Pacific Journal of Mathematics

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Vol. 70, No. 2 October 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol 70, No. 2, 1977

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This paper is concerned with certain functional analytic and functional theoretic questions concerning the spaces of bounded analytic and bounded harmonic functions in the unit disk.

Specifically, a characterization is given of those weak-star closed, invariant subspaces of L^∞, on the unit circle, whose vector space sum with the space of continuous function is uniformly closed. This generalizes Sarason's result that $H^\infty + C$ is a closed subspace. The characterization involves the notion of local distances to H^∞. In addition, a partial solution is given to a problem raised by Sarason concerning the structure of functions in $H^\infty + C$.

1. Introduction. Let H^∞ denote the closed subspace of L^∞ on the unit circle T, consisting of all $F \in L^\infty$ whose negative Fourier coefficients are all zero. Denote by C, the space of all continuous functions on T.

Sarason [8], [11], has shown that $H^\infty + C$ is a closed subalgebra of L^∞. Sarason's theorem answered a question raised by Devinatz [2] who asked for a characterization of those functions on T which belong to the smallest closed subalgebra of L^∞ containing H^∞ and the set of all trigonometric polynomials on T. By showing that $H^\infty + C$ is a closed subspace of L^∞ these additional algebraic properties follow easily. Sarason proves that $H^\infty + C$ is closed by considering the bidual of the quotient space C/A, where $A = H^\infty \cap C$ (the disk algebra), by means of the F. and M. Riesz theorem.

The intent of this paper is to generalize Sarason's theorem by replacing H^∞ with an arbitrary weak star closed, invariant subspace M, and ask under what conditions $M + C$ is closed. The word invariant means that M is closed under multiplication by the identity function on T, namely z. We approach this problem by considering the bidual of the space $M/M \cap C$ as was the case in Sarason's proof, however, we find that $M + C$ is not closed in general. The simplest counterexample arises by taking $M = \varphi H^\infty$, where φ is a certain type of inner function. An inner function is any $\varphi \in H^\infty$ with $|\varphi| = 1$ almost everywhere with respect to Lebesgue measure. This situation can be compared with the fact $\bar{\varphi} H^\infty + C$ is closed for all inner functions φ. These two examples are typical of the general case when M is of the form φH^∞, where φ is any unimodular function in L^∞. Our characterization involves the relationship between φ and H^∞ at each point of T.

In §4 we show that a question raised by Sarason concerning the structure of unimodular functions in $H^\infty + C$ is equivalent to showing that $\bar{\varphi}H^\infty + C$ is closed for all unimodular functions φ in $H^\infty + C$. We do not answer Sarason's question in general but we do answer an important special case.

I would like to express my thanks to Patrick Ahern for many valuable discussions. I would also like to thank Allen Shields for many helpful suggestions in preparing the final manuscript.

2. The doubly invariant case. If M is a weak star closed invariant subspace of L^∞ then it is known that either $M = \varphi H^\infty$ where φ is a unimodular function in L^∞ ($|\varphi| = 1$ a.e.), or else M is determined by a measurable subset E of T and M consists of all functions in L^∞ which vanish almost everywhere on the complement of E. Denote this second set of functions by L^∞_E. These facts can be established by noting that since M is weak star closed it suffices to obtain the analogous characterization for closed invariant subspaces of L^1. Now in Helson's book **[6**, pg. 26] he proves that if M is a closed invariant subspace of L^1 then $M \cap L^2$ is dense in M (actually M is simply invariant in this proof, but the argument is the same). Since $M \cap L^2$ is a closed invariant subspace of L^2 it suffices to obtain the analogous characterization in this setting. But these facts are well-known and follow from a theorem of Wiener and a generalization of a theorem of Beurling **[1]** by Helson and Lowenslager **[7]**. Modern proofs for both of these theorems can be found in Helson's book [**6**, pp. 7–9]. M is simply invariant if zM is contained in, but not equal to M and doubly invariant if zM is equal to M. The simply invariant weak star closed subspaces of L^∞ are of the form φH^∞ and the doubly invariant ones are of the form L^∞_E.

