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## **ASYMPTOTIC PROPERTIES OF NONOSCILLATORY SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT**

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# ASYMPTOTIC PROPERTIES OF NONOSCILLATORY SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Recently, there is an increasing interest in studying the  $n$ th order differential equations involving the so called  $n$ th order  $r$ -derivative of  $x$

$$(r_{n-1}(t)(r_{n-2}(t)(\cdots(r_1(t)(r_0(t)x(t)))'\cdots))')'$$

which causes damped terms. Here, the asymptotic behavior of nonoscillatory solutions of such general differential equations with deviating argument is studied and, more precisely, sufficient conditions which guarantee that

$$\lim_{t \rightarrow \infty} x(t) = 0$$

for the bounded nonoscillatory solutions  $x(t)$  are established. A basic theorem is obtained for the general case and then it is specialized into four corollaries concerning the particular case

$$r_j = 1 \quad \text{for } j \neq n - N \quad \text{and} \quad r_{n-N} = r \quad (1 \leq N \leq n - 1)$$

which is of special interest. Finally, some examples are given to illustrate the significance of the results.

In this paper we consider the  $n$ th order ( $n > 1$ ) differential equation with deviating argument of the form

$$(E) \quad \begin{aligned} & (r_{n-1}(t)(r_{n-2}(t)(\cdots(r_1(t)(r_0(t)x(t)))'\cdots))')' \\ & + a(t)F(x[\sigma(t)]) = b(t), \quad t \geq t_0 \end{aligned}$$

where the functions  $r_i$  ( $i = 0, 1, \cdots, n - 1$ ) are supposed positive at least on the interval  $[t_0, \infty)$ . The continuity of the functions involved in the above equation (E) as well as sufficient smoothness to guarantee the existence of solutions of (E) on an infinite subinterval of  $[t_0, \infty)$  will be assumed without mention. In what follows the term "solution" is always used only for such solutions  $x(t)$  of (E) which are defined for all large  $t$ . The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form  $[T, \infty)$

is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*.

Furthermore, the conditions (i) and (ii) below are assumed to hold throughout the paper:

- (i)  $\lim_{t \rightarrow \infty} \sigma(t) = \infty$
- (ii)  $y \neq 0 \Rightarrow yF(y) > 0$ .

The results of this paper are included in a general theorem given in §2, which establishes sufficient conditions in order that all bounded nonoscillatory solutions  $x(t)$  of the differential equation (E) tend to zero as  $t \rightarrow \infty$ . This theorem extends a recent result given by the authors in [9, Theorem 3] concerning also the equation (E). Also, it unifies and extends two results by Kusano and Onose [5, Theorems 3 and 4] concerning, in particular, the differential equation

$$(D_N) \quad [r(t)x^{(n-N)}(t)]^{(N)} + a(t)F(x[\sigma(t)]) = b(t), \quad t \geq t_0,$$

where  $1 \leq N \leq n-1$  and the function  $r$  is continuous and positive at least on the interval  $[t_0, \infty)$ .

The technique used in the proof of our theorem is based on three lemmas which are given in §1. Lemma 1 is fundamental and has been proved by the authors in [9], Lemma 2 is proved here and it is an extension of a result due to Hardy and Littlewood [4], while Lemma 3 is new and it is rather technical.

In §3 our main theorem is specialized for the differential equation  $(D_N)$ , which is obtained from the equation (E) by setting

$$r_j = 1 \quad \text{for } j \neq n-N \quad \text{and} \quad r_{n-N} = r$$

and which is of special interest. Thus, four corollaries are obtained, from which Corollary 1 is the main result of a recent paper by Kusano and Onose [5] while Corollaries 2, 3 and 4 are new and are illustrated by examples.

**1. Preliminaries.** Let  $q_i$  ( $i = 0, 1, \dots, m$ ) be positive continuous functions on an interval  $I$  of the real line. For a real-valued function  $h$  on  $I$  and any  $\mu = 0, 1, \dots, m$  we define the  $\mu$ th  $q$ -derivative of  $h$  by the formula

$$D_q^{(\mu)} h = q_\mu (q_{\mu-1} (q_{\mu-2} (\dots (q_1 (q_0 h))' \dots)')')$$

when obviously we have

$$D_q^{(0)} h = q_0 h$$

and

$$D_q^{(j)}h = q_j(D_q^{(j-1)}h)' \quad (j = 1, 2, \dots, m).$$

Moreover, if  $D_q^{(m)}h$  is defined as a continuous function on  $I$ , then  $h$  is said to be  $m$ -times *continuously  $q$ -differentiable*. We note that in the case where  $q_0 = q_1 = \dots = q_m = 1$  the above notion of  $q$ -differentiability specializes to the usual one.

By using this shorthand notation, the differential equation (E) can be written

$$(E) \quad (D_{r_n}^{(n)}x)(t) + a(t)F(x[\sigma(t)]) = b(t), \quad t \geq t_0$$

where  $r_n = 1$ .

Now, let  $\rho$  be a real-valued function which is defined and positive at least on the interval  $(t_0, \infty)$  and let  $R_i$  ( $i = 0, 1, \dots, n$ ) be the functions defined as follows:

$$R_n = \rho$$

and for every  $j = n-1, n-2, \dots, 0$

$$R_j = r_j R'_{j+1}.$$

The function  $\rho$  is said to be *of the type  $r[k]$* ,  $0 \leq k \leq n-1$ , if:

- ( $\alpha$ ) the functions  $R_j$  ( $j = k+1, \dots, n$ ) are defined at least on  $(t_0, \infty)$ ,
- ( $\beta$ )  $R_{k+1}$  is a constant nonzero function on  $(t_0, \infty)$ ,
- ( $\gamma$ ) if  $k < n-1$ , then for every  $j = k+2, \dots, n$

$$\lim_{t \rightarrow \infty} R_j(t) \text{ exists in } \{-\infty, 0, +\infty\}$$

and

- ( $\delta$ ) if  $k < n-2$ , then for every  $j = k+2, \dots, n-1$

$$R_j(t) \neq 0 \text{ for all } t > t_0.$$

For some interesting examples of functions of the above type we refer to [9].

