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UNBOUNDED REPRESENTATIONS OF $*$ -ALGEBRAS

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Basic results on unbounded operator algebras are given, a general class of representations, called adjointable representations is introduced and irreducibility of representations is considered. A characterization of self-adjointness for closed, strongly cyclic *-representations is presented.

1. Introduction. Algebras of unbounded operators and unbounded representations of *-algebras have been important in quantum field theory [1, 3, 9, 10] and certain studies of Lie algebras [5, 7]. The present paper proceeds along the lines initiated and developed by Robert Powers [6, 7] and much of the notation and definitions follow [6]. In §2, we present some basic results concerning unbounded operator algebras, introduce a class of representations called adjointable representations, and consider irreducibility of representations. Section 3 characterizes the self-adjointness of closed, strongly cyclic *-representations.

2. Adjointable representations. Let M and N be subspaces (linear manifolds) in a Hilbert space H . Let $L(M, N)$ and $L_c(M, N)$ denote the collection of linear operators and closable linear operators, respectively with domain M and range in N . For simplicity we use the notation $L(M) = L(M, M)$ and $L_c(M) = L_c(M, M)$. Notice that $L_c(H)$ is the set of bounded linear operators on H . We denote the domain of an operator A by $D(A)$ and if A is closable we denote the closure of A by \bar{A} . A collection of operators \mathcal{B} is an *op-algebra* if there exists a subspace M such that $\mathcal{B} \subseteq L(M)$ and $A, B \in \mathcal{B}$ implies $AB, (\alpha A + B) \in \mathcal{B}$ for all $\alpha \in \mathbb{C}$. A set $\mathcal{B} \subseteq L(M)$ is *symmetric* if M is dense and $A \in \mathcal{B}$ implies $D(A^*) \supseteq M$ and $A^*|_M \in \mathcal{B}$. A symmetric op-algebra $\mathcal{B} \subseteq L(M)$ that contains $I|_M$ is called an *op*-algebra*. It is easy to see that if $\mathcal{B} \subseteq L(M)$ is an op*-algebra, then the map $A \rightarrow A^*|_M$ is an involution so \mathcal{B} is a *-algebra. Also, if π is a representation of a *-algebra \mathcal{A} , then $\pi(\mathcal{A}) = \{\pi(A): A \in \mathcal{A}\}$ is an op-algebra and if π is a *-representation of \mathcal{A} , then $\pi(\mathcal{A})$ is an op*-algebra (we always assume that a *-algebra contains an identity I).

A set $\mathcal{B} \subseteq L(M, N)$ is *directed* if for any $B_1, B_2 \in \mathcal{B}$ there exists a $B_3 \in \mathcal{B}$ such that $\|B_1 x\|, \|B_2 x\| \leq \|B_3 x\|$ for all $x \in M$. For example, if $\mathcal{B} \subseteq L_c(H)$ and $\{\lambda I: \lambda \geq 0\} \subseteq \mathcal{B}$, then \mathcal{B} is directed. Indeed, just let $B_3 = (\|B_1\| + \|B_2\|)I$. For an example of an unbounded directed set, let $\mathcal{B} \subset L(M, H)$ and suppose $B_1, B_2 \in \mathcal{B}$ implies $B_3 =$

$I \mid M + B_1^* B_1 + B_2^* B_2 \in \mathcal{B}$. Then for any $x \in M$ we have

$$\begin{aligned} \|B_3 x\|^2 &= \|x\|^2 + \|(B_1^* B_1 + B_2^* B_2)x\|^2 + 2\langle (B_1^* B_1 + B_2^* B_2)x, x \rangle \\ &\cong 2(\|B_1 x\|^2 + \|B_2 x\|^2) \\ &\cong \|B_1 x\|^2, \|B_2 x\|^2. \end{aligned}$$

In particular, any $\text{op } *$ -algebra is directed.

An extension \mathcal{B}_1 of $\mathcal{B} \subseteq L(M, N)$ is a set of operators $\mathcal{B}_1 \subseteq L(M_1, N_1)$ where $M \subseteq M_1$, $N \subseteq N_1$ and for which there exists a bijection $\phi: \mathcal{B} \rightarrow \mathcal{B}_1$ such that $\phi(B) \mid M = B$ for every $B \in \mathcal{B}$. If $\mathcal{B} \subseteq L(M, N)$, the \mathcal{B} -topology on M is the topology generated by the set of seminorms $\{\|x\|, \|Bx\|: B \in \mathcal{B}\}$. The completion of M in the \mathcal{B} -topology is denoted by $\hat{M}_{\mathcal{B}}$ or simply \hat{M} if no confusion can arise. We say that $\mathcal{B} \subseteq L(M, N)$ is *collectively closed* if for any net $x_\alpha \in M$ satisfying $x_\alpha \rightarrow x \in H$, $Bx_\alpha \rightarrow y(B) \in H$ for every $B \in \mathcal{B}$, then $x \in M$ and $Bx = y(B)$. Clearly if all $B \in \mathcal{B}$ are closed then \mathcal{B} is collectively closed; the converse need not hold.

THEOREM 1.

- (1) $\mathcal{B} \subseteq L(M, N)$ is collectively closed if and only if $M = \hat{M}_{\mathcal{B}}$.
- (2) If $\mathcal{B} \subseteq L_c(M, N)$, then the set $\mathcal{B}_1 = \{\bar{B} \mid M_1: B \in \mathcal{B}\}$ where $M_1 = \cap \{D(\bar{B}): B \in \mathcal{B}\}$ is collectively closed.
- (3) If $\mathcal{B} \subseteq L_c(M, N)$, then the set $\bar{\mathcal{B}} = \{\bar{B} \mid \hat{M}_{\mathcal{B}}: B \in \mathcal{B}\}$ is the minimal collectively closed extension of \mathcal{B} . Moreover, if $\mathcal{B} \subseteq L_c(M)$ and $A, B \in \mathcal{B}$ implies $AB \in \mathcal{B}$, then $\bar{\mathcal{B}} \subseteq L_c(\hat{M}_{\mathcal{B}})$.
- (4) If $\mathcal{B} \subseteq L_c(M, N)$ is directed, then $\hat{M}_{\mathcal{B}} = \cap \{D(\bar{B}): B \in \mathcal{B}\}$.
- (5) If $\mathcal{B} \subseteq L_c(M)$ is an op -algebra, then $\bar{\mathcal{B}}$ is an op -algebra. If $\mathcal{B} \subseteq L_c(M)$ is an $\text{op } *$ -algebra, then $\bar{\mathcal{B}}$ is an $\text{op } *$ -algebra and $\hat{M}_{\mathcal{B}} = \cap \{D(\bar{B}): B \in \mathcal{B}\}$.

Proof.