We consider the doubly invariant case first. Given a measurable subset E of T, define the essential interior of E to be the union of all open sets with the property that their intersection with $E^c (E^c = T \backslash E)$ has measure zero. Denote this set by E_0, then clearly $E_0 \cap E^c$ has measure zero and E_0 is the largest open set with this property. Note that if two measurable subsets are essentially equal (symmetric difference has measure zero) then they have the same essential interiors.

LEMMA 2.1. *A function* $g \in L^\infty_E \cap C$ *if and only if* $g \in C$ *and* $\{|g| > 0\} \subset E_0$.

Proof. For $g \in C$ the set $\{|g| > 0\}$ is open and since $g \in L^\infty_E$ if and only if the set $\{|g| > 0\} \cap E^c$ has measure zero, it is clear from the maximality of E_0 that this happens if and only if $\{|g| > 0\} \subset E_0$. The proof of the lemma is complete.

LEMMA 2.2. *For $g \in C$ the following three quantities are equal*:
(i) $\|g + L_E^\infty\| = \inf_{F \in L_E^\infty} \|g + F\|$.
(ii) $\|g + L_E^\infty \cap C\| = \inf_{F \in L_E^\infty \cap C} \|g + F\|$.
(iii) $\|g\|_{E_0^c} = \sup_{z \in E_0^c} |g(z)|$.

Proof. For $g \in C$ we have $\|g + L_E^\infty\| = \|X_{E^c} g\|_\infty$ where X_{E^c} is the characteristic function of E^c. Since g is continuous and E_0^c is compact there exists an $\alpha \in E_0^c$ with $\|g\|_{E_0^c} = |g(\alpha)|$. Let $\epsilon > 0$ then there is an open neighborhood U of α with $|g(z)| \geqq \|g\|_{E_0^c} - \epsilon$ for all $z \in U$. Now U is not contained in E_0 so $U \cap E^c$ has positive measure and so $\|X_{E^c} g\|_\infty \geqq \|g\|_{E_0^c} - \epsilon$. But ϵ being arbitrary yields the inequality $\|g\|_{E_0^c} \leqq \|g + L_E^\infty\|$.

Now let K be a compact subset of E_0, then we can find a continuous function φ with $0 \leqq \varphi \leqq 1$, $\varphi = 1$ on K, and $\{\varphi > 0\} \subset E_0$. By the previous lemma $\varphi g \in L_E^\infty \cap C$, thus

$$\|g + L_E^\infty \cap C\| \leqq \|g - \varphi g\|_\infty \leqq \|g\|_{K^c}.$$

Since K is arbitrary we conclude that $\|g + L_E^\infty \cap C\| \leqq \|g\|_{E_0^c}$. Combining these two inequalities we have

$$|g + L_E^\infty \cap C\| \leqq \|g\|_{E_0^c} \leqq \|g + L_E^\infty\|$$

but inequality $\|g + L_E^\infty\| \leqq \|g + L_E^\infty \cap C\|$ is trivially true so all three quantities are equal and the proof is complete.

The following lemma is basic and essentially appears in Zalcman's proof that $H^\infty + C$ is closed, see Theorem 6.1 **[14]**.

LEMMA 2.3. *Suppose X and Y are closed subspaces of a Banach space Z. Then a necessary and sufficient condition for $X + Y$ to be closed is that there exist a $K < \infty$ with*

$$\|y + X \cap Y\| \leqq K\|y + X\| \tag{1}$$

for all y in Y.

Proof. By elementary Banach space techniques we know that $X + Y$ being closed in Z is equivalent to $X/X \cap Y + Y/x \cap Y$ being closed in the quotient space $Z/X \cap Y$. But $X/X \cap Y$ and $Y/X \cap Y$ intersect trivially in $Z/X \cap Y$. As a simple consequence of the closed graph theorem we have $X/X \cap Y + Y/X \cap Y$ is closed if and only if the natural projection operator taking this sum onto $Y/X \cap Y$ is bounded. Thus $X + Y$ is closed in Z if and only if there is a constant $K < \infty$ with

$$\|y + X \cap Y\| \leqq K \|y + x + X \cap Y\|$$

for all $y \in Y$ and $x \in X$. But this last relation is easily seen to be equivalent with (1) and the proof is complete.