The technique used here is based on the following lemma, which has been proved by the authors in [9].

**LEMMA 1.** *Let  $\rho$  be a function of the type  $r[k]$ ,  $0 \leq k \leq n-1$ , and  $h$  an  $n$ -times continuously  $r$ -differentiable function of  $[T, \infty)$ ,  $T > t_0$ . If the improper integral*

$$\int_T^\infty \rho(t)(D_r^{(n)}h)(t)dt$$

exists in the extended real line  $\mathbf{R}^*$ , then so does the  $\lim_{t \rightarrow \infty} (D_r^{(k)}h)(t)$ . Moreover,

$$\int_T^\infty \rho(t)(D_r^{(n)}h)(t)dt = \pm \infty \quad \text{implies} \quad \lim_{t \rightarrow \infty} |(D_r^{(k)}h)(t)| = \infty.$$

In order to obtain our results we need further the following lemma which is an extension of a result due to Hardy and Littlewood [4].

LEMMA 2. Let  $q_i$  ( $i = 0, 1, \dots, m$ ), where  $m > 1$ , be positive continuous functions on an interval  $[T, \infty)$  such that

$$\liminf_{t \rightarrow \infty} q_i(t) > 0 \quad (i = 1, 2, \dots, m)$$

and

$$\limsup_{t \rightarrow \infty} q_i(t) < \infty \quad (i = 1, 2, \dots, m-1).$$

Moreover, let  $h$  be an  $m$ -times continuously  $q$ -differentiable function on  $[T, \infty)$ .

If  $D_q^{(0)}h$  is bounded on  $[T, \infty)$  and

$$\lim_{t \rightarrow \infty} (D_q^{(m)}h)(t) = 0,$$

then

$$\lim_{t \rightarrow \infty} (D_q^{(j)}h)(t) = 0 \quad (j = 1, 2, \dots, m-1).$$

The above lemma follows immediately from the following proposition, which in the particular case  $q_0 = q_1 = \dots = q_m = 1$  is an improved version of a result due to Landau [6] (cf. also Coppel [2, p. 140]).

PROPOSITION. Let  $q_i$  ( $i = 0, 1, \dots, m$ ), where  $m > 1$ , be positive continuous functions on an interval  $I$  such that

$$A \equiv \min_{i=1, \dots, m} \inf_{t \in I} q_i(t) > 0 \quad \text{and} \quad B \equiv \max_{i=1, \dots, m-1} \sup_{t \in I} q_i(t) < \infty.$$

Moreover, let  $h$  be an  $m$ -times continuously  $q$ -differentiable function on  $I$  with

$$|(D_q^{(0)}h)(t)| \leq K \quad \text{and} \quad |(D_q^{(m)}h)(t)| \leq M \quad \text{for every } t \in I.$$

If  $L$  is the length of the interval  $I$  and

$$(a) \quad I \text{ is closed and } L \geq 2A \left( \frac{K}{M} \right)^{1/m}$$

or

$$(b) \quad L > 2A \left( \frac{K}{M} \right)^{1/m},$$

then for every  $t \in I$

$$|(D_q^{(j)}h)(t)| \leq c_m K^{1-j/m} M^{j/m} \quad (j = 1, 2, \dots, m-1),$$

where

$$c_m \equiv \left( \frac{2B}{A} \right)^{2^{m-2}}.$$

*Proof.* It suffices to prove the proposition in the case where (a) is satisfied. Indeed, if (b) holds, then for any  $t \in I$  we can choose a closed subinterval  $J$  of  $I$  with length  $L' \geq 2A(K/M)^{1/m}$  and  $t \in J$ . So, applying the proposition for the closed interval  $J$  we obviously obtain

$$|(D_q^{(j)}h)(t)| \leq c_m K^{1-j/m} M^{j/m} \quad (j = 1, 2, \dots, m-1).$$

Now, we suppose that (a) is satisfied and define

$$S = \max \left\{ c_m, \max_{0 < j < m} \max_{t \in I} \frac{|(D_q^{(j)}h)(t)|}{K^{1-j/m} M^{j/m}} \right\}.$$

Obviously, for every  $j = 0, 1, \dots, m-1$

$$1 \leq \left( \frac{2B}{A} \right)^{2^{j+1}-2} \leq \left( \frac{2B}{A} \right)^{2^m-2} = c_m \leq S,$$

by which, after some manipulations, we derive that

$$(1) \quad \left( \frac{4B^2}{A^2} S \right)^{\gamma_j} \leq S,$$

where  $\gamma_j = 1 - 2^{-j}$ .

We shall prove next that for  $j = 0, 1, \dots, m-1$

$$(2) \quad |(D_q^{(j)}h)(t)| \leq \left(\frac{4B^2}{A^2} S\right)^{j_l} K^{1-j/m} M^{j/m} \equiv K_j \text{ for every } t \in I.$$

Indeed, (2) is valid for  $j = 0$ , since

$$|(D_q^{(0)}h)(t)| \leq K = K_0 \text{ for every } t \in I.$$

We suppose that (2) is satisfied for  $j = l$ ,  $0 \leq l < m - 1$ , i.e.

