} \leq M(*)$.
  (b) Let $f \in \Lambda(\alpha; p, q)$. The proof of this part is essentially the same as that for Lemma 8 of Taibleson [18; p. 438].
  The proofs of the following Propositions 1.3, 1.4, 1.5, and 1.6 are analogous to the proofs of Theorems 6, 7, 8, and 9 of Taibleson [7; p. 437–443]. The appearence of the $(n-1)$ in Proposition 1.6 comes from the estimate $\|P((1+r)/2x)\|_p \leq A(1-r)^{(1-n)/t'}$ where $1/t + 2/t' = 1$. See Greenwald [1; (1.8)].

PROPOSITION 1.3. *Let* α, β *be real and* $1 \leq p, q \leq \infty$. *Then* J^β *maps* $\Lambda(\alpha; p, q)$ *isomorphically onto* $\Lambda(\alpha + \beta; p, q)$.

PROPOSITION 1.4. *Let* $f \in S'$ *and* $f(rx)$ *be its Poisson integral. Then for each integer* $k \geq \bar{\alpha}$ *and real number* $\beta < \alpha$, *the norm*

$$\|(1 - r)^{k-\alpha} T^k f(rx)\|_{pq} + \|f\|_{p,\beta}$$

is equivalent to $\|f\|_{\alpha,p,q}$ *where* $1 \leq p, q \leq \infty$.

PROPOSITION 1.5. *Suppose* $f \in \Lambda(\alpha_i; p_i, q_i)$ *where* $i = 0, 1$ *and* $1 \leq p_i$, $q_i \leq \infty$. *Let* $0 \leq t \leq 1$ *and define* $\alpha = (1 - t)\alpha_0 + t\alpha_1$, $1/p = (1 - t)/p_0 + t/p_1$, *and* $1/q = (1 - t)q_0 + t/q_1$. *Then* $f \in \Lambda(\alpha; p, q)$ *and*

$$\|f\|_{\alpha,p,q} \leq M_{\alpha_0,\alpha_1}(\|f\|_{\alpha_0,p_0,q_0})^{1-t}(\|f\|_{\alpha_1;p_1,q_1})^t.$$

Also, (i) $\|f\|_{p,\beta} \leq \|f\|_{p_0,\beta}^{1-t}\|f\|_{p_1,\beta}^t$ *for* $\beta < \alpha_0, \alpha_1$.
(ii) $\|(1 - r)^{k-\alpha} T^k f(rx)\|_{pq}$
$\leq (\|(1 - r)^{k-\alpha_0} T^k f(rx)\|_{p_0 q_0})^{1-t} (\|(1 - r)^{k-\alpha_1} T^k f(rx)\|_{p_1 q_1})^t$ *for* $k < \alpha_0, \alpha_1$.

PROPOSITION 1.6. *Let* $1 \leq p_1 \leq q_1 \leq \infty$. *Then* $\Lambda(\alpha_1; p_1, q_1) \subset \Lambda(\alpha_2; p_2 q_2)$ *if* $\alpha_1 - (n - 1)/p_1 > \alpha_2 - (n - 1)/p_2$ *or if* $\alpha_1 - (n - 1)/p_1 = \alpha_2 - (n - 1)/p_2$ *and* $1 \leq q_1 \leq q_2 \leq \infty$. *Moreover, the inclusion map is continuous.*

PROPOSITION 1.7. (i) $\Lambda(\alpha; p, q)$ *is complete if* $1 \leq p, q < \infty$. (ii) S *is dense in* $\Lambda(\alpha; p, q)$ *if* $1 \leq p \leq \infty$, $1 \leq q < \infty$. *We shall need the following lemma.*

LEMMA 1.1. *If* $f \in \Lambda(\alpha; \infty, q)$, $0 < \alpha < 1$, *then* f *is uniformly continuous.*