(1) Suppose $\mathcal{B} \subseteq L(M, N)$ is collectively closed and $x_\alpha \in M$ is a Cauchy net in the \mathcal{B} -topology. Then x_α and Bx_α are Cauchy in H so there exist $x, y(B) \in H$ such that $x_\alpha \rightarrow x$, $Bx_\alpha \rightarrow y(B)$ in H for every $B \in \mathcal{B}$. Since \mathcal{B} is collectively closed, $x \in M$ and $Bx_\alpha \rightarrow Bx$, so $x_\alpha \rightarrow x$ in the \mathcal{B} -topology and M is complete in the \mathcal{B} -topology. Hence $M = \hat{M}_{\mathcal{B}}$. Conversely, suppose $M = \hat{M}_{\mathcal{B}}$, and x_α is a net in M such that $x_\alpha \rightarrow x$ and $Bx_\alpha \rightarrow y(B)$ in H for every $B \in \mathcal{B}$. Then x_α is Cauchy in the \mathcal{B} -topology. Since M is complete in the \mathcal{B} -topology there exists an $x' \in M$ such that $x_\alpha \rightarrow x'$ and $Bx_\alpha \rightarrow Bx'$ in H for every $B \in \mathcal{B}$. Hence $x = x' \in M$ and $Bx = Bx' = y(B)$.

(2) This is straightforward.

(3) It is clear that

$\hat{M}_{\mathcal{B}} = \{x \in \cap \{D(\bar{B}): B \in \mathcal{B}\}: M \ni x_\alpha \rightarrow x, Bx_\alpha \rightarrow \bar{B}x \text{ for all } B \in \mathcal{B}\}$. We now show that $\hat{M}_{\mathcal{B}}$ is complete in the $\bar{\mathcal{B}} | \hat{M}_{\mathcal{B}}$ -topology. Suppose $x_\alpha \in \hat{M}_{\mathcal{B}}$ is Cauchy in the $\bar{\mathcal{B}} | \hat{M}_{\mathcal{B}}$ -topology. Then x_α and Bx_α are Cauchy in H for every $B \in \mathcal{B}$. Hence there exists an $x \in H$ such that $x_\alpha \rightarrow x$ and $\bar{B}x_\alpha \rightarrow \bar{B}x$ for every $B \in \mathcal{B}$. Since $x_\alpha \in \hat{M}_{\mathcal{B}}$ there exists a net $x_{\alpha\beta} \in M$ such that $x_{\alpha\beta} \rightarrow x_\alpha$ and $Bx_{\alpha\beta} \rightarrow \bar{B}x_\alpha$ in H for every $B \in \mathcal{B}$. Now $x_{\alpha\beta}$ is a net in M and $x_{\alpha\beta} \rightarrow x$, $Bx_{\alpha\beta} \rightarrow \bar{B}x$ in H for every $B \in \mathcal{B}$. Hence $x \in \hat{M}_{\mathcal{B}}$. It follows from (1) that $\bar{\mathcal{B}}$ is collectively closed. Clearly $\bar{\mathcal{B}}$ is an extension of \mathcal{B} . Moreover, $\bar{\mathcal{B}}$ is a minimal collectively closed extension since any collectively closed extension of \mathcal{B} must contain $\hat{M}_{\mathcal{B}}$ in its domain. Now suppose $\mathcal{B} \subseteq L_c(M)$ and $A, B \in \mathcal{B}$. If $x \in \hat{M}_{\mathcal{B}}$, then there exists a net $x_\alpha \in M$ such that $x_\alpha \rightarrow x$ and $Bx_\alpha \rightarrow \bar{B}x$ for every $B \in \mathcal{B}$. For fixed $A \in \mathcal{B}$ we have $Ax_\alpha \in M$ and $Ax_\alpha \rightarrow \bar{A}x$ and for every $B \in \mathcal{B}$, since $BA \in \mathcal{B}$, $BAx_\alpha \rightarrow \bar{B}\bar{A}x = \bar{B}\bar{A}x$. Hence $\bar{A}x \in \hat{M}_{\mathcal{B}}$ and $\bar{\mathcal{B}} \subseteq L_c(\hat{M}_{\mathcal{B}})$.

(4) Suppose that $\mathcal{B} \subseteq L_c(M, N)$ is directed. We have seen that $\hat{M}_{\mathcal{B}} \subseteq \cap \{D(\bar{B}): B \in \mathcal{B}\}$. If $x \in \cap \{D(\bar{B}): B \in \mathcal{B}\}$, then for each $B \in \mathcal{B}$ there exists a sequence $x(B, i) \in M$ such that $x(B, i) \rightarrow x$ and $Bx(B, i) \rightarrow \bar{B}x$. For each $B \in \mathcal{B}$ and for each integer $n > 0$ there exists an integer $n_B > 0$ such that $\|x(B, n_B) - x\| < n^{-1}$ and $\|Bx(B, n_B) - \bar{B}x\| < n^{-1}$. For $A, B \in \mathcal{B}$, define the order $(A, n_A) < (B, m_B)$ if $\|Az\| \leq \|Bz\|$ for every $z \in M$ and $n < m$. Since \mathcal{B} is directed, $\{(B, n_B)\}$ is a directed partially ordered set and $x(B, n_B)$ is a net. Notice that if $\|Az\| \leq \|Bz\|$ for every $z \in M$ then $\|\bar{A}y\| \leq \|\bar{B}y\|$ for every $y \in \cap \{D(\bar{B}): B \in \mathcal{B}\}$. Indeed let $z_i \in M$ be a sequence such that $z_i \rightarrow y$ and $Bz_i \rightarrow \bar{B}y$. Since $\|Az_i - Az_j\| \leq \|Bz_i - Bz_j\|$, Az_i is Cauchy and hence $Az_i \rightarrow \bar{A}y$. Therefore,

$$\|\bar{A}y\| = \lim \|Az_i\| \leq \lim \|Bz_i\| = \|\bar{B}y\|.$$

Clearly, $x(B, m_B) \rightarrow x$ and to show that $Ax(B, m_B) \rightarrow \bar{A}x$ let $\epsilon > 0$ and let $n > 0$ be an integer such that $n^{-1} < \epsilon$. Then for $(B, m_B) > (A, n_A)$ we have

$$\begin{aligned} \|Ax(B, m_B) - \bar{A}x\| &= \|\bar{A}x(B, m_B) - \bar{A}x\| \\ &\leq \|\bar{B}x(B, m_B) - \bar{B}x\| \\ &< m^{-1} < n^{-1} < \epsilon. \end{aligned}$$

It follows that $x \in \hat{M}_{\mathcal{B}}$.

(5) This is a straightforward consequence of (2) and (3).

In the work of R. Powers [6] only hermitian representations are considered. But there are important representations that are not hermitian. For example, even if π is hermitian, π^* need not be. We therefore treat a larger class of representations, which we call adjointable, that includes π^* whenever π is hermitian.