$$(3) \quad |(D_q^{(l)}h)(t)| \leq K_l \text{ for every } t \in I.$$

For this  $l$  we have that

$$(4) \quad |(D_q^{(l+2)}h)(t)| \leq SK^{1-(l+2)/m} M^{(l+2)/m} \equiv M_l \text{ for every } t \in I$$

which, by the definition of  $S$ , is obvious for  $l < m - 2$  and follows from the inequality

$$|(D_q^{(m)}h)(t)| \leq M \leq MS = M_{m-2} \text{ for every } t \in I,$$

when  $l = m - 2$ . Since, by (1),

$$\left(\frac{K_l}{M_l}\right)^{1/2} \leq \left(\frac{K}{M}\right)^{1/m}$$

and because of (a), there exist  $t_1, t_2$  in  $I$  with  $t_2 - t_1 = 2A(K_l/M_l)^{1/2}$  and  $t_1 \leq T_0 \leq t_2$ , where  $T_0$  is such that

$$|(D_q^{(l+1)}h)(T_0)| = \max_{t \in I} |(D_q^{(l+1)}h)(t)|.$$

It is easy to verify that

$$\begin{aligned} (D_q^{(l)}h)(t_2) - (D_q^{(l)}h)(T_0) &= (D_q^{(l+1)}h)(T_0) \int_{T_0}^{t_2} \frac{ds}{q_{l+1}(s)} \\ &\quad + \int_{T_0}^{t_2} \frac{1}{q_{l+1}(s)} \int_{T_0}^s \frac{1}{q_{l+2}(w)} (D_q^{(l+2)}h)(w) dw ds \end{aligned}$$

and

$$\begin{aligned} (D_q^{(l)}h)(t_1) - (D_q^{(l)}h)(T_0) &= (D_q^{(l+1)}h)(T_0) \int_{T_0}^{t_1} \frac{ds}{q_{l+1}(s)} \\ &\quad + \int_{T_0}^{t_1} \frac{1}{q_{l+1}(s)} \int_{T_0}^s \frac{1}{q_{l+2}(w)} (D_q^{(l+2)}h)(w) dw ds, \end{aligned}$$

when, by subtraction, we get

$$\begin{aligned} (D_q^{(l)}h)(t_2) - (D_q^{(l)}h)(t_1) &= (D_q^{(l+1)}h)(T_0) \int_{t_1}^{t_2} \frac{ds}{q_{l+1}(s)} \\ &\quad + \int_{T_0}^{t_2} \frac{1}{q_{l+1}(s)} \int_{T_0}^s \frac{1}{q_{l+2}(w)} (D_q^{(l+2)}h)(w) dw ds \\ &\quad - \int_{t_1}^{T_0} \frac{1}{q_{l+1}(s)} \int_s^{T_0} \frac{1}{q_{l+2}(w)} (D_q^{(l+2)}h)(w) dw ds. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} |(D_q^{(l+1)}h)(T_0)| \int_{t_1}^{t_2} \frac{ds}{q_{l+1}(s)} &\leq |(D_q^{(l)}h)(t_1)| + |(D_q^{(l)}h)(t_2)| \\ &\quad + \int_{t_1}^{T_0} \frac{1}{q_{l+1}(s)} \int_s^{T_0} \frac{1}{q_{l+2}(w)} |(D_q^{(l+2)}h)(w)| dw ds \\ &\quad + \int_{T_0}^{t_2} \frac{1}{q_{l+1}(s)} \int_{T_0}^s \frac{1}{q_{l+2}(w)} |(D_q^{(l+2)}h)(w)| dw ds \end{aligned}$$

and consequently, by (3) and (4),

$$\begin{aligned} |(D_q^{(l+1)}h)(T_0)| &\leq \frac{1}{\int_{t_1}^{t_2} \frac{ds}{q_{l+1}(s)}} \left\{ 2K_l + M_l \left[ \int_{t_1}^{T_0} \frac{1}{q_{l+1}(s)} \int_s^{T_0} \frac{1}{q_{l+2}(w)} dw ds \right. \right. \\ &\quad \left. \left. + \int_{T_0}^{t_2} \frac{1}{q_{l+1}(s)} \int_{T_0}^s \frac{1}{q_{l+2}(w)} dw ds \right] \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} |(D_q^{(l+1)}h)(T_0)| &\leq \frac{B}{t_2 - t_1} \left\{ 2K_l + \frac{M_l}{2A^2} [(T_0 - t_1)^2 + (t_2 - T_0)^2] \right\} \\ &\leq \frac{B}{t_2 - t_1} \left[ 2K_l + \frac{M_l}{2A^2} (t_2 - t_1)^2 \right] \\ &= B \left[ \frac{2K_l}{t_2 - t_1} + \frac{M_l}{2A^2} (t_2 - t_1) \right] \\ &= \frac{2B}{A} (K_l M_l)^{1/2}. \end{aligned}$$

But, it is a matter of calculations to verify that

$$\frac{2B}{A} (K_l M_l)^{1/2} = K_{l+1},$$



when

$$(D_q^{(l+1)}h)(T_0) \leq K_{l+1},$$

which proves (2) for  $j = l + 1$  and consequently for all  $j = 0, 1, \dots, m - 1$ .

From (2) it follows that

$$\max_{0 < j < m} \max_{t \in I} \frac{|(D_q^{(j)}h)(t)|}{K^{1-j/m} M^{j/m}} \leq \left( \frac{4B^2}{A^2} S \right)^{\gamma_{m-1}}.$$

Also,

$$c_m = \left( \frac{2B}{A} \right)^{2^{m-2}} = \left( \frac{4B^2}{A^2} c_m \right)^{\gamma_{m-1}} \leq \left( \frac{4B^2}{A^2} S \right)^{\gamma_{m-1}}.$$

Therefore, by the definition of  $S$ ,

$$S \leq \left( \frac{4B^2}{A^2} S \right)^{\gamma_{m-1}},$$

which gives

$$S \leq \left( \frac{2B}{A} \right)^{2^{m-2}} = c_m, \quad \text{i.e.} \quad S = c_m.$$

Hence, by the definition of  $S$ , we have that for every  $t \in I$

$$|(D_q^{(j)}h)(t)| \leq c_m K^{1-j/m} M^{j/m} \quad (j = 1, 2, \dots, m - 1).$$

Finally, for our purpose we need one more lemma.