Proof. It suffices to show that $\|f(rx) - f(x)\|_{\infty,dx} \to 0$ as $r \to 1^-$. By Proposition 1.6, $f \in \Lambda(\alpha; \infty, \infty)$. $f(rx) - f(x) = \int_r^1 f_\rho(\rho x) d\rho$ for almost every $x \in \Sigma_{n-1}$. Hence $\|f(rx) - f(x)\|_{\infty,dx} \leq 2\int_r^1 \|\rho f_\rho(\rho x)\|_{\infty,dx} d\rho$ if $r \geq \frac{1}{2}$. Thus

$$(1 - r)^{-\alpha} \|f(rx) - f(x)\|_{\infty,dx} \leq 2(1 - r)^{-\alpha} \int_r^1 A(1 - \rho)^{\alpha-1} d\rho \leq M_\alpha.$$

Hence $\|f(rx) - f(x)\|_{\infty,dx} \leq M_\alpha(1 - r)^\alpha$ and thus tends to zero as $r \to 1^-$.

Proof of Proposition 1.7.   (i) The proof is similar to the proof of the corresponding part of Theorem 11 of Taibleson [7; p. 444].   (ii) It is easy to see that $J^{\alpha}s = S$.   Hence, it suffices to consider the case $0 < \alpha < 1$. We claim that $S \subset \Lambda(\alpha; p, q)$.   Let $\phi \in S$ with $\phi = \Sigma_{k,l} a_{kl} Y_l^{(k)}$.   Then $\phi(rx) = \Sigma_{k,l} a_{kl} r^k Y_l^{(k)}(x)$ *and* $T\phi(rx) = \Sigma_{k,l} a_{kl}(k+1) r^k Y_l^{(k)}(x)$.

Since $a_{kl} = O(k^{-s})$ for every $s > 0$, $|T\phi(rx)| \leq \Sigma_{k,l} |a_{kl}(k+1) Y_l^{(k)}| r^k \leq M < \infty$.   So, $\phi \in \Lambda(\alpha; p, q)$.   Let $f \in \Lambda(\alpha; p, q)$.   If $1 \leq p < \infty$, $f(rx) \to f(x)$ in $L_p(\Sigma_{n-1})$ as $r \to 1^-$.

If $p = \infty$, f is continuous by Lemma 1.1 and so $f(rx) \to f(x)$ in $L_\infty(\Sigma_{n-1})$ as $r \to 1^-$.   We claim that for each r, $(1-r)^{1-\alpha} \| Tf(rsx) - Tf(rx) \|_{p, dx} \to 0$ as $s \to 1^-$.

Let $g(x) = Tf(rx)$.   Then $g(sx) = Tf(rsx)$.   By the above, $g(sx) \to g(x)$ in $L_p(\Sigma_{n-1})$ as $s \to 1^-$.   Also, $\| Tf(rsx) \|_{p, dx} \leq \| Tf(rx) \|_{p, dx}$.

If $q < \infty$, by applying the Dominated Convergence Theorem we have that $\| (1-r)^{1-\alpha} [Tf(rsx) - Tf(rx)] \|_{pq} \to 0$ as $s \to 1^-$.

Thus $f(sx) \to f(x)$ in $\Lambda(\alpha; p, q)$ if $q < \infty$.   For fixed $s < 1$, $f(sx)$ is clearly in S.   This finishes the proof.

REMARKS.   Let $A = \{\phi \in S : a_{kl}$ are rational$\}$.   It is clear that A is dense in $\Lambda(\alpha; p, q)$ if $q < \infty$.   Hence $\Lambda(\alpha; p, q)$ is separable if $q < \infty$.

Let $B = \{\alpha \in S : \Sigma_{k,l} a \Sigma_{k,l} a_{kl} Y_l^{(k)}$ consists only of a finite number of terms$\}$.   It is clear that B is dense $\Lambda(\alpha; p, q)$ if $q < \infty$.

Lastly, before finishing this chapter we would like to relate the Lipschitz spaces defined here to those defined by Ragozin [2].   For this we will need the notion of a derivative in the distribution sense.

DEFINITION.   If f is differentiable on Σ_{n-1} and D is a skew-symmetric $n \times n$ matrix then we define $Df(x) = (d/dt)f[(\exp tD)(x)]|_{t=0}$.