By putting the information of the last two lemmas together we have proved the following:

THEOREM 2.4. *$L^\infty_E + C$ is closed in L^∞ for all measurable subsets E of T.*

In other words, if M is weak star closed and doubly invariant subspace of L^∞ then $M + C$ is closed.

3. The simply invariant case. With the doubly invariant case disposed of we consider the more interesting simply invariant case where $M = \varphi H^\infty$ for some unimodular $\varphi \in L^\infty$.

For f and g in L^∞ and λ a point in T, define

$$\operatorname{dist}_\lambda (f, g) = \text{ess.} \limsup_{\substack{z \to \lambda \\ |z|=1}} |f(z) - g(z)|.$$

If f and g are extended harmonically for $|z| < 1$ then we also have

$$\operatorname{dist}_\lambda (f, g) = \limsup_{\substack{z \to \lambda \\ |z|<1}} |f(z) - g(z)|.$$

For f in L^∞ define

$$\operatorname{dist}_\lambda (f, H^\infty) = \inf \left\{ \operatorname{dist}_\lambda (f, h) \colon h \in H^\infty \right\}.$$

We might note that if X_λ denotes the L^∞ fiber over λ then $\operatorname{dist}_\lambda (f, g) = \|f - g\|_{X_\lambda}$.

We first consider the case that $\varphi H^\infty \cap C = \{0\}$. For instance, φ could be a Blaschke product whose zeros cluster on the entire unit circle, or even on a set of positive measure.

THEOREM 3.1. *Suppose φ is a unimodular function in L^∞ for which $\varphi H^\infty \cap C = \{0\}$ holds. The following are equivalent:*

(a) *$\varphi H^\infty + C$ is closed in L^∞.*

(b) *There is an $\epsilon > 0$ with $\operatorname{dist}_\lambda (\bar{\varphi}, H^\infty) \geqq \epsilon$ for all λ in T.*

Proof. Assume (b) is true and let $g \in C$. Suppose $\lambda \in T$, $g(\lambda) \neq 0$, and $h \in H^\infty$ then for $z \in T$

$$\begin{aligned}|g(\lambda)||\bar{\varphi}(z)-h(z)| &= |\bar{\varphi}(z)g(\lambda)-h(z)g(\lambda)| \\ &\leqq |g(\lambda)-g(z)| + \|g-g(\lambda)\varphi h\|_\infty\end{aligned}$$

and so $|g(\lambda)|\,\mathrm{dist}_\lambda(\bar{\varphi}, h) \leqq \|g-g(\lambda)\varphi h\|_\infty$. Thus

$$|g(\lambda)|\,\mathrm{dist}_\lambda(\bar{\varphi}, H^\infty) \leqq \|g+\varphi H^\infty\|$$

holds for all $\lambda \in T$. Clearly then,

$$\|g\|_\infty \leqq \epsilon^{-1}\|g+\varphi H^\infty\|$$

and $\varphi H^\infty + C$ is closed by Lemma 2.3. This gives (b) implies (a).

Suppose that (a) holds then for some $\epsilon > 0$

$$\|g\|_\infty \leqq \epsilon^{-1}\|g+\varphi H^\infty\|$$

for all $g \in C$, once again by Lemma 2.3. Fix $\lambda \in T$ and suppose $\mathrm{dist}_\lambda(\bar{\varphi}, H^\infty) < \epsilon$, then there is a function $h \in H^\infty$ and a neighborhood U of λ such that

$$\operatorname*{ess\,sup}_{z \in U} |\bar{\varphi}(z)-h(z)| < \epsilon.$$

Let $g(z) = (2\lambda - z)^{-1}$ then $g \in H^\infty \cap C$, $|g(\lambda)| = 1$, and $|g(z)| < 1$ for $z \neq \lambda$ but in T. By considering powers of g we can make $|g^n|$ as small as we like off the set U. Hence

$$\|\bar{\varphi}g^n - hg^n\|_\infty < \epsilon$$

for n sufficiently large. However, $|g^n(\lambda)| = 1$ for all n. Thus,

$$\begin{aligned}1 = \|g^n\|_\infty &\leqq \epsilon^{-1}\|g^n+\varphi H^\infty\| \leqq \epsilon^{-1}\|g^n - \varphi h g^n\| \\ &\leqq \epsilon^{-1}\|\bar{\varphi}g^n - hg^n\|_\infty < 1\end{aligned}$$

and we arrive at a contradiction. So we must have $\mathrm{dist}_\lambda(\bar{\varphi}, H^\infty) \geqq \epsilon$ for all λ and the proof is complete.