Let \mathcal{A} be a $*$ -algebra and let π, π_1 be two representations of \mathcal{A} with domains $D(\pi), D(\pi_1) \subseteq H$. We say that π and π_1 are *adjoint* and write $\pi a \pi_1$, if $\langle \pi(A)x, y \rangle = \langle x, \pi_1(A^*)y \rangle$ for every $A \in \mathcal{A}$ and $x \in D(\pi)$, $y \in D(\pi_1)$. Notice that a is a symmetric relation; that is $\pi a \pi_1$ if and only if $\pi_1 a \pi$. Also, $\pi a \pi$ if and only if π is hermitian. Furthermore, if $\pi a \pi_1$ and $\pi_1 a \pi_2$ then $\pi(A) = \pi_2(A)$ on $D(\pi) \cap D(\pi_2)$ for every $A \in \mathcal{A}$ and if $D(\pi) = D(\pi_2)$ then $\pi = \pi_2$. We say that a representation π is *adjointable* if there exists a representation π_1 such that $\pi a \pi_1$.

If π is a representation of a $*$ -algebra \mathcal{A} , we define $D(\pi^*) = \cap \{D(\pi(A)^*): A \in \mathcal{A}\}$ and $\pi^*(A) = \pi(A^*)^*|_{D(\pi^*)}$ for all $A \in \mathcal{A}$. (To save parentheses we use the notation $\pi(A)^* = [\pi(A)]^*$.) In general, π^* need not be a representation since, for one thing, $D(\pi^*)$ need not be dense. If π is hermitian, then π^* is a representation [6]. Hence, if π is hermitian, then

$$\langle \pi(A)x, y \rangle = \langle x, \pi(A)^*y \rangle = \langle x, \pi^*(A^*)y \rangle$$

for every $A \in \mathcal{A}$ and $x \in D(\pi)$, $y \in D(\pi^*)$ so $\pi a \pi^*$ and each is adjointable.

THEOREM 2.

- (1) π is adjointable if and only if $D(\pi^*)$ is dense.
- (2) If π is adjointable, then π^* is a closed representation and is the largest representation adjoint to π .
- (3) Suppose $\pi \subset \pi_1$. If $\pi_1 a \pi_2$, then $\pi a \pi_2$. If π_1 is adjointable, then so is π and $\pi_1^* \subset \pi^*$.
- (4) If π is adjointable, then there exists a smallest closed representation $\bar{\pi}$ which extends π . If $\pi a \pi_1$, then $\bar{\pi} a \pi_1$.
- (5) If π is adjointable, then $\pi^*, \bar{\pi}$ are adjointable, π^{**} is a closed representation and $\pi \subset \bar{\pi} \subset \pi^{**}$, $\pi^{***} = \pi^*$, $\bar{\pi}^* = \pi^*$.
- (6) If π is hermitian and π_1 is an hermitian extension of π , then $\pi \subset \pi_1 \subset \pi^*$.
- (7) If π is hermitian, then π^{**} and $\bar{\pi}$ are hermitian and $\pi \subset \bar{\pi} \subset \pi^{**} \subset \pi^*$.

Proof.

- (1) If π is adjointable and $\pi a \pi_1$ then $\pi_1(A^*) \subset \pi(A)^*$ for every $A \in \mathcal{A}$ so $D(\pi_1) \subseteq D(\pi^*)$ and $D(\pi^*)$ is dense. Conversely, suppose

$D(\pi^*)$ is dense. For $x \in D(\pi)$, $y \in D(\pi^*)$ we have

$$\begin{aligned}\langle \pi(A^*)x, \pi^*(B)y \rangle &= \langle \pi(A^*)x, \pi(B^*)^*y \rangle \\ &= \langle \pi(B^*)\pi(A^*)x, y \rangle \\ &= \langle \pi(B^*A^*)x, y \rangle \\ &= \langle x, \pi(B^*A^*)^*y \rangle.\end{aligned}$$

Hence $\pi^*(B)y \in D(\pi(A^*)^*)$ and $\pi(A^*)^*\pi^*(B)y = \pi(B^*A^*)^*y$ for every $A, B \in \mathcal{A}$. It follows that $\pi^*(B): D(\pi^*) \rightarrow D(\pi^*)$ and $\pi^*(A)\pi^*(B) = \pi((AB)^*)^* = \pi^*(AB)$. Moreover, π^* is linear since for $x \in D(\pi)$, $y \in D(\pi^*)$ we have

$$\begin{aligned}\langle \pi^*(\alpha A + B)y, x \rangle &= \langle \pi(\bar{\alpha}A^* + B^*)^*y, x \rangle \\ &= \langle y, \bar{\alpha}\pi(A^*)x \rangle + \langle y, \pi(B^*)x \rangle \\ &= \langle [\alpha\pi^*(A) + \pi^*(B)]y, x \rangle.\end{aligned}$$

It follows that π^* is a representation and $\pi a \pi^*$.

(2) It was shown in (1) that π^* is a representation if π is adjointable. It follows from Theorem 1 (2) that π^* is closed. If $\pi a \pi_1$ then $\langle \pi(A)x, y \rangle = \langle x, \pi_1(A^*)y \rangle$ for all $x \in D(\pi)$, $y \in D(\pi_1)$. Hence, $D(\pi_1) \subseteq D(\pi^*)$ and $\pi_1(A^*) \subset \pi(A)^* = \pi^*(A^*)$ for every $A \in \mathcal{A}$ so $\pi_1 \subset \pi^*$.

(3) Suppose $\pi \subset \pi_1$ and $\pi_1 a \pi_2$. Then for every $x \in D(\pi)$, $y \in D(\pi_2)$ we have $\langle \pi(A)x, y \rangle = \langle \pi_1(A)x, y \rangle = \langle x, \pi_2(A^*)y \rangle$. Hence $\pi a \pi_2$. For all $x \in D(\pi)$, $y \in D(\pi_1^*)$ we have $\langle \pi(A)x, y \rangle = \langle x, \pi_1^*(A^*)y \rangle$. Hence $\pi a \pi_1^*$ and by (2) we have $\pi_1^* \subset \pi^*$.

(4) If π is adjointable, then by (1), $D(\pi^*)$ is dense. Then $D(\pi(A)^*)$ is dense so $\pi(A)$ is closable for every $A \in \mathcal{A}$. Define $D(\bar{\pi}) = \hat{D}(\pi)_{\mathcal{B}}$ where $\mathcal{B} = \{\pi(A): A \in \mathcal{A}\}$ and $\bar{\pi}(A) = \overline{\pi(A)}|_{D(\bar{\pi})}$. It follows from Theorem 1 (3) that $\{\bar{\pi}(A): A \in \mathcal{A}\}$ is the minimal collectively closed extension of \mathcal{B} . It is straightforward to show that $\bar{\pi}$ is a representation and that $\pi a \pi_1$ implies $\bar{\pi} a \pi_1$.