See Rogazin [2] for a more complete discussion of this topic.

Let $f, \phi \in S$.   Then an easy argument shows that $\int_{\Sigma_{n-1}} Df(x)\phi(x)dx = \int_{\Sigma_{n-1}} f(x)(-D)\phi(x)dx$.   This leads to the following definition.

DEFINITION.   Let $f \in S'$.   We define Df to be the distribution defined by $Df(\phi) = f(-D\phi)$ for $\phi \in S$.

If $i < j$, let D_{ij} be the $n \times n$ matrix with a 1 in the (i, j) place, a (-1) in the (j, i) place, and zeros everywhere else.   See Ragozin [2].

DEFINITION.   We define $(D_{ij})^k$ to be the result of applying some k-fold product of D_{ij}'s to f in the distribution sense where $f \in S'$.   This is clearly ambiguous but we shall be summing over all possible k-fold products.   Hence, no problem will arise.

PROPOSITION 1.8.   *Let $\alpha > 0$ and $1 \leq p,\ q \leq \infty$.   Then the norm* $\|f(x)\|_p + \Sigma \|(D_{ij})^k f\|_{\alpha,p,q}$ *is equivalent to* $\|f\|_{\alpha+k,p,q}$ *where the sum is over all k-fold products of the D_{ij}'s.*

Proof.   The proof is essentially the same as that used by Taibleson for the analogous result in $\mathbf{R}^n$.   See Taibleson [7; Theorem 10, p. 444].   The proof uses (1.3) and Proposition 5.1 of Greenwald [1].

CHAPTER II.   The Relation of Lipschitz Spaces to Lebesgue Spaces.

In this chapter certain inclusion relations between Lipschitz Spaces and Lebesgue Spaces are proved.   The chief tool is an analogue of the Littlewood–Paley function.   The proofs are, for the most part, direct analogues of those used by Stein [4], [5] for the Littlewood–Paley function in a compact Lie group.   The reader is referred to Zygmund [8; Chapter XIV] for the one dimensional version of the Littlewood–Paley function.

DEFINITION.   Let $\qquad f^*(x) = \sup_{0 < t \leq \pi} t^{-n+1} \int_{d(x,y) \leq t} |f(y)| \, dy \qquad$ if $f \in L_1(\Sigma_{n-1})$.

REMARK.   If $f \in L_p(\Sigma_{n-1});\ 1 < p \leq \infty$, then $f^* \in L_p(\Sigma_{n-1})$ and $\|f^*\|_p \leq A_{p,n}\|f\|_p$.

(2.1)   Let $f \in L_1(\Sigma_{n-1})$.   Then $\sup_{0 \leq r < 1} |f(rx)| \leq A_n f^*(x)$ for all $x \in \Sigma_{n-1}$.

Proof.   We may assume that $f(x) \geq 0$.   Let $x \in \Sigma_{n-1}$ and let $g(\theta) = \int_{x \cdot y = \cos\theta} f(y) dy$ and $G(t) = \int_0^t \sin^{n-2}\theta g(\theta) d\theta$.

The proof now proceeds in a manner similar to the proof of Theorem 3.4 of Stein and Weiss [6; p. 101].

(2.2)   If $f \in L_1(\Sigma_{n-1})$, then $\sup_{0 < r < 1} r \ln(1/r) |(rf)_r(rx)| \leq A_n f^*(x)$.

Proof.

$$r \ln(1/r) |(rf)_r(rx)| \leq r \ln(1/r) \int_{\Sigma_{n-1}} |f(y)| |(rP)_r(rx \cdot y)| \, dy$$

$$\leq B_n \int_{\Sigma_{n-1}} |f(y)| |P(rx \cdot y)| \, dy \leq A_n f^*(x)$$

by (2.1) above.

(2.3)   If $f \in L_1(\Sigma_{n-1})$, then $|\nabla f(rx)| \leq A_n f^*(x)$ if $r \leq \frac{1}{2}$ where ∇ is the gradient in $\mathbf{R}^n$.