LEMMA 3. *Let  $q_i$  ( $i = 0, 1, \dots, \lambda$ ) be positive continuous functions on an interval  $[T, \infty)$  such that for  $\lambda > 0$ ,*

$$(I) \quad \int_T^\infty \frac{dt}{q_i(t)} = \infty \quad (i = 1, 2, \dots, \lambda)$$

and let

$$Q_{0\lambda}(t; T) = \begin{cases} 1, & \text{if } \lambda = 0 \\ \int_T^t \frac{1}{q_1(s_1)} \int_T^{s_1} \frac{1}{q_2(s_2)} \cdots \int_T^{s_{\lambda-1}} \frac{1}{q_\lambda(s_\lambda)} ds_\lambda \cdots ds_2 ds_1, & \text{if } \lambda > 0. \end{cases}$$

Moreover, let  $h$  be a  $\lambda$ -times continuously  $q$ -differentiable function on the interval  $[T, \infty)$ . Then we have:

( $\alpha$ ) If

$$\liminf_{t \rightarrow \infty} (D_q^{(\lambda)} h)(t) > 0 \quad \text{or} \quad \limsup_{t \rightarrow \infty} (D_q^{(\lambda)} h)(t) < 0$$

and

$$(II) \quad \int^\infty \frac{Q_{0\lambda}(t; T)}{q_0(t)} dt = \infty,$$

then

$$\int^\infty h(t) dt = \pm \infty.$$

( $\beta$ ) If

$$\lim_{t \rightarrow \infty} (D_q^{(\lambda)} h)(t) = 0$$

and

$$(III) \quad \limsup_{t \rightarrow \infty} \frac{Q_{0\lambda}(t; T)}{q_0(t)} < \infty,$$

then

$$\lim_{t \rightarrow \infty} h(t) = 0.$$

*Proof.* Since the lemma is obvious for  $\lambda = 0$ , we suppose that  $\lambda > 0$ . For any integers  $i$  and  $j$ ,  $0 \leq i \leq j \leq \lambda$ , and for every  $u$  and  $v$  with  $v \geq u \geq T$ , we define

$$Q_{ij}(v; u) = \begin{cases} 1, & \text{if } i = j \\ \int_u^v \frac{1}{q_{i+1}(s_{i+1})} \int_u^{s_{i+1}} \frac{1}{q_{i+2}(s_{i+2})} \cdots \int_u^{s_{j-1}} \frac{1}{q_j(s_j)} ds_j \cdots ds_{i+2} ds_{i+1}, & \text{if } i < j. \end{cases}$$

Then, taking into account condition (I), we have that for every  $T^* \geq T$ ,

$$\lim_{t \rightarrow \infty} Q_{ij}(t; T^*) = \infty \quad (0 \leq i < j \leq \lambda)$$

and consequently, by L'Hospital's rule,

$$(5) \quad \lim_{t \rightarrow \infty} \frac{Q_{0j}(t; T^*)}{Q_{0\lambda}(t; T^*)} = 0 \quad (j = 0, 1, \dots, \lambda - 1)$$

and

$$(6) \quad \lim_{t \rightarrow \infty} \frac{Q_{0\lambda}(t; T^*)}{Q_{0\lambda}(t; T)} = 1.$$

Moreover, for every  $t$  and  $T^*$  with  $t \geq T^* \geq T$ , it is easy to derive the following generalized Taylor's formula

$$(7) \quad \begin{aligned} (D_q^{(0)}h)(t) &= \sum_{j=0}^{\lambda-1} (D_q^{(j)}h)(T^*) Q_{0j}(t; T^*) \\ &+ \int_{T^*}^t \frac{1}{q_1(s_1)} \int_{T^*}^{s_1} \frac{1}{q_2(s_2)} \cdots \int_{T^*}^{s_{\lambda-1}} \frac{1}{q_\lambda(s_\lambda)} (D_q^{(\lambda)}h)(s_\lambda) ds_\lambda \cdots ds_2 ds_1, \end{aligned}$$

( $\alpha$ ) Let  $d > 0$  and  $T^* > T$  be chosen so that

$$(D_q^{(\lambda)}h)(t) \geq d \text{ for every } t \geq T^*$$

or

$$(D_q^{(\lambda)}h)(t) \leq -d \text{ for every } t \geq T^*.$$

Then from the formula (7) we have respectively

$$\frac{(D_q^{(0)}h)(t)}{Q_{0\lambda}(t; T)} \geq \sum_{j=0}^{\lambda-1} (D_q^{(j)}h)(T^*) \frac{Q_{0j}(t; T^*)}{Q_{0\lambda}(t; T)} + d \frac{Q_{0\lambda}(t; T^*)}{Q_{0\lambda}(t; T)}, \quad t \geq T^*$$

or

$$\frac{(D_q^{(0)}h)(t)}{Q_{0\lambda}(t; T)} \leq \sum_{j=0}^{\lambda-1} (D_q^{(j)}h)(T^*) \frac{Q_{0j}(t; T^*)}{Q_{0\lambda}(t; T)} - d \frac{Q_{0\lambda}(t; T^*)}{Q_{0\lambda}(t; T)}, \quad t \geq T^*$$

and hence, by virtue of (5) and (6), we obtain

$$\liminf_{t \rightarrow \infty} \frac{(D_q^{(0)}h)(t)}{Q_{0\lambda}(t; T)} > 0 \quad \text{or} \quad \limsup_{t \rightarrow \infty} \frac{(D_q^{(0)}h)(t)}{Q_{0\lambda}(t; T)} < 0.$$

But this, by condition (II), gives

$$\int^{\infty} h(t) dt = \pm \infty.$$

( $\beta$ ) For any arbitrary  $\epsilon > 0$ , we consider a  $T^* > T$  so that

$$|(D_q^{(\lambda)} h)(t)| < \epsilon \text{ for every } t \geq T^*,$$

when from the formula (7) we obtain

$$\frac{|(D_q^{(0)} h)(t)|}{Q_{0\lambda}(t; T)} \leq \sum_{j=0}^{\lambda-1} |(D_q^{(j)} h)(T^*)| \frac{Q_{0j}(t; T^*)}{Q_{0\lambda}(t; T)} + \epsilon \frac{Q_{0\lambda}(t; T^*)}{Q_{0\lambda}(t; T)}$$

for every  $t \geq T^*$ . Thus, by (5) and (6), we have

$$\limsup_{t \rightarrow \infty} \frac{|(D_q^{(0)} h)(t)|}{Q_{0\lambda}(t; T)} \leq \epsilon$$

and, since  $\epsilon$  is arbitrary,

$$\lim_{t \rightarrow \infty} \frac{(D_q^{(0)} h)(t)}{Q_{0\lambda}(t; T)} = 0,$$

which, by condition (III), gives

$$\lim_{t \rightarrow \infty} h(t) = 0.$$

**2. The main result.** Our main theorem establishes conditions which essentially guarantee that

$$\lim_{t \rightarrow \infty} x(t) = 0$$

for the bounded nonoscillatory solutions  $x$  of the equation (E).