(5) If π is adjointable then so is π^* and from (2) π^{**} is a closed representation. If $x \in D(\pi)$, $y \in D(\pi^*)$ then for all $A \in \mathcal{A}$ we have $\langle \pi^*(A^*)y, x \rangle = \langle \pi(A)^*y, x \rangle = \langle y, \pi(A)x \rangle$. Hence $x \in \cap \{D[\pi^*(A^*)^*]: A \in \mathcal{A}\} = D(\pi^{**})$ and $\pi^{**}(A)x = \pi^*(A^*)^*x = \pi(A)x$ so $\pi \subset \pi^{**}$. Since $\pi \subset \bar{\pi}$ we have by (3) that $\bar{\pi}^* \subset \pi^*$. Since $\pi a \pi^*$ from (4) we have $\bar{\pi} a \pi^*$. Hence by (2) $\pi^* \subset \bar{\pi}^*$ so $\pi^* = \bar{\pi}^*$. By (3) $\pi^{***} \subset \pi^*$. Since $\pi^* a \pi^{**}$, by (2) we have $\pi^* \subset \pi^{***}$ so $\pi^{***} = \pi^*$.

(6) For all $x \in D(\pi)$, $y \in D(\pi_1)$, $A \in \mathcal{A}$ we have

$$\langle \pi(A)x, y \rangle = \langle \pi_1(A)x, y \rangle = \langle x, \pi_1(A^*)y \rangle.$$

Hence $\pi_1 a \pi$ and by (2) $\pi_1 \subset \pi^*$.

(7) It is shown in [6] that $\bar{\pi}$ is hermitian if π is hermitian. Since π is hermitian we have $\pi \subset \pi^*$. Applying (3) twice gives $\pi^{**} \subset \pi^{***}$ so π^{**} is hermitian. Since π^{**} is closed we have from (2) that $\pi \subset \bar{\pi} \subset \pi^{**}$ and from (6) $\pi^{**} \subset \pi^*$.

We now show that the extensions in (7) can be distinct. Let \mathcal{A} be the free commutative $*$ -algebra on one hermitian generator A . Define the representation π of \mathcal{A} on the Hilbert space $H = L^2[0, 1]$ as follows: $D(\pi) = \{f \in C^\infty[0, 1]: f^{(n)}(0) = f^{(n)}(1) = 0, n = 0, 1, 2, \dots\}$ $\pi(A) = -i d/dt$. It is straightforward to show that π is hermitian and that $\pi = \bar{\pi} = \pi^{**} \subsetneq \pi^*$ [8]. Now let π_1 be the representation of \mathcal{A} on H defined by:

$$D(\pi_1) = \{f \in C^\infty[0, 1]: f(0) = f(1), f^{(n)}(0) = f^{(n)}(1), n = 1, 2, \dots\}$$

$$\pi_1(A) = -i d/dt.$$

It is straightforward to show that π_1 is hermitian and that $\pi_1 = \bar{\pi}_1 \subsetneq \pi_1^{**} = \pi_1^*$ [8].

We now consider commutants and irreducibility. If $\pi a \pi_1$, define $C(\pi, \pi_1)$ to be the set of operators $C \in L_c(H)$ satisfying $\langle C\pi(A)x, y \rangle = \langle Cx, \pi_1(A^*)y \rangle$ for every $x \in D(\pi)$, $y \in D(\pi_1)$, $A \in \mathcal{A}$. The proof of the following lemma is straightforward.

LEMMA 3.

- (1) $C(\pi, \pi_1)$ is a weakly closed subspace of $L_c(H)$ containing I .
- (2) $C(\pi, \pi_1)$
 $= \{C \in L_c(H): C: D(\pi) \rightarrow D(\pi_1^*), C\pi(A) = \pi_1^*(A)C \mid D(\pi)\}.$
- (3) $C \in C(\pi, \pi_1)$ if and only if $C^* \in C(\pi_1, \pi)$.

The *commutant* of a $*$ -representation π is defined as $\pi(\mathcal{A})' = C(\pi, \pi)$. It follows from Lemma 3 that $\pi(\mathcal{A})'$ is a weakly closed, symmetric subspace of $L_c(H)$ containing I . However, $\pi(\mathcal{A})'$ need not be a von Neumann algebra [6]. If π is self-adjoint then $\pi(\mathcal{A})'$ is a von Neumann algebra [6]. If π is a $*$ -representation, the *strong commutant* is defined by

$$\pi(\mathcal{A})'_s = \{C \in \pi(\mathcal{A})': C: D(\pi) \rightarrow D(\pi)\}.$$

Hence

$$\begin{aligned}\pi(\mathcal{A})'_s &= \{C \in L_c(H): C: D(\pi) \rightarrow D(\pi), C\pi(A) \\ &= \pi(A)C \mid D(\pi), \forall A \in \mathcal{A}\}.\end{aligned}$$

It is easy to see that $\pi(\mathcal{A})'_s$ is an op-algebra in $L_c(H)$ containing I and if π is closed, then $\pi(\mathcal{A})'_s$ is weakly closed [1]. Again $\pi(\mathcal{A})'_s$ need not be a von Neumann algebra but if π is self-adjoint, then $\pi(\mathcal{A})'_s$ is a von Neumann algebra and $\pi(\mathcal{A})'_s = \pi(\mathcal{A})'$.

LEMMA 4. *A *-representation π is self-adjoint if and only if $\pi(\mathcal{A})' = \pi(\mathcal{A})'_s$ and $D(\pi^*) = \cup \{Cx: x \in D(\pi), C \in \pi(\mathcal{A})'\}$.*

Proof. Necessity follows from our previous observations. For sufficiency, if $\pi(\mathcal{A})' = \pi(\mathcal{A})'_s$ then $C: D(\pi) \rightarrow D(\pi)$ for all $C \in \pi(\mathcal{A})'$. Hence $D(\pi^*) = \cup \{Cx: x \in D(\pi), C \in \pi(\mathcal{A})'\} \subset D(\pi)$.

For a bounded *-representation π of a *-algebra \mathcal{A} on a Hilbert space H the following conditions are equivalent [2, 4].

- (i) $\pi(\mathcal{A})' = \{\lambda I: \lambda \in C\}$.
- (ii) The only invariant closed subspaces of H are $\{0\}$ and H .
- (iii) Every nonzero vector in $H = D(\pi)$ is cyclic.

A bounded *-representation π is said to be *irreducible* if π satisfies any one (and hence all) of these three conditions.

For unbounded self-adjoint representations one can give examples [6, 8] which show that no two of the above conditions are equivalent. Also, there is more than one natural way to extend some of the above conditions for unbounded self-adjoint representations. Let π be a self-adjoint representation. We say that a subspace M is a *self-adjoint invariant subspace* for π if M is invariant and $\pi|_M$ is self-adjoint. The following are natural conditions that one might use to define irreducibility for a self-adjoint representation π of a *-algebra \mathcal{A} with domain $D(\pi) \subseteq H$.