Proof.   The proof is similar to proof of (2.2) above.

(2.4)   Let $f \in L_p(\Sigma_{n-1})$ and $F(rx) = [f(rx)]^p$ where $1 < p < \infty$. Then $|F_r(rx)| \leq A_{pn}(1-r)^{-1}(f^*(x))^p$.

Proof. $F_r(rx) = p[f(rx)]^{p-1} f_r(rx)$. As in (2.2), $|f_r(rx)| \leq A_n(1-r)^{-1} f^*(x)$. The result now easily follows.

DEFINITION.   Let $f \in L_1(\Sigma_{n-1})$. Define

$$g(f)(x) = \left[\int_0^1 r \ln(1/r) |\nabla f(rx)|^2 dr \right]^{1/2}.$$

REMARK.   The map $f \to g(f)$ is sublinear. This will enable us to apply the Marcinkiewicz Interpolation Theorem later in the chapter.

PROPOSITION 2.1.   *Let* $f \in L_p(\Sigma_{n-1})$, $1 < p < 2$. *Then* $g(f) \in L_p(\Sigma_{n-1})$ *and* $\|g(f)(x)\|_p \leq A_{p,n} \|f\|_p$.

Proof.   It suffices to prove this for f strictly positive. In view of (2.3) we need only consider $\left[\int_{1/2}^1 r \ln(1/r) |\nabla f(rx)|^2 dr \right]^{1/2}$.

Let $F(rx) = [f(rx)]^p$. An easy calculation shows that $|\nabla f(rx)|^2 = p^{-1}(p-1)^{-1}[f(rx)]^{2-p} \Delta[(f(rx))^p]$. So

$$\int_{1/2}^1 r \ln(1/r) |\nabla f(rx)|^2 dr = p^{-1}(p-1)^{-1} \int_{1/2}^1 r \ln(1/r) [f(rx)]^{-p} \Delta F(rx) dr$$

$$\leq [f^*(x)]^{2-p} A_{p,n} \int_{1/2}^1 r \ln(1/r) \Delta F(rx) dr.$$

Let $I(x) = \int_{1/2}^1 r \ln(1/r) \Delta F(rx) dr.$

$$\int_{\Sigma_{n-1}} I(x) dx = \int_{1/2}^1 r \ln(1/r) \int_{\Sigma_{n-1}} \Delta F(rx) dx dr$$

$$= \int_{1/2}^1 r \ln(1/r) \int_{\Sigma_{n-1}} r^{-n+1} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} F(rx) \right) dx dr$$

$$= \int_{\Sigma_{n-1}} \int_{1/2}^1 r \ln(1/r) r^{-n+1} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} F(rx) \right) dr dx.$$

Now

$$\int_{1/2}^{1} r \ln(1/r) r^{-n+1} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} F(rx) \right) dr$$

$$= \int_{1/2}^{1} r \ln(1/r) F_{rr}(rx) dr + (n-1) \int_{1/2}^{1} \ln(1/r) F_r(rx) dr.$$

The second integral is bounded by $B_{p,n} f^*(x)^p$ by (2.4) since $\ln(1/r) \leq A(1-r)$ if $r \geq 1/2$. Upon integrating the first by parts, one gets

$$\left| \int_{1/2}^{1} r \ln(1/r) r^{-n+1} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} F(rx) \right) dr \right| \leq C_{p,n} [f^*(x)]^p.$$

Thus

$$\int_{\Sigma_{n-1}} I(x) dx \leq C_{p,n} \int_{\Sigma_{n-1}} (f^*(x))^p dx \leq C'_{p,n} \|f\|_p^p.$$

Hence

$$\left\| \left[\int_{1/2}^{1} r \ln(1/r) |\nabla f(rx)|^2 dr \right]^{1/2} \right\|_p \leq A_{p,n} \left[\int_{\Sigma_{n-1}} [f^*(x)]^{1/2(2-p)p} [I(x)]^{p/2} dx \right]^{1/p}$$

$$\leq A_{p,n} \|f^*(x)\|_p^{1/2(2-p)p} \|f\|_p^{p/2}$$

by Hölder's inequality.