**THEOREM.** Consider the differential equation (E) subject to the conditions (i), (ii) and

(iii)  $\limsup_{t \rightarrow \infty} r_0(t) < \infty$ .

Let  $m$  and  $k$  be integers with  $1 \leq m \leq k \leq n - 1$  so that the conditions (iv), (v) and (vi) below are satisfied:

(iv) If  $m < k$ , then for every  $j = m + 1, \dots, k$

$$\int^{\infty} \frac{dt}{r_j(t)} = \infty.$$

$$(v) \quad \int^{\infty} \frac{R_{mk}(t)}{r_m(t)} dt = \infty \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{R_{mk}(t)}{r_m(t)} < \infty,$$

where

$$R_{mk}(t) = \begin{cases} 1, & \text{if } m = k \\ \int_{t_0}^t \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{k-1}} \frac{1}{r_k(s_k)} ds_k \cdots ds_{m+2} ds_{m+1}, & \text{if } m < k. \end{cases}$$

(vi) If  $m > 1$ , then for every  $i = 1, 2, \dots, m-1$

$$0 < \liminf_{t \rightarrow \infty} r_i(t) \leq \limsup_{t \rightarrow \infty} r_i(t) < \infty.$$

Moreover, let there exist a function  $\rho$  of the type  $r[k]$  such that:

$$(C_1) \quad \int^{\infty} \rho(t) |b(t)| dt < \infty$$

and

(C<sub>2</sub>) For some  $\delta > 0$ ,

$$\liminf_{t \rightarrow \infty} \int_t^{t+\delta} \rho(s) a^+(s) ds > 0 \quad \text{and} \quad \int^{\infty} \rho(t) a^-(t) dt < \infty$$

or

$$\liminf_{t \rightarrow \infty} \int_t^{t+\delta} \rho(s) a^-(s) ds > 0 \quad \text{and} \quad \int^{\infty} \rho(t) a^+(t) dt < \infty,$$

where

$$a^+(t) = \max\{a(t), 0\} \quad \text{and} \quad a^-(t) = \max\{-a(t), 0\}.$$

If the function  $\sigma$  is differentiable with bounded derivative on  $[t_0, \infty)$ , then for all bounded nonoscillatory solutions  $x$  of the equation (E),

$$\lim_{t \rightarrow \infty} (D_r^{(k)} x)(t) = 0 = \lim_{t \rightarrow \infty} (D_r^{(m-1)} x)'(t)$$

and

$$\lim_{t \rightarrow \infty} (D_r^{(l)}x)(t) = 0 \quad (l = 0, 1, \dots, m-1).$$

*Proof.* Let  $x$  be a bounded nonoscillatory solution of (E). Without loss of generality, we assume that  $x$  is a solution on the whole interval  $[t_0, \infty)$ . Moreover, this solution is supposed positive on  $[t_0, \infty)$ , since the substitution  $u = -x$  transforms (E) into an equation of the same form satisfying the assumptions of the theorem.

If, by (i),  $T > t_0$  is chosen so that for every  $t \geq T$

$$\sigma(t) \geq t_0,$$

then from equation (E) we obtain

$$(8) \quad \int_T^t \rho(s)(D_r^{(n)}x)(s)ds = \int_T^t \rho(s)b(s)ds - \int_T^t \rho(s)a^+(s)F(x[\sigma(s)])ds \\ + \int_T^t \rho(s)a^-(s)F(x[\sigma(s)])ds$$

for every  $t \geq T$ . Because of (ii) and the boundedness of  $x$ , the function  $F \circ x \circ \sigma$  is positive and bounded on  $[T, \infty)$ . Thus, by conditions (C<sub>1</sub>) and (C<sub>2</sub>), we conclude from (8) that the improper integral

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt$$

exists in  $\mathbf{R}^*$  and consequently, by virtue of Lemma 1, the  $\lim_{t \rightarrow \infty} (D_r^{(k)}x)(t)$  also exists in  $\mathbf{R}^*$ . Moreover,

$$(9) \quad \lim_{t \rightarrow \infty} (D_r^{(k)}x)(t) = 0.$$

Indeed, in the opposite case we have

$$\liminf_{t \rightarrow \infty} (D_r^{(k)}x)(t) > 0 \quad \text{or} \quad \limsup_{t \rightarrow \infty} (D_r^{(k)}x)(t) < 0.$$

Thus, because of conditions (iv) and (v), we can apply Lemma 3(a) for  $h = (D_r^{(m-1)}x)'$ ,  $\lambda = k - m$  and  $q_j = r_{m+j}$  ( $j = 0, 1, \dots, \lambda$ ), when we obtain

$$\int^\infty (D_r^{(m-1)}x)'(t)dt = \pm\infty, \quad \text{i.e.} \quad \lim_{t \rightarrow \infty} (D_r^{(m-1)}x)(t) = \pm\infty.$$

From this, by condition (vi), it is easy to derive

$$\lim_{t \rightarrow \infty} (D_r^{(0)}x)(t) = \pm \infty$$

a contradiction, since, by condition (iii),  $D_r^{(0)}x = r_0x$  is bounded.