We now come to the most interesting part of this problem, that is, we consider the possibility that $\varphi H^\infty \cap C$ is not trivial. So we are assuming that there is a F in H^∞ and a g in C with $\varphi F = g$ and g is not identically zero. A well-known property of functions in H^∞ is that $\log|F| \in L^1$ for all nontrivial $F \in H^\infty$. As a result, the set where g is zero must have measure zero since $|F| = |g|$. Define $Z(\varphi)$ as the closed set formed by the intersection of the zero sets of each g in $\varphi H^\infty \cap C$. In other words

$Z(\varphi)$ is the set of common zeros for the functions in $\varphi H^\infty \cap C$. Thus $Z(\varphi)$ is closed and has measure zero. Suppose $\lambda \notin Z(\varphi)$ then $\varphi F = g$ for some $F \in H^\infty$, $g \in C$, with $g(\lambda) \neq 0$. Thus

$$|g(\lambda)| \operatorname{dist}_\lambda\left(\bar{\varphi}, \frac{1}{g(\lambda)} F\right) = \text{ess.} \limsup_{\substack{z\to\lambda \\ |z|=1}} |\bar{\varphi}(z)g(\lambda) - F(z)|$$

$$= \text{ess.} \limsup |\overline{\varphi(z)}g(\lambda) - \overline{\varphi(z)}g(z)|$$

$$= \text{ess.} \limsup |g(\lambda) - g(z)| = 0$$

and so $\{\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) > 0\} \subset Z(\varphi)$. Our aim is to show that if for some $\epsilon > 0$ the function $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty)$ is greater than ϵ whenever it doesn't vanish, then $Z(\varphi)$ is equal to the set $\{\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) > 0\}$ and $\varphi H^\infty + C$ is closed. The converse is also true.

We start by stating a sharpened version of the Rudin–Carleson Theorem, due to Rudin and Stout [**10**, Thm. 4.1] which will be extremely useful to use. Let A denote the intersection of H^∞ with C; A is usually referred to as the disk algebra.

THEOREM 3.2. *Suppose K is a compact subset of measure zero in T and g is a continuous function defined on K. Then there is a function F in A with the property that $F \equiv g$ on K and $|F(z)| < \|g\|_K$ for all $z \notin K$.*

In the proof of Theorem 3.1 we used a trivial special case of this theorem, where K was a single point.

Let φ be an inner function; then the harmonic extension of φ into the open unit disc (also denoted by φ) has the form

$$\varphi(z) = B(z)S(z) \qquad (|z| < 1)$$

where

$$B(z) = \prod^{\infty} \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z}$$

with $|a_n| < 1$ for all n, and $\Sigma(1 - |a_n|) < \infty$ $(a_n/|a_n| \equiv 1$, if $a_n = 0)$ and

$$S(z) = \exp\left[-\int_T \frac{\lambda + z}{\lambda - z}\, d\mu(\lambda)\right]$$

where μ is a finite, nonnegative measure on T, singular with respect to Lebesgue measure.