The above is bounded by $A''_{p,n} \|f\|_p$. The result for an arbitrary f now follows from an easy limiting argument.

PROPOSITION 2.2. *Let* $f \in L_p(\Sigma_{n-1})$, $2 \leq p < \infty$. *Then* $g(f) \in L_p(\Sigma_{n-1})$ *and* $\|g(f)\|_p \leq A_{p,n} \|f\|_p$.

Proof. By the Marcinkiewicz Interpolation Theorem it suffices to prove this for $p \geq 4$. Choose q such that $q^{-1} + (p/2)^{-1} = 1$. Then $1 < q \leq 2$. $\|g(f)\|_p^2 = \sup \int_{\Sigma_{n-1}} g^2(f)(x) h(x) dx$ where the sup is over all $h \geq 0$ such that $h \in L_q(\Sigma_{n-1})$ and $\|h\|_q = 1$.

$$\int_{\Sigma_{n-1}} g^2(f)(x) h(x) dx = \int_{\Sigma_{n-1}} \left[\int_0^1 r \ln(1/r) |\nabla f(rx)|^2 dr \right] h(x) dx.$$

Let $w(rx) = |\nabla f(rx)|^2$. Since each partial derivative is harmonic, w is subharmonic. Thus $w(r\rho x) \leq \int_{\Sigma_{n-1}} w(\rho y) P(rx \cdot y) dy$ since the right

side is harmonic in rx and converges to the continuous function $w(\rho x)$ as $r \to 1^-$. Hence $w(r^2 x) \leqq \displaystyle\int_{\Sigma_{n-1}} w(ry)P(rx \cdot y)dy$. Hence

$$\int_{\Sigma_{n-1}} g^2(f)(x)h(x)dy \leqq 4 \int_{\Sigma_{n-1}} \left[\int_0^1 r^3 \ln(1r)|\nabla f(rx)|^2 h(rx)dr \right] dx.$$

Now $|\nabla f(rx)|^2 = \tfrac{1}{2}\Delta f^2(rx)$ and

$$\Delta[f^2(rx)h(rx)] = f^2(rx)\Delta h(rx) + h(rx)\Delta f^2(rx) + \sum_{i=1}^n \frac{\partial}{\partial x_i} f^2(rx) \frac{\partial}{\partial x_i} h(rx)$$

where $x_1, \cdots, x_n$ are the Cartesian Coordinates. Since $h(rx)$ is harmonic, $\Delta h = 0$. Thus,

$$4 \int_{\Sigma_{n-1}} \left[\int_0^1 r^3 \ln(1/r)|\nabla f(rx)|^2 h(rx)dr \right] dx$$

$$\leqq 2 \left| \int_{\Sigma_{n-1}} \left[\int_0^1 r^3 \ln(1/r)\Delta[f^2(rx)h(rx)]dr \right] dx \right|$$

$$+ 4n \int_{\Sigma_{n-1}} \left[\int_0^1 r^3 \ln(1/r)|f(rx)||\nabla f(rx)||\nabla h(rx)| dr \right] dx.$$

The second integral is bounded by $A_n \displaystyle\int_{\Sigma_{n-1}} f^*(x)g(f)(x)g(h)(x)dx$ by Schwartz's Inequality since $|f(rx)| \leqq A_n' f^*(x)$. By Hölder's Inequality, this integral is bounded by $A_{p,n} \|f^*\|_p \|g(f)\|_p \|g(h)\|_q \leqq A_{p,n}' \|f\|_p \|g(f)\|_p$ since $\|g(h)\|_q \leqq B_{n,q} \|h\|_q = B_{n,q}$ for $1 < q \leqq 2$. The first integral equals

$$I = \left| \int_{\Sigma_{n-1}} \left[\int_0^1 r^3 \ln(1/r)r^{-n+1} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} [f^2(rx)h(rx)]dr \right) \right] dx \right|$$

since spherical part of the Laplacian vanishes.