- (1) $\pi(\mathcal{A})' = \{\lambda I: \lambda \in C\}$.
- (2) The only invariant subspaces for π which are complete in the $\pi(\mathcal{A})$ -topology are $\{0\}$ and $D(\pi)$.
- (2') The only self-adjoint invariant subspaces for π are $\{0\}$ and $D(\pi)$.
- (3) Every nonzero vector in $D(\pi)$ is strongly cyclic.
- (3') Every nonzero vector in $D(\pi)$ is cyclic.

THEOREM 5. *If π is self-adjoint representation of the *-algebra \mathcal{A} on the Hilbert space H , then*

$$\begin{array}{ccc}
 (2') & \leftrightarrow (1) & \leftarrow (2) \\
 \uparrow & & \downarrow \\
 (3') & \longleftarrow & (3)
 \end{array}$$

Proof. $(2) \rightarrow (3)$. Suppose (2) holds, $0 \neq \phi \in D(\pi)$ and $M = \{\pi(A)\phi : A \in \mathcal{A}\}$. Clearly, $M \neq \{0\}$ and M is an invariant subspace of H for π . Let \hat{M} be the completion of M in the $\pi(\mathcal{A})$ -topology. Since π is closed, $\hat{M} \subseteq D(\pi)$ and clearly \hat{M} is a subspace of H . We now show that \hat{M} is invariant under π . If $x \in \hat{M}$, then there exists a net $x_\alpha \in M$ such that $x_\alpha \rightarrow x$ in the $\pi(\mathcal{A})$ -topology. Fix an $A \in \mathcal{A}$. Then for every $B \in \mathcal{A}$ we have

$$\pi(B)\pi(A)x_\alpha = \pi(BA)x_\alpha \rightarrow \pi(BA)x = \pi(B)\pi(A)x.$$

Hence $\pi(A)x_\alpha \rightarrow \pi(A)x$ in the $\pi(\mathcal{A})$ -topology so $\pi(A)x \in \hat{M}$ and $\pi(A)\hat{M} \subseteq \hat{M}$. Since (2) holds, $\hat{M} = D(\pi)$. Hence M is dense in $D(\pi)$ in the $\pi(\mathcal{A})$ -topology so ϕ is a strongly cyclic vector for π .

$(3) \rightarrow (2)$. Suppose (2) does not hold. Then there exists a $\pi(\mathcal{A})$ -complete invariant subspace M of H with $M \neq \{0\}$, $D(\pi)$. If $0 \neq \phi \in M$, then clearly ϕ is not a strongly cyclic vector for π .

$(1) \rightarrow (2')$. Suppose $(2')$ does not hold. Then there exists a nontrivial self-adjoint invariant subspace M for π . Now M is not dense in H since otherwise $\pi|_M$ is a $*$ -representation of \mathcal{A} on $\bar{M} = H$ with domain $M \subseteq D(\pi)$. Then $\pi|_M \subset \pi = \pi^* \subset (\pi|_M)^*$. Since $\pi|_M$ is self-adjoint, $\pi|_M = \pi$ and $D(\pi) = M$ which is a contradiction. By Theorem 4.7 [6] the projection E on \bar{M} satisfies $E \in \pi(\mathcal{A})'$. Since $E \neq 0, I$, (1) does not hold.

$(2') \rightarrow (1)$. Suppose (1) does not hold. Since π is self-adjoint, $\pi(\mathcal{A})'$ is a von Neumann algebra so there exists a nontrivial projection $E \in \pi(\mathcal{A})'$. By Theorem 4.7 [6], $ED(\pi)$ is a nontrivial self-adjoint invariant subspace for π . Thus $(2')$ does not hold.

$(3') \rightarrow (1)$. Suppose $(3')$ holds. Let $0 \neq E \in \pi(\mathcal{A})'$ be a projection. By Theorem 4.7 [6], $ED(\pi) = M$ is a self-adjoint invariant subspace for π . Let $0 \neq \phi \in M$. Since ϕ is cyclic and $\{\pi(A)\phi : A \in \mathcal{A}\} \subseteq M$, M is dense in H . As in $(1) \rightarrow (2')$ above, $M = D(\pi)$ and hence $E = I$. Since 0 and I are the only projections in $\pi(\mathcal{A})'$, we have $\pi(\mathcal{A})' = \{\lambda I : \lambda \in C\}$.

$(3) \rightarrow (3')$. This is trivial. $(2) \rightarrow (1)$. Since $(2) \rightarrow (2')$ trivially, this follows from $(2') \rightarrow (1)$ above.

3. Closed strongly cyclic $*$ -representations. In this section we shall mainly be concerned with characterizing self-

adjointness for closed strongly cyclic *-representations. Let π be a *-representation of a *-algebra \mathcal{A} with domain $D(\pi) \subseteq H$. The *unbounded commutant* $\pi(\mathcal{A})^c$ of π is defined as the set of operators $C \in L(D(\pi), H)$ such that $\langle C\pi(A)x, y \rangle = \langle Cx, \pi(A^*)y \rangle$ for all $x, y \in D(\pi)$ and $A \in \mathcal{A}$. The *strong unbounded commutant* is defined by $\pi(\mathcal{A})_s^c = \{C \in \pi(\mathcal{A})^c : C D(\pi) \rightarrow D(\pi)\}$. Notice that $\pi(\mathcal{A})' \mid D(\pi) \subseteq \pi(\mathcal{A})^c$ and $\pi(\mathcal{A})_s' \mid D(\pi) \subseteq \pi(\mathcal{A})_s^c$. In fact,

$$\pi(\mathcal{A})' = \{\bar{C} : C \in \pi(\mathcal{A})^c, C \text{ bounded}\}$$

$$\pi(\mathcal{A})_s' = \{\bar{C} : C \in \pi(\mathcal{A})_s^c, C \text{ bounded}\}.$$

We say that a net $B_\alpha \in L(M, N)$ *converges weakly* to $B \in L(M, N)$ if $\langle B_\alpha x, y \rangle \rightarrow \langle Bx, y \rangle$ for every $x, y \in M$. Moreover, $\mathcal{B} \subseteq L(M, N)$ is *weakly closed* if for any net $B_\alpha \in \mathcal{B}$ which converges weakly to some $B \in L(M, N)$ we have $B \in \mathcal{B}$. The proof of the next lemma is straightforward.

LEMMA 6.