Now, taking into account (9) and conditions (iv) and (v), we apply again Lemma 3( $\beta$ ) for  $h = (D_r^{(m-1)}x)'$ ,  $\lambda = k - m$  and  $q_j = r_{m+j}$  ( $j = 0, 1, \dots, \lambda$ ), when we get

$$(10) \quad \lim_{t \rightarrow \infty} (D_r^{(m-1)}x)'(t) = 0.$$

So, if  $m > 1$ , then, by virtue of condition (vi), we can apply Lemma 2 for  $q_j = r_j$  ( $j = 0, 1, \dots, m-1$ ) and  $q_m = 1$ , to obtain

$$(11) \quad \lim_{t \rightarrow \infty} (D_r^{(j)}x)(t) = 0 \quad (j = 1, 2, \dots, m-1).$$

Thus, it remains to prove that

$$(12) \quad \lim_{t \rightarrow \infty} (D_r^{(0)}x)(t) = 0.$$

To do this, we first observe that

$$(13) \quad \liminf_{t \rightarrow \infty} (D_r^{(0)}x)(t) = 0.$$

Indeed, in the opposite case for some positive constant  $d_1$  and for every  $t \geq t_0$  we have

$$(D_r^{(0)}x)(t) \geq d_1$$

and consequently, by (iii),

$$x(t) \geq d_2 \text{ for every } t \geq t_0$$

where  $d_2 = d_1 / \sup_{t \geq t_0} r_0(t)$ . From this and the boundedness of  $x$  we conclude that

$$F(x[\sigma(t)]) \geq d \text{ for every } t \geq T,$$

where  $d$  is a positive constant. Hence, from (8) and on account of conditions ( $C_1$ ) and ( $C_2$ ), it follows that

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt = \pm \infty$$

and consequently, by applying Lemma 1,

$$\lim_{t \rightarrow \infty} (D_r^{(k)}x)(t) = \pm \infty,$$

which contradicts (9).

To complete the proof of (12), we have verify that

$$\limsup_{t \rightarrow \infty} (D_r^{(0)}x)(t) = 0.$$

Indeed, in the opposite case we have

$$\limsup_{t \rightarrow \infty} (D_r^{(0)}x)[\sigma(t)] \geq K$$

for some positive constant  $K$ . Hence, on account of (13) and based on the arguments of Hammett [3] (cf. also Singh [7, 8] and Staikos and Philos [9]), we derive that there exist three sequences  $(\alpha_\nu)$ ,  $(\beta_\nu)$  and  $(\gamma_\nu)$  with  $\lim \alpha_\nu = \infty$  and such that for every  $\nu = 1, 2, \dots$

$$T \leq \alpha_\nu < \gamma_\nu < \beta_\nu \leq \alpha_{\nu+1}$$

$$(D_r^{(0)}x)[\sigma(\alpha_\nu)] = \frac{K}{2} = (D_r^{(0)}x)[\sigma(\beta_\nu)]$$

$$(D_r^{(0)}x)[\sigma(\gamma_\nu)] > K$$

$$(D_r^{(0)}x)[\sigma(t)] > \frac{K}{2} \text{ for every } t \in (\alpha_\nu, \beta_\nu).$$

By mean-value theorem, we have

$$\frac{(D_r^{(0)}x)[\sigma(\gamma_\nu)] - (D_r^{(0)}x)[\sigma(\alpha_\nu)]}{\gamma_\nu - \alpha_\nu} = \sigma'(\xi_\nu)(D_r^{(0)}x)'[\sigma(\xi_\nu)]$$

and consequently

$$\frac{K}{2(\beta_\nu - \alpha_\nu)} < \sigma'(\xi_\nu)(D_r^{(0)}x)'[\sigma(\xi_\nu)],$$

where obviously  $\lim \xi_\nu = \infty$ . But, because of (10), (11) and (vi), it is easy to see that

$$\lim_{t \rightarrow \infty} (D_r^{(0)}x)'(t) = 0.$$



Thus, we obtain

$$(14) \quad \lim (\beta_\nu - \alpha_\nu) = \infty.$$

Next, we observe that for every  $\nu = 1, 2, \dots$

$$(D_r^{(0)}x)[\sigma(t)] \geq \frac{K}{2} \text{ for every } t \in [\alpha_\nu, \beta_\nu]$$

and consequently, by (iii),

$$x[\sigma(t)] \geq \frac{K}{2r_0[\sigma(t)]} \geq \frac{K}{2\sup_{t \geq t_0} r_0(t)} > 0 \text{ for every } t \in [\alpha_\nu, \beta_\nu].$$

We have thus proved that the bounded function  $x \circ \sigma$  has a positive lower bound on the set  $\bigcup_{\nu=1}^{\infty} [\alpha_\nu, \beta_\nu]$ . Hence, because of (ii), we have

$$F(x[\sigma(t)]) \geq M \text{ for every } t \in \bigcup_{\nu=1}^{\infty} [\alpha_\nu, \beta_\nu],$$

where the constant  $M$  is positive.

Obviously,

$$\begin{aligned} \int_T^\infty \rho(t) a^\pm(t) F(x[\sigma(t)]) dt &\geq \sum_{\nu=1}^\infty \int_{\alpha_\nu}^{\beta_\nu} \rho(t) a^\pm(t) F(x[\sigma(t)]) dt \\ &\geq M \sum_{\nu=1}^\infty \int_{\alpha_\nu}^{\beta_\nu} \rho(t) a^\pm(t) dt. \end{aligned}$$

But, by virtue of (14) and condition  $(C_2)$ , we have

$$\sum_{\nu=1}^\infty \int_{\alpha_\nu}^{\beta_\nu} \rho(t) a^+(t) dt = \infty \quad \text{or} \quad \sum_{\nu=1}^\infty \int_{\alpha_\nu}^{\beta_\nu} \rho(t) a^-(t) dt = \infty$$

and consequently

$$\int_T^\infty \rho(t) a^+(t) F(x[\sigma(t)]) dt = \infty \quad \text{or} \quad \int_T^\infty \rho(t) a^-(t) F(x[\sigma(t)]) dt = \infty.$$

Thus, from (8) it follows that

$$\int_T^\infty \rho(t) (D_r^{(n)}x)(t) dt = \pm \infty.$$

Finally, by Lemma 1, we obtain

$$\lim_{t \rightarrow \infty} (D_r^{(k)} x)(t) = \pm \infty,$$

which contradicts (9).