The support of φ is the set of points $\lambda \in T$ for which there is a sequence $\{z_n\}$ of points ($|z_n| < 1$) such that $z_n \to \lambda$ and $\varphi(z_n) \to 0$. This set, denoted by $\operatorname{supp} \varphi$, is known to be the union of the support of the measure μ and the cluster set of the sequence $\{a_n\}$. It is also well-known that φ is analytic on the complement of $\operatorname{supp} \varphi$. Now if $\{a_{n_i}\}$ is a subsequence of $\{a_n\}$ and μ_0 is a measure with $0 \leqq \mu_0 \leqq \mu$ then the corresponding inner function φ_0 is a divisor of φ, i.e., φ / φ_0 is a bounded analytic function for $|z| < 1$. In terms of boundary values this says that $\varphi \bar{\varphi}_0 \in H^\infty$. All divisors of φ are of this form. See Hoffman's book [**9**, Chapter 5] for details. We need to know the following simple fact about inner functions: if φ is a nontrivial inner function then φ has a nontrivial divisor φ_0 with $m(\operatorname{supp} \varphi_0) = 0$ (m refers to normalized Lebesgue measure on T). This fact is obvious unless $\varphi = \underset{\bar{}}{S}$. However, if this is the case then there is a compact subset K of measure zero and $\mu(K) > 0$. If we take φ_0 to be the inner function determined by restricting the measure μ to the set K then $\operatorname{supp} \varphi_0 \subset K$ and we are done.

LEMMA 3.4. *Suppose φ is unimodular and $\varphi H^\infty \cap C \neq \{0\}$ then $\varphi H^\infty \cap C$ is weak star dense in φH^∞.*

Proof. First note that $\varphi H^\infty \cap C$ is weak star dense in φH^∞ if and only if $\bar{\varphi} C \cap H^\infty$ is weak star dense in H^∞. Since the weak star closed, invariant subspaces of H^∞ are of the form ψH^∞ where ψ is an inner function we may assume that $\bar{\varphi} C \cap H^\infty \subset \psi H^\infty$ for some inner function ψ. We must show that $\psi \equiv 1$, so assume that $\psi \not\equiv 1$. By the above discussion ψ has a nontrivial divisor ψ_0 where $m(\operatorname{supp} \psi_0) = 0$. Now we are assuming that $\bar{\varphi} g = F$ for some $F \in H^\infty$ and $0 \not\equiv g \in C$. It is easy to show that for some positive integer n, $\bar{\psi}_0^n F \in H^\infty$ but $\bar{\psi}_0^{n+1} F \notin H^\infty$. By Theorem 3.2 there is a function $h \in A$ with $h \equiv 1$ on $\operatorname{supp} \psi_0$ and $|h| < 1$ on the complement of $\operatorname{supp} \psi_0$. Clearly, $\bar{\psi}_0^n (1 - h^m) \in C$ for all positive integers m and

$$\bar{\psi}_0^n (1 - h^m) \to \bar{\psi}_0^n$$

as $m \to \infty$ in the weak star topology. Now we have

$$\bar{\varphi}[g \bar{\psi}_0^n (1 - h^m)] = F \bar{\psi}_0^n (1 - h^m) \in H^\infty$$

for all m. But then $F \bar{\psi}_0^n (1 - h^m) \in \psi H^\infty$ since $\bar{\varphi} C \cap H^\infty \subset \psi H^\infty$ by assumption and hence $F \bar{\psi}_0^n \in \psi H^\infty$. This is a contradiction since it implies that $F \bar{\psi}_0^{n+1} \in \psi \bar{\psi}_0 H^\infty \subset H^\infty$. Thus $\varphi \equiv 1$ and the proof is complete.

LEMMA 3.5. *For functions φ as in Lemma 3.4 we have*

$$\|g\|_{Z(\varphi)} \leqq \|g + \varphi H^\infty \cap C\| \leqq \|g + \varphi H^\infty\| + \|g\|_{Z(\varphi)}$$

for all $g \in C$.

Proof. The first inequality is trivial. Let $M(T)$ denote the Banach space of all complex measures on T, with the total variation norm. Now $M(T)$ is the dual of C by the Riesz representation theorem and each μ in $M(T)$ has the form $d\mu = hdm + d\mu_s$ where $h \in L^1$ and μ_s is singular with respect to m. Also we have $\|\mu\| = \|h\|_1 + \|\mu_s\|$. Let X denote the annihilator subspace of $\varphi H^\infty \cap C$ in $M(T)$, i.e.,

$$X = \left\{\mu \in M(T) \colon \int_T F d\mu = 0 \text{ for all } F \in \varphi H^\infty \cap C\right\}$$

then a simple functional analysis argument shows that X can be identified with the dual of the quotient space $C/\varphi H^\infty \cap C$. Let $F \in \varphi H^\infty \cap C$ and $\mu \in X$ then, since $z^n F \in \varphi H^\infty \cap C$ for $n = 0, 1, 2, \cdots$ we have