$$\int_0^1 r^3 \ln(1/r)r^{-n+1} \frac{\partial}{\partial r} \left[r^{n-1} \frac{\partial}{\partial r} f^2(rx)h(rx) \right] dr$$

$$= \int_0^1 r^3 \ln(1/r) \frac{\partial^2}{\partial r^2} (f^2(rx)h(rx))dr$$

$$+ (n-1) \int_0^1 r^2 \ln(1/r) \frac{\partial}{\partial r} (f^2(rx)h(rx))dr.$$

In the second integral use $|r \ln(1/r)f_r(rx)| \leqq B_n f^*(x)$, $|r \ln(1/r)h_r(rx)| \leqq B_n h^*(x)$, $|f(rx)| \leqq B_n f^*(x)$, and $|h(rx)| \leqq B_n h^*(x)$.

Hence, the second integral is bounded by $B'_n f^*(x)^2 h^*(x)$. The first integral equals

$$r^3 \ln(1/r) \frac{\partial}{\partial r} (f^2(rx)h(rx))|_0^1 - \int_0^1 [3r^2 \ln(1/r) - r^2] \frac{\partial}{\partial r} (f^2(rx)h(rx))dr$$

$$= r^3 \ln(1/r) \frac{\partial}{\partial r} [f^2(rx)h(rx)]|_0^1$$

$$- [3r^2 \ln(1/r) - r^2][f^2(rx)h(rx)]|_0^1$$

$$+ \int_0^1 [6r \ln(1/r) - 5r][f^2(rx)h(rx)]dr.$$

Using the above estimates one gets that the absolute value is bounded by $B''_n f^*(x)^2 h^*(x)$. So

$$I \leqq B'''_n \int_{\Sigma_{n-1}} f^*(x)^2 h(x)dx \leqq B_{n,p} \|f^*\|_p^2 \|h^*\|_q \leqq C_{n,p} \|f\|_p^2.$$

Thus $\quad \|g(f)\|_p^2 \leqq A'_{p,n} \|f\|_p \|g(f)\|_p + C_{n,p} \|f\|_p^2$. Hence $\quad \|g(f)\|_p \leqq$ $C'_{n,p} \|f\|_p$.

DEFINITION.

$$g_p(f)(x) = \left[\int_0^1 (r \ln(1/r)|(rf)_r(rx)|)^p dr/(r \ln(1/r)) \right]^{1/p}$$

for $f(rx)$ harmonic on B_n if $1 \leqq p < \infty$.

$$g_\infty(f)(x) = \sup_{0 < r < 1} r \ln(1/r)|(rf)_r(rx)|.$$

PROPOSITION 2.3. *Let* $f \in L_p(\Sigma_{n-1})$ *and* $q = \max[p, 2]$. *Then for* $1 < p \leqq \infty$, $\|g_q(f)\|_p \leqq A_{p,n} \|f\|_p$.

Proof. *Case I.* $1 < p \leqq 2$. Then $q = 2$ and

$$g_2(f)(x) \leqq \left[\int_0^1 r^2 \ln(1/r)|f_r(rx)|^2 dr \right]^{1/2} + \left[\int_0^1 r \ln(1/r)|f(rx)|^2 dr \right]^{1/2}.$$

The second integral is bounded by $B_{p,n}f^*(x)$. Since $|f_r(rx)| \leqq$ $|\nabla f(rx)|$, the first integral is bounded by $g(f)(x)$.

Thus $g_2(f)(x) \leqq B_{p,n}f^*(x) + g(f)(x)$ and so by Proposition 2.1, $g_2(f)(x) \in L_p(\Sigma_{n-1})$ and $\|g_2(f)(x)\|_p \leqq B_{p,n}\|f^*\|_p + \|g(f)(x)\|_p \leqq A_{p,n}\|f\|_p$.