**3. Applications.** We shall give now some interesting applications of our main result for the particular case

$$r_j = 1 \quad \text{for } j \neq n - N \quad \text{and} \quad r_{n-N} = r,$$

where  $N$  is an integer with  $1 \leq N \leq n - 1$ . More precisely, we shall derive some corollaries concerning the differential equation

$$(D_N) \quad [r(t)x^{(n-N)}(t)]^{(N)} + a(t)F(x[\sigma(t)]) = b(t), \quad t \geq t_0.$$

All corollaries are new except Corollary 1, which is the main result of a recent paper by Kusano and Onose [5].

**COROLLARY 1.** *Consider the differential equation  $(D_N)$  subject to the conditions (i) and (ii). Moreover, let  $k$ ,  $0 \leq k \leq N - 1$ , be an integer such that:*

$$(\alpha) \quad \int_t^\infty \frac{t^{N-1-k}}{r(t)} dt = \infty \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{t^{N-1-k}}{r(t)} < \infty,$$

$$(\beta) \quad \int_t^\infty t^k |b(t)| dt < \infty$$

and

$$(\gamma) \quad \text{For some } \delta > 0,$$

$$\liminf_{t \rightarrow \infty} \int_t^{t+\delta} s^k a^+(s) ds > 0 \quad \text{and} \quad \int_t^\infty t^k a^-(t) dt < \infty$$

or

$$\liminf_{t \rightarrow \infty} \int_t^{t+\delta} s^k a^-(s) ds > 0 \quad \text{and} \quad \int_t^\infty t^k a^+(t) dt < \infty.$$

*If the function  $\sigma$  is differentiable with bounded derivative on  $[t_0, \infty)$ , then for all bounded nonoscillatory solutions  $x$  of the differential equation  $(D_N)$ ,*

$$\lim_{t \rightarrow \infty} [r(t)x^{(n-N)}(t)]^{(N-1-k)} = 0$$

and

$$\lim_{t \rightarrow \infty} x^{(i)}(t) = 0 \quad (i = 0, 1, \dots, n - N).$$

*Proof.* It is easy to see that the function  $\rho$ ,

$$\rho(t) = (t - t_0)^k$$

is of the type  $r[n - 1 - k]$ . Thus, the corollary follows by applying our theorem for  $m = n - N$  and with  $n - 1 - k$  in place of  $k$ .

**COROLLARY 2.** Consider the differential equation  $(D_N)$  with  $N > 1$ , subject to the conditions (i), (ii) and

$$0 < \liminf_{t \rightarrow \infty} r(t) \leq \limsup_{t \rightarrow \infty} r(t) < \infty.$$

Moreover, let  $k, 0 \leq k < N - 1$ , be an integer such that  $(\beta)$  and  $(\gamma)$  are satisfied.

If the function  $\sigma$  is differentiable with bounded derivative on  $[t_0, \infty)$ , then for all bounded nonoscillatory solutions  $x$  of the differential equation  $(D_N)$ ,

$$\lim_{t \rightarrow \infty} [r(t)x^{(n-N)}(t)]^{(j)} = 0 \quad (j = 0, 1, \dots, N - 1 - k)$$

and

$$\lim_{t \rightarrow \infty} x^{(i)}(t) = 0 \quad (i = 0, 1, \dots, n - N).$$

*Proof.* It follows from our theorem, by taking the same function  $\rho$  as in the proof of Corollary 1,  $m = n - 1 - k > n - N$  and  $n - 1 - k$  in place of  $k$ .

**COROLLARY 3.** Consider the differential equation  $(D_N)$  with  $N < n - 1$ , subject to the conditions (i) and (ii). Moreover, let  $k, 1 \leq k \leq n - N - 1$ , be an integer such that:

$$(\delta) \quad \int_{t_0}^{\infty} \frac{t^{n-N-1-k}}{r(t)} dt = \infty,$$

$$(\epsilon) \quad \int^{\infty} \rho_k(t) |b(t)| dt < \infty$$

and for some  $\delta > 0$  either

$$\liminf_{t \rightarrow \infty} \int_t^{t+\delta} \rho_k(s) a^+(s) ds > 0 \quad \text{and} \quad \int^{\infty} \rho_k(t) a^-(t) dt < \infty$$

or

$$\liminf_{t \rightarrow \infty} \int_t^{t+\delta} \rho_k(s) a^-(s) ds > 0 \quad \text{and} \quad \int^{\infty} \rho_k(t) a^+(t) dt < \infty,$$

where

$$\rho_k(t) = \int_{t_0}^t \frac{(t-s)^{N-1} (s-t_0)^{n-N-1-k}}{r(s)} ds.$$

If the function  $\sigma$  is differentiable with bounded derivative on  $[t_0, \infty)$ , then for all bounded nonoscillatory solutions  $x$  of the differential equation  $(D_N)$ ,

$$\lim_{t \rightarrow \infty} x^{(i)}(t) = 0 \quad (i = 0, 1, \dots, k).$$

*Proof.* Here, we have to apply our theorem for  $m = k$  and  $\rho = \rho_k$ , since, as it is easy to see, the function  $\rho_k$  is of the type  $r[k]$ .