- (1) If π is self-adjoint, then $\pi(\mathcal{A})^c = \pi(\mathcal{A})_s^c$.
- (2) $\pi(\mathcal{A})^c = \{C \in L(D(\pi), D(\pi^*)) : C\pi(A) = \pi^*(A)C, \forall A \in \mathcal{A}\}$.
- (3) $\pi(\mathcal{A})_s^c = \{C \in L(D(\pi)) : C\pi(A) = \pi(A)C, \forall A \in \mathcal{A}\}$.
- (4) $\pi(\mathcal{A})^c$ is a weakly closed subspace of $L(D(\pi), D(\pi^*))$ containing $I \mid D(\pi)$.
- (5) $\pi(\mathcal{A})_s^c$ is an op-algebra in $L(D(\pi))$.
- (6) $\pi(\mathcal{A})^c = \pi(\mathcal{A})_s^c$ if and only if $\pi(\mathcal{A})^c$ is an op-algebra.

Let \mathcal{A} be a *-algebra and let π, π_1 be *-representation of \mathcal{A} on Hilbert spaces H, H_1 , respectively. We say that π and π_1 are *equivalent*, and write $\pi \cong \pi_1$, if there exists a unitary transformation V from H onto H_1 such that $VD(\pi) = D(\pi_1)$ and $\pi(A) = V^*\pi_1(A)V$ for every $A \in \mathcal{A}$.

Let ω be a state on \mathcal{A} . Then by the GNS construction for *-algebras [6], there exists a closed, strongly cyclic *-representation π_ω of \mathcal{A} with strongly cyclic vector x_0 such that $\omega(A) = \langle \pi_\omega(A)x_0, x_0 \rangle$ for every $A \in \mathcal{A}$. Moreover, if π is any closed, strongly cyclic *-representation of \mathcal{A} with strongly cyclic vector y_0 such that $\langle \pi(A)y_0, y_0 \rangle = \omega(A)$ for every $A \in \mathcal{A}$ then $\pi \cong \pi_\omega$ [6].

We now characterize states ω such that π_ω is self-adjoint. A linear functional $F : \mathcal{A} \rightarrow C$ is ω -*bounded* if for every $B \in \mathcal{A}$ there exists an $M_B \geq 0$ such that $|F(BA)| \leq M_B \omega(A^*A)^{1/2}$ for every $A \in \mathcal{A}$. For example, if $A_\alpha \in \mathcal{A}$ is a net such that $\omega(A_\alpha^* A^* A A_\alpha)$ is Cauchy for every $A \in \mathcal{A}$, then the functional $F(A) = \lim \omega(A_\alpha^* A)$ is ω -bounded. Indeed, for every $B \in \mathcal{A}$ we have

$$\begin{aligned}
 |F(BA)| &= \lim |\omega(A^*BA)| = \lim |\omega(A^*B^*A_\alpha)| \\
 &\leq \omega(A^*A)^{1/2} \lim \omega(A^*BB^*A_\alpha)^{1/2}.
 \end{aligned}$$

If every ω -bounded linear functional has the above form, then we call ω a *Riesz state*.

THEOREM 7. *Let ω be a state on the $*$ -algebra \mathcal{A} . Then π_ω is self-adjoint if and only if ω is a Riesz state.*

Proof. Recall that π_ω is constructed as follows. Let \mathcal{I} be the left ideal $\mathcal{I} = \{A \in \mathcal{A} : \omega(A^*A) = 0\}$ and let H_0 be the inner product space consisting of equivalence classes $[A]$ in \mathcal{A}/\mathcal{I} with inner product $\langle [A], [B] \rangle = \omega(B^*A)$. Let H be the Hilbert space completion of H_0 . Define a $*$ -representation π_0 of \mathcal{A} with domain $D(\pi_0) = H_0$ by $\pi_0(A)[B] = [AB]$. If $\pi_\omega = \bar{\pi}_0$, then π_ω is a closed, strongly cyclic $*$ -representation with domain $D(\pi_\omega) = \bar{H}_{0\pi_0(\mathcal{A})}$ and strongly cyclic vector $[I]$. Now suppose π_ω is self-adjoint and $F: \mathcal{A} \rightarrow C$ is ω -bounded. If $\omega(A^*A) = 0$, then $F(A) = 0$ so $F: \mathcal{I} \rightarrow 0$. Hence F can be considered as a linear functional on H_0 . Since $|F([A])| \leq M_I \|[A]\|$, F is a continuous linear functional on H_0 and by the Riesz theorem there exists a $z \in H$ such that $F([A]) = \langle [A], z \rangle$ for every $[A] \in H_0$. Now for every $B \in \mathcal{A}$ we have

$$\begin{aligned}
 |\langle \pi_0(B)[A], z \rangle| &= |\langle [BA], z \rangle| = |F([BA])| \\
 &= |F(BA)| \leq M_B \|[A]\|.
 \end{aligned}$$

Hence $z \in D(\pi_0^*) = D(\pi_\omega^*) = D(\pi_\omega)$, so there exists a net $[A_\alpha] \in H_0$ which converges to z in the $\pi_0(\mathcal{A})$ -topology. Thus $[AA_\alpha]$ is Cauchy for every $A \in \mathcal{A}$. Finally, for every $A \in \mathcal{A}$ we have

$$F(A) = \lim \langle [A], [A_\alpha] \rangle = \lim \omega(A^*A).$$

Conversely, suppose ω is a Riesz state and $x \in D(\pi_\omega^*)$. Define the linear functional $F: \mathcal{A} \rightarrow C$ by $F(A) = \langle [A], x \rangle$. Then for every $A, B \in \mathcal{A}$ we have

$$|F(BA)| = |\langle \pi(B)[A], x \rangle| \leq M_B \|[A]\| = M_B \omega(A^*A)^{1/2}$$

so F is ω -bounded. Hence there exists a net $A_\alpha \in \mathcal{A}$ such that $\omega(A^*A^*AA_\alpha)$ is Cauchy for every $A \in \mathcal{A}$ and $F(A) = \lim \omega(A^*A)$ for every $A \in \mathcal{A}$. It follows that $[A_\alpha]$ is Cauchy in the $\pi_0(\mathcal{A})$ -topology and hence there exists a $y \in D(\pi_\omega)$ such that $[A_\alpha] \rightarrow y$. Furthermore, for every $A \in \mathcal{A}$ we have $F(A) = \lim \omega(A^*A) = \lim \langle [A], [A_\alpha] \rangle = \langle [A], y \rangle$. Hence $x = y \in D(\pi_\omega)$ and π_ω is self-adjoint.

COROLLARY. *A closed, strongly cyclic *-representation π with strongly cyclic vector x_0 is self-adjoint if and only if the state $A \rightarrow \langle \pi(A)x_0, x_0 \rangle$ is a Riesz state.*

A state ω is *faithful* if $\omega(A^*A) = 0$ implies $A = 0$. A vector $x_0 \in D(\pi)$ is *separating* if $\pi(A)x_0 = 0$ implies $\pi(A) = 0$. If ω is faithful then the strongly cyclic vector x_0 for π_ω is separating. Conversely, if x_0 is separating, then $\omega(A^*A) = 0$ implies $\pi_\omega(A) = 0$. A representation π of \mathcal{A} is *ultra-cyclic* if there exists an $x_0 \in D(\pi)$ such that $D(\pi) = \{\pi(A)x_0; A \in \mathcal{A}\}$. We then call x_0 an *ultra-cyclic vector*. Ultra-cyclic representations are important because of the following result.