$$\int z^n F(z) d\mu(z) = 0 \qquad (n = 0, 1, 2, \cdots)$$

and by the F. and M. Riesz theorem the measure $F d\mu$ is absolutely continuous with respect to m. If $d\mu = hdm + d\mu_s$ this amounts to $\operatorname{supp} \mu_s \subset \{z \colon F(z) = 0\}$, since $F d\mu_s$ is the zero measure. Hence

$$\int_T F h dm = 0.$$

Since this is true for all $F \in \varphi H^\infty \cap C$ we conclude that $\operatorname{supp} \mu_s \subset Z(\varphi)$ and the measure hdm is in X. Conversely, if $h \in L^1$ and it annihilates $\varphi H^\infty \cap C$, and μ_s is a measure whose support is contained in $Z(\varphi)$, then $hdm + d\mu_s$ is in X.

Now, if $h \in L^1 \cap X$ then

$$(*) \qquad \int F h dm = 0 \qquad (F \in \varphi H^\infty \cap C)$$

and so by Lemma 3.4 we have that $(*)$ holds for all $F \in \varphi H^\infty$. Thus

$$\int F(\varphi h) dm = 0$$

for all $F \in H^\infty$ and an elementary argument shows that this is equivalent to the vanishing of the Fourier coefficients $(\varphi h)^\wedge(n)$ for $n = 0, -1, -2, \cdots$. Let H_0^1 denote the subspace of L^1 consisting of those functions whose nonpositive Fourier coefficients are zero, then the above discussion shows that $X \cap L^1 = \bar{\varphi} H_0^1$.

Let X_s denote the subspace of $M(T)$ consisting of all measures with support contained in $Z(\varphi)$. We now have $X = \bar{\varphi}H_0^1 \oplus X_s$ where we view $\bar{\varphi}H_0^1$ as a subspace of $M(T)$. Thus, for $g \in C$

$$\|g + \varphi H^\infty \cap C\| = \sup_{F,\mu} \left| \int_T g\bar{\varphi}Fdm + \int_T gd\mu \right| \geqq \sup_{\mu} \left| \int gd\mu \right| = \|g\|_{Z(\varphi)}$$

where the supremum is taken over $F \in H_0^1$ and $\mu \in X_s$ with $\|\bar{\varphi}Fdm + d\mu\| = \|F\|_1 + \|\mu\| = 1$. Clearly then,

$$\|g\|_{Z(\varphi)} \leqq \|g + \varphi H^\infty \cap C\| \leqq \sup_{\substack{F \in H_0 \\ \|F\|_1 = 1}} \left| \int (g\bar{\varphi})Fdm \right| + \|g\|_{Z(\varphi)}$$

for all g in C. If we consider the function $g\bar{\varphi}$ as determining a bounded linear functional on H_0^1 then the first supremum on the right hand side of the above inequality is the norm of this linear functional. Using the well-known fact that the dual of H_0^1 can be identified with the quotient space L^∞/H^∞ we have

$$\sup_{\substack{F \in H_0^1 \\ \|F\|_1 = 1}} \left| \int (g\bar{\varphi})Fdm \right| = \|g\bar{\varphi} + H^\infty\| = \|g + \varphi H^\infty\|$$

since φ is unimodular. This completes the proof.

The above lemma is a direct generalization of Sarason's proof that $\|g + A\| = \|g + H^\infty\|$ for g in C, which is all that is needed to show that $H^\infty + C$ is closed in L^∞. Take $\varphi = 1$ to obtain Sarason's result. The above lemma and Lemma 2.3 show that if $Z(\varphi)$ is empty then $\varphi H^\infty + C$ is closed. This applies in particular to the case when $\bar{\varphi}$ is an inner function.