**COROLLARY 4.** Consider the differential equation  $(D_N)$  with  $N < n-1$ , subject to the conditions (i) and (ii). Moreover, let  $k$ ,  $1 \leq k \leq n-N-1$  be an integer such that:

$$(\zeta) \quad \int^{\infty} \frac{t^{n-N-1-k}}{r(t)} dt < \infty,$$

$(\eta)$  If  $N > 1$ , then for every  $j = 1, 2, \dots, N-1$

$$\lim_{t \rightarrow \infty} \int_{t_0}^t (t-s)^{j-1} \int_s^{\infty} \frac{(u-t_0)^{n-N-1-k}}{r(u)} du ds \text{ exists in } \{0, \infty\}$$

$$(\vartheta) \quad \int^{\infty} \rho_k(t) |b(t)| dt < \infty$$

and for some  $\delta > 0$  either

$$\liminf_{t \rightarrow \infty} \int_t^{t+\delta} \rho_k(s) a^+(s) ds > 0 \quad \text{and} \quad \int^{\infty} \rho_k(t) a^-(t) dt < \infty$$

or

$$\liminf_{t \rightarrow \infty} \int_t^{t+\delta} \rho_k(s) a^-(s) ds > 0 \quad \text{and} \quad \int_t^\infty \rho_k(t) a^+(t) dt < \infty,$$

where

$$\rho_k(t) = \begin{cases} \int_t^\infty \frac{(s-t_0)^{n-2-k}}{r(s)} ds, & \text{if } N = 1 \\ \int_{t_0}^t (t-s)^{N-2} \int_s^\infty \frac{(u-t_0)^{n-N-1-k}}{r(u)} du ds, & \text{if } N > 1. \end{cases}$$

If the function  $\sigma$  is differentiable with bounded derivative on  $[t_0, \infty)$ , then for all bounded nonoscillatory solutions  $x$  of the differential equation  $(D_N)$ ,

$$\lim_{t \rightarrow \infty} x^{(i)}(t) = 0 \quad (i = 0, 1, \dots, k).$$

*Proof.* It is easy to verify that the function  $\rho_k$  is of the type  $r[k]$ , when the corollary follows immediately by applying our theorem for  $m = k$ .

The significance of Corollaries 2, 3 and 4 is illustrated by the three examples below, where in each case exactly one of these corollaries can be applied and in addition Corollary 1 fails.

EXAMPLE 1. The retarded differential equation

$$[(1 + e^{-t})x'(t)]'' + \frac{1}{t} x^2(\log t) \operatorname{sgn} x(\log t) = \frac{1}{t^3} + e^{-t}(1 + 8e^{-t}), \quad t \geq 1$$

has the bounded nonoscillatory solution  $x(t) = e^{-t}$  with the property

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} x'(t) = \lim_{t \rightarrow \infty} [(1 + e^{-t})x'(t)]' = 0.$$

Further, as it follows from Corollary 2, every bounded nonoscillatory solution  $x$  of the above equation has this property.

EXAMPLE 2. By Corollary 3, all bounded nonoscillatory solutions of the differential equation

$$[t^{1/2} x''(t)]' + t^{-1/2} x^5(\gamma t) = (\gamma^{-5/2} - 3/2)t^{-3}, \quad t \geq 1,$$

where  $\gamma$  is a positive constant, tend to zero as  $t \rightarrow \infty$  together with their derivatives. For example,  $x(t) = t^{-1/2}$  is a bounded nonoscillatory solution of the above equation with  $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} x'(t) = 0$ .

EXAMPLE 3. Consider the retarded differential equation

$$[t^2 x''(t)]' + tx^6(\sqrt{t}) \operatorname{sgn} x(\sqrt{t}) = -\frac{1}{t^2}, \quad t \geq 1.$$

This equation has the bounded nonoscillatory solution  $x(t) = 1/t$  with  $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} x'(t) = 0$ . Further, from Corollary 4 it follows that all bounded nonoscillatory solutions of the considered equation tend to zero as  $t \rightarrow \infty$  together with their derivatives.

REMARK. The results of this paper can be formulated in the case of more general differential equations of the form

$$(E^*) \quad (D_r^{(n)} x)(t) + a(t)F(x[\sigma(t)]) + G(t; x\langle\tau_0(t)\rangle; (D_r^{(1)} x)\langle\tau_1(t)\rangle, \dots, (D_r^{(n-1)} x)\langle\tau_{n-1}(t)\rangle) = b(t), \quad t \geq t_0$$

where

$$g\langle\tau_i(t)\rangle = (g[\tau_{i1}(t)], g[\tau_{i2}(t)], \dots, g[\tau_{i\mu_i}(t)]).$$

From the proof of our theorem, this is obvious under additional conditions on  $G$ , which ensure that for every bounded nonoscillatory and  $n$ -times continuously  $r$ -differentiable function  $u$  on an interval  $[T, \infty)$ ,

$$\int_T^\infty \rho(t) |G(t; u\langle\tau_0(t)\rangle; (D_r^{(1)} u)\langle\tau_1(t)\rangle, \dots, (D_r^{(n-1)} u)\langle\tau_{n-1}(t)\rangle)| dt < \infty,$$

where  $\rho$  is the function introduced in conditions  $(C_1)$  and  $(C_2)$  of our theorem. For a such related result to Corollary 1 see Chen [1].

## REFERENCES

1. Lu-San Chen, *On the nonoscillatory properties of solutions of a functional differential equation*, Bull. Soc. Math. Grèce, to appear.
2. W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath, Boston, 1965.
3. M. E. Hammett, *Nonoscillation properties of a nonlinear differential equation*, Proc. Amer. Math. Soc., **30** (1971), 92-96.

4. G. H. Hardy and J. E. Littlewood, *Contributions to the arithmetic theory of series*, Proc. London Math. Soc., **11** (1913), 411–478.
5. T. Kusano and H. Onose, *Asymptotic behavior of nonoscillatory solutions of nonlinear differential equations with forcing term*, Ann. Mat. Pura Appl., **112** (1977), 231–240.
6. E. Landau, *Über einen Satz von Herrn Esclangon*, Math. Ann., **102** (1930), 177–188.
7. B. Singh, *Nonoscillation of forced fourth order retarded equations*, SIAM J. Appl. Math., **28** (1975), 265–269.
8. ———, *Asymptotic nature of nonoscillatory solutions of  $n$ th order retarded differential equations*, SIAM J. Math. Anal., **6** (1975), 784–795.
9. V. A. Staikos and Ch. G. Philos, *On the asymptotic behavior of nonoscillatory solutions of differential equations with deviating arguments*, Hiroshima Math. J., **7** (1977), 9–31.

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