Case II. $2 \leqq p \leqq \infty$.   Then $q = p$.   If $p = \infty$,

$$\|g(f)(x)\|_\infty = \left\| \sup_{0 < r < 1} r \ln(1/r) |(rf)_r(rx)| \right\|_{\infty, dx} \leqq A_{p,n} \|f\|_\infty.$$

If $p < \infty$, then

$$g_p(f)(x) = \left[\int_0^1 |r \ln(1/r)(rf)_r(rx)|^2 |r \ln(1/r)(rf)_r(rx)|^{p-2} dr/r \ln(1/r) \right]^{1/2}$$

$$\leqq C_{p,n}[f^*(x)]^{(p-2)/2} \left[\int_0^1 |r \ln(1/r)(rf)_r(rx)|^2 dr/r \ln(1/r) \right]^{1/p}$$

$$= C_{p,n}[f^*(x)]^{(p-2)/2} g_2(f)(x)^{2/p}.$$

Hence $\|g_p(f)\|_p^p \leqq C_{p,n} \|f^*\|_p^{(p-2)/2} \|g_2(f)\|_p^{2/p} \leqq A_{p,n} \|f\|_p$ by Hölder's inequality, Proposition 2.2, and the argument above in Case I.   We have the following converse.

PROPOSITION 2.4.   *Let $f(rx)$ be harmonic on B_n.   Let $q = \min[p, 2]$ and $g_q(f) \in L_p(\Sigma_{n-1})$. $f(rx)$ is the Poisson integral of a function $f(x) \in L_p(\Sigma_{n-1})$ and $\|f\|_p \leqq A_{p,n} \|g_q(f)\|_p$ for $1 \leqq p < \infty$.*

We will need the following lemmas.

LEMMA 2.1.   *Let $f \in L_p(\Sigma_{n-1})$ and $h \in L_{p'}(\Sigma_{n-1})$ where $1/p + 1/p' = 1$ and $1 < p < \infty$.   Then*

$$\int_{\Sigma_{n-1}} f(x)g(x)dx = 4 \int_{\Sigma_{n-1}} \int_0^1 r \ln(1/r)(rf)_r(rx)(rh)_r(rx)drdx.$$

Proof.   It suffices to prove this for $f \in L_p(\Sigma_{n-1})$ and $g \in C^\infty(\Sigma_{n-1})$ since $C^\infty(\Sigma_{n-1})$ is dense in $L_{p'}(\Sigma_{n-1})$.   Let $0 < s < 1$.   Then

$$\int_{\Sigma_{n-1}} f(x)h(\dot{x})dx = \lim_{s \to 1^-} \int_{\Sigma_{n-1}} f(sx)h(sx)dx.$$

Let $f(sx) = \Sigma_{k,l} a_{kl} S^k Y_l^{(k)}(x)$ and $h(sx) = \Sigma_{k,l} b_{kl} S^k Y_l^{(k)}(x)$ where $b_{kl} = O(k^{-t})$ for all t since $h \in C^\infty(\Sigma_{n-1})$.

Thus $\int_{\Sigma_{n-1}} f(x)g(x)dx = \Sigma_{k,l} a_{kl} b_{kl}$.

An easy argument shows that

$$4 \int_{\Sigma_{n-1}} \int_0^1 r \ln(1/r)(rf)_r(rx)(rh)_r(rx)drdx = \sum_{k,l} a_{kl} b_{kl}.$$

REMARK. This Lemma shows that the map $f \to \frac{1}{2} g_2(f)$ is an isometry of $L_2(\Sigma_{n-1})$.

LEMMA 2.2. *Let $f \in L_p(\Sigma_{n-1})$ and $f(rx)$ be its Poisson integral. Let $q = \min[p, 2]$ and $g_q(f) \in L_p(\Sigma_{n-1})$. Then $\|f\|_p \leq A_{p,n} \|g_p(f)\|_p$ for $1 \leq p < \infty$.*

Proof. Case I. $p = 1$. $rf(rx) = \int_0^r (\rho f)_\rho (\rho x) d\rho$.