LEMMA 8. *π is a closed, strongly cyclic *-representation if and only if π is the closure of an ultra-cyclic *-representation π^0 .*

Proof. Suppose π is a closed, strongly cyclic *-representation of \mathcal{A} with strongly cyclic vector x_0 . Define $D(\pi^0) = \{\pi(A)x_0; A \in \mathcal{A}\}$ and $\pi^0(B)\pi(A)x_0 = \pi(BA)x_0$. Then π^0 is an ultra-cyclic *-representation and $\bar{\pi}^0 = \pi$. Conversely, if π is the closure of an ultra-cyclic *-representation π^0 with ultra-cyclic vector x_0 , then π is a closed *-representation. Moreover, since $D(\pi)$ is the completion of $D(\pi^0)$ in the $\pi(\mathcal{A})$ -topology, x_0 is a strongly cyclic vector for π .

We call π^0 in the proof of Lemma 8 the *underlying* ultra-cyclic *-representation for π . We can obtain information about π by studying the simpler representation π^0 . For example, a condition characterizing the essential self-adjointness of π^0 characterizes the self-adjointness of π . Moreover, $\pi^{0*} = \pi^*$ and $\pi^0(\mathcal{A})' = \pi(\mathcal{A})'$.

Let π be an arbitrary ultra-cyclic *-representation of \mathcal{A} with a separating ultra-cyclic vector x_0 . For $x \in D(\pi^*)$ define $\pi^c(x) \in L(D(\pi), D(\pi^*))$ by $\pi^c(x)\pi(A)x_0 = \pi^*(A)x$. This is a well-defined operator since $\pi(A)x_0 = \pi(B)x_0$ implies $\pi(A) = \pi(B)$. Then for every $y, z \in D(\pi)$ we have

$$\langle \pi(A^*)y, z \rangle = \langle y, \pi(A)z \rangle = \langle y, \pi(B)z \rangle = \langle \pi(B^*)y, z \rangle.$$

Hence $\pi(A^*) = \pi(B^*)$, so $\pi(A^*)^* = \pi(B^*)^*$ and finally

$$\pi^*(A) = \pi(A^*)^*|D(\pi^*) = \pi(B^*)^*|D(\pi^*) = \pi^*(B).$$

It is straightforward to see that $D(\pi)$ is a *-algebra with identity x_0 under the product $(\pi(A)x_0) \circ (\pi(B)x_0) = \pi(AB)x_0$ and involution $(\pi(A)x_0)^* =$

$\pi(A^*)x_0$. Moreover, for every $x, y, z \in D(\pi)$ we have $\langle x \circ y, z \rangle = \langle y, x^* \circ z \rangle$.

THEOREM 9. *Let π be an ultra-cyclic $*$ -representation of \mathcal{A} with a separating, ultra-cyclic vector x_0 .*

(1) π^c is a weakly continuous linear bijection from $D(\pi^*)$ into $\pi(\mathcal{A})^c$.

(2) *The following statements are equivalent.*

(a) $Cx_0 \in D(\pi)$ for every $C \in \pi(\mathcal{A})^c$.

(b) $\pi(\mathcal{A})^c$ is an op -algebra.

(c) π is self-adjoint.

(3) $\pi(\mathcal{A})^c$ is an $op*$ -algebra if and only if π is self-adjoint and there exists an involution b on the $*$ -algebra $D(\pi)$ satisfying

$$(3.1) \quad \langle x^*, y \rangle = \langle y^b, x \rangle$$

for every $x, y \in D(\pi)$.

(4) If $\pi(\mathcal{A})^c$ is an $op*$ -algebra, then π^c is a weakly continuous b -anti-isomorphism of $D(\pi)$ onto $\pi(\mathcal{A})^c$.

Proof.

(1) Clearly, π^c is linear. To show that π^c maps $D(\pi^*)$ into $\pi(\mathcal{A})^c$, for $x \in D(\pi^*)$, $A \in \mathcal{A}$, $z \in D(\pi)$ and $y = \pi(B)x_0 \in D(\pi)$ we have

$$\begin{aligned} \langle \pi^c(x)\pi(A)y, z \rangle &= \langle \pi^c(x)\pi(AB)x_0, z \rangle \\ &= \langle \pi^*(AB)x, z \rangle = \langle \pi^*(B)x, \pi(A^*)z \rangle \\ &= \langle \pi^c(x)\pi(B)x_0, \pi(A^*)z \rangle = \langle \pi^c(x)y, \pi(A^*)z \rangle. \end{aligned}$$

To show that π^c is surjective, let $C \in \pi(\mathcal{A})^c$. Then $Cx_0 \in D(\pi^*)$ and for any $y = \pi(A)x_0 \in D(\pi)$ we have

$$\begin{aligned} \pi^c(Cx_0)y &= \pi^c(Cx_0)\pi(A)x_0 = \pi^*(A)Cx_0 \\ &= C\pi(A)x_0 = Cy. \end{aligned}$$

To show that π^c is injective, suppose that $x, x_1 \in D(\pi^*)$ and $\pi^c(x) = \pi^c(x_1)$. Then

$$x = \pi^*(1)x = \pi^c(x)x_0 = \pi^c(x_1)x_0 = \pi^*(1)x_1 = x_1.$$

To show that π^c is weakly continuous, suppose that $x_i, x \in D(\pi^*)$ and

$x_i \rightarrow x$ in norm. Then for any $y = \pi(B)x_0 \in D(\pi)$ and $z \in D(\pi)$ we have

$$\begin{aligned} \lim \langle \pi^c(x_i)y, z \rangle &= \lim \langle \pi^*(B)x_i, z \rangle \\ &= \lim \langle x_i, \pi(B^*)z \rangle = \langle x, \pi(B^*)z \rangle \\ &= \langle \pi^*(B)x, z \rangle = \langle \pi^c(x)\pi(B)x_0, z \rangle \\ &= \langle \pi^c(x)y, z \rangle. \end{aligned}$$

(2) (a) \rightarrow (b). Suppose that (a) holds and $C \in \pi(\mathcal{A})^c$, $y = \pi(A)x_0 \in D(\pi)$. We then have $Cy = C\pi(A)x_0 = \pi^*(A)Cx_0 = \pi(A)Cx_0 \in D(\pi)$. Hence, by Lemma 6(6), $\pi(\mathcal{A})^c$ is an op-algebra.

(b) \rightarrow (c). If $x \in D(\pi^*)$, then by (1) $\pi^c(x) \in \pi(\mathcal{A})^c$. If $\pi(\mathcal{A})^c$ is an op-algebra, then $x = \pi^*(1)x = \pi^c(x)x_0 \in D(\pi)$. Hence $D(\pi^*) = D(\pi)$ and π is self-adjoint.