So $|rf(rx)| \leq \int_0^1 |(\rho f)_\rho (\rho x)| d\rho \leq g_1(f)(x)$. Thus $\|rf(rx)\|_{1,dx} \leq \|g_1(f)\|_1$ and so $\|f\|_1 \leq \|g_1(f)\|_1$.

Case II. $1 < p \leq \infty$. Then $q = p$. $\|f\|_p = \sup \int_{\Sigma_{n-1}} f(x) h(x) dx$ where the sup is over all $h \in L_{p'}(\Sigma_{n-1})$ where $\|h\|_{p'} = 1$ and $1/p + 1/p' = 1$.
By Lemma 2.1,

$$\left| \int_{\Sigma_{n-1}} f(x) h(x) dx \right| = \left| 4 \int_{\Sigma_{n-1}} \int_0^1 r \ln(1/r) (rf)_r (rx) (rh)_r (rx) dr dx \right|$$

$$\leq 4 \int_{\Sigma_{n-1}} g_p(f)(x) g_{p'}(f)(x) dx \leq 4 \|g_p(f)\|_p \|g_{p'}(h)\|_{p'}$$

$$\leq A_{p,n} \|g_p(f)\|_p \|h\|_{p'} = A_{n,p} \|g_p(f)\|_p$$

by Proposition 2.3 since $p' \geq 2$.

Case III. $2 \leq p < \infty$. Then $q = 2$. An argument similar to the one above applies here.

Proof of Proposition 2.4.

Case I. $p = 1$. In view of Lemma 2.2, it suffices to show that $f(rx)$ is the Poisson Integral for a function in $L_1(\Sigma_{n-1})$. $rf(rx) = \int (\rho f)_\rho (\rho x) d\rho$ and so $\|rf(rx)\|_{1,dx} \leq \|g_1(f)(x)\|_1$. The result now follows from an easy argument.

Case II. $1 < p < \infty$. In view of Lemma 2.2, it suffices to show that $f(rx)$ is the Poisson Integral of a function in $L_p(\Sigma_{n-1})$. An easy variant of Lemma 2.2 shows that if $f \in L_p(\Sigma_{n-1})$, then $\|f\|_p \leq A_{p,n} \|g_2(f)\|_p$ for $1 < p < \infty$. A straightforward argument finishes this part of the proof.

We have the following easy corollaries.

COROLLARY 2.1. *Let $f(rx)$ be harmonic for $x \in \Sigma_{n-1}$, $0 \leqq r < 1$, and let $g_2(f) \in L_p(\Sigma_{n-1})$ for $1 < p < \infty$. Then $f(rx)$ is the Poisson integral of a function $f(x) \in L_p(\Sigma_{n-1})$ and $\|f(x)\|_p \leqq A_{p,n} \|g_2(f)(x)\|_p$.*

COROLLARY 2.2. *Let $f(rx)$ be harmonic for $x \in \Sigma_{n-1}$, $0 \leqq r < 1$, and let $g(f) \in L_p(\Sigma_{n-1})$ for $1 < p < \infty$. Then $f(rx)$ is the Poisson integral of a function $f(x) \in L_p(\Sigma_{n-1})$ and $\|f(x)\|_p \leqq A_{p,n} \|g(f)(x)\|_p$.*

PROPOSITION 2.5. (a) $L_{p,\alpha} \subset \Lambda(\alpha; p, q)$, $q = \max[p, 2]$, $1 < p \leqq \infty$.
  (b) $\Lambda(\alpha; p, q) \subset L_{p,\alpha}$, $q = \min[p, 2]$, $1 \leqq p < \infty$.
  (c) $L_{1,\alpha} \subset \Lambda(\alpha; 1, \infty)$
  (d) $\Lambda(\alpha; \infty, 1) \subset L_{\infty,\alpha}$.
The inclusion maps are continuous.

Proof. With minor changes the proof of Theorem 15 of Taibleson [7; p. 452] can be apllied here to give the result.

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CALIFORNIA POLYTECHNIC STATE UNIVERSITY — SAN LUIS OBISPO

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