(c) \rightarrow (a). If π is self-adjoint, then $\pi(\mathcal{A})^c \subseteq L(D(\pi))$.

(3) Suppose $\pi(\mathcal{A})^c$ is an op*-algebra. Then, by (2), π is self-adjoint. If $C \in \pi(\mathcal{A})^c$, then $D(\pi) \subseteq D(C^*)$ and $C^*|D(\pi) \in \pi(\mathcal{A})^c$ so $C^*: D(\pi) \rightarrow D(\pi)$. For $x \in D(\pi)$, by (1) $\pi^c(x) \in \pi(\mathcal{A})^c$ so $x^b \equiv \pi^c(x)^*x_0 \in D(\pi)$. For $x = \pi(A)x_0 \in D(\pi)$ and $y \in D(\pi)$ we have

$$\begin{aligned} \langle y^b, x \rangle &= \langle \pi^c(y)^*x_0, x \rangle = \langle x_0, \pi^c(y)\pi(A)x_0 \rangle \\ &= \langle x_0, \pi(A)y \rangle = \langle \pi(A^*)x_0, y \rangle = \langle x^*, y \rangle \end{aligned}$$

so (3.1) holds. That b is an involution now follows from (3.1). For example,

$$\begin{aligned} \langle (y \circ z)^b, x \rangle &= \langle x^*, y \circ z \rangle = \langle y^* \circ x^*, z \rangle \\ &= \langle z^b, x \circ y \rangle = \langle x^* \circ z^b, y \rangle = \langle y^b, z^{b^*} \circ x \rangle \\ &= \langle z^b \circ y^b, x \rangle. \end{aligned}$$

The other properties of an involution follow in a similar way. Conversely, suppose π is self-adjoint and there exists an involution b on $D(\pi)$ satisfying (3.1). Then by (2), $\pi(\mathcal{A})^c$ is an op-algebra. If $C \in \pi(\mathcal{A})^c$, then for any $x = \pi(B)x_0 \in D(\pi)$ and $y = \pi(A)x_0 \in D(\pi)$ we have

$$\begin{aligned} \langle Cy, x \rangle &= \langle C\pi(A)x_0, x \rangle = \langle Cx_0, \pi(A^*)x \rangle \\ &= \langle Cx_0, \pi(A^*B)x_0 \rangle = \langle Cx_0, [\pi(B^*A)x_0]^* \rangle \\ &= \langle \pi(B^*A)x_0, (Cx_0)^b \rangle = \langle \pi(A)x_0, \pi(B)[(Cx_0)^b] \rangle \\ &= \langle y, \pi^c[(Cx_0)^b]x \rangle. \end{aligned}$$

Hence $D(\pi) \subseteq D(C^*)$, $C^*|D(\pi) = \pi^c[(Cx_0)^b] \in \pi(\mathcal{A})^c$ and so $\pi(\mathcal{A})^c$ is an op*-algebra.

(4) Suppose $\pi(\mathcal{A})^c$ is an op*-algebra. It follows from (1) that $\pi^c: D(\pi) \rightarrow \pi(\mathcal{A})^c$ is a weakly continuous linear bijection. For $x = \pi(A)x_0 \in D(\pi)$ and $y = \pi(B)x_0 \in D(\pi)$ we have

$$\pi^c(x)y = \pi^c(x)\pi(B)x_0 = \pi(B)x = \pi(B)x_0 \circ \pi(A)x_0 = y \circ x.$$

It is now clear that π^c is an anti-isomorphism. To show that π^c is a b -anti-isomorphism, for $x \in D(\pi)$ and $y = \pi(A)x_0 \in D(\pi)$ we have

$$\begin{aligned} \pi^c(x^b)y &= y \circ x^b = y \circ [\pi^c(x)^*x_0] = \pi(A)\pi^c(x^*)x_0 \\ &= \pi^c(x)^*\pi(A)x_0 = \pi^c(x)^*y. \end{aligned}$$

COROLLARY. *Let π be a closed, strongly cyclic *-representation of \mathcal{A} with separating, strongly cyclic vector x_0 and let π^0 be the underlying ultra-cyclic representation. Then π is self-adjoint if and only if $Cx_0 \in D(\pi)$ for every $C \in \pi^0(\mathcal{A})^c$.*

Proof. If π is self-adjoint and $C \in \pi^0(\mathcal{A})^c$, then $Cx_0 \in D(\pi^{0*}) = D(\pi^*) = D(\pi)$. Conversely, suppose $Cx_0 \in D(\pi)$ for every $C \in \pi^0(\mathcal{A})^c$. If $x \in D(\pi^*)$, then $x \in D(\pi^{0*})$ so by Theorem 9(1), $\pi^c(x) \in \pi^0(\mathcal{A})^c$. Hence $x = \pi^c(x)x_0 \in D(\pi)$, so $D(\pi) = D(\pi^*)$ and π is self-adjoint.

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B. Arazi, <i>A generalization of the Chinese remainder theorem</i>	289
Thomas E. Armstrong, <i>Polyhedrality of infinite dimensional cubes</i>	297
Yoav Benyamini, Mary Ellen Rudin and Michael L. Wage, <i>Continuous images of weakly compact subsets of Banach spaces</i>	309
John Thomas Burns, <i>Curvature functions on Lorentz 2-manifolds</i>	325
Dennis F. De Raggi and Nelson Groh Markley, <i>Shear distality and equicontinuity</i>	337
Claes Fernström, <i>Rational approximation and the growth of analytic capacity</i>	347
Pál Fischer, <i>On some new generalizations of Shannon's inequality</i>	351
Che-Kao Fong, <i>Quasi-affine transforms of subnormal operators</i>	361
Stanley P. Gudder and W. Scruggs, <i>Unbounded representations of *-algebras</i>	369
Chen F. King, <i>A note on Drazin inverses</i>	383
Ronald Fred Levy, <i>Countable spaces without points of first countability</i>	391
Eva Lowen-Colebunders, <i>Completeness properties for convergence spaces</i>	401
Calvin Cooper Moore, <i>Square integrable primary representations</i>	413
Stanisław G. Mrówka and Jung-Hsien Tsai, <i>On preservation of E-compactness</i>	429
Yoshiomi Nakagami, <i>Essential spectrum $\Gamma(\beta)$ of a dual action on a von Neumann algebra</i>	437
L. Alayne Parson, <i>Normal congruence subgroups of the Hecke groups $G(2^{(1/2)})$ and $G(3^{(1/2)})$</i>	481
Louis Jackson Ratliff, Jr., <i>On the prime divisors of zero in form rings</i>	489
Caroline Series, <i>Ergodic actions of product groups</i>	519
Robert O. Stanton, <i>Infinite decomposition bases</i>	549
David A. Stegenga, <i>Sums of invariant subspaces</i>	567