By Lemma 2.3 and Lemma 3.5 we see that $\varphi H^\infty + C$ is closed if and only if there is a constant $K < \infty$ with $\|g\|_{Z(\varphi)} \leqq K\|g + \varphi H^\infty\|$ for all g in C. Using this fact and an argument paralleling the proof of Theorem 3.1 we have the following:

LEMMA 3.6. *Let φ be a unimodular function. Then a necessary and sufficient condition for $\varphi H^\infty + C$ to be closed is that there exists an $\varepsilon > 0$ such that*

$$\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) \geqq \epsilon$$

for all λ in $Z(\varphi)$.

If $\varphi H^\infty \cap C = \{0\}$ then $Z(\varphi) = T$. Thus, Lemma 3.6 and Theorem 3.1 can be combined to give the above statement in this case.

We might also remark that if $\varphi \in H^\infty + C$ is unimodular, then $\varphi H^\infty + C$ is closed if and only if $Z(\varphi)$ is the empty set. This follows from the fact that $\operatorname{dist}_\lambda(\bar\varphi, H^\infty) = 0$ for all λ, since $\bar\varphi$ is in $H^\infty + C$.

We mentioned earlier the fact that $\operatorname{dist}_\lambda(\bar\varphi, H^\infty)$ vanishes for all $\lambda \notin Z(\varphi)$, and now we see that $\operatorname{dist}_\lambda(\bar\varphi, H^\infty)$ must be bounded away from zero on $Z(\varphi)$ in order for $\varphi H^\infty + C$ to be closed. Thus a natural question is what relationship does the support of the function $\lambda \to \operatorname{dist}_\lambda(\bar\varphi, H^\infty)$ bear to the set $Z(\varphi)$, aside from the fact that $Z(\varphi)$ is always the larger of the two.

If φ is an inner function with $0 < m(\operatorname{supp}\varphi) < 1$ then $Z(\varphi) = T$. To prove this let $F \in H^\infty$, $g \in A$, and assume that $\varphi(z)F(z) = g(z)$ for all $|z| < 1$ and consequently g must vanish on $\operatorname{supp}\varphi$. But $\operatorname{supp}\varphi$ has positive measure so g must vanish identically. Since φ is continuous on $T\backslash\operatorname{supp}\varphi$ we clearly have $\operatorname{dist}_\lambda(\bar\varphi, H^\infty) = 0$ on this set. Now if $\lambda \in \operatorname{supp}\varphi$ and $F \in H^\infty$ then

$$
\begin{aligned}
\operatorname{ess.}\limsup_{\substack{z\to\lambda\\|z|=1}} |\bar\varphi(z) - F(z)| &= \operatorname{ess.}\limsup_{\substack{z\to\lambda\\|z|=1}} |1 - \varphi(z)F(z)| \\
&= \limsup_{\substack{z\to\lambda\\|z|<1}} |1 - \varphi(z)F(z)| \\
&\geqq 1
\end{aligned}
$$

since φ tends to zero on some sequence tending to λ. Thus $\operatorname{dist}_\lambda(\bar\varphi, H^\infty) \geqq 1$ on $\operatorname{supp}\varphi$. Since $\operatorname{dist}_\lambda(\bar\varphi, H^\infty) \leqq 1$ for all λ, we actually *have* $\operatorname{dist}_\lambda(\bar\varphi, H^\infty)$ *equal to the characteristic function of* $\operatorname{supp}\varphi$. *So the* support of the function $\operatorname{dist}_\lambda(\bar\varphi, H^\infty)$ is equal to the set $\operatorname{supp}\varphi$ which is strictly smaller than $Z(\varphi)$.

The next lemma shows that the situation changes if $\varphi H^\infty \cap C \neq \{0\}$.

LEMMA 3.7. *Let φ be unimodular and suppose $\varphi H^\infty \cap C \neq \{0\}$ then the support of the function $\lambda \to \operatorname{dist}_\lambda(\bar\varphi, H^\infty)$ is equal to $Z(\varphi)$.*

Proof. We first note that $Z(\varphi)$ is a totally disconnected compact subset of the circle since $m(Z(\varphi)) = 0$. Assuming that the support of $\operatorname{dist}_\lambda(\bar\varphi, H^\infty)$ is not equal to $Z(\varphi)$ then it follows that there is a compact subset K of $Z(\varphi)$ and an open subset U of T containing K with $U \cap Z(\varphi) = K$ and $\operatorname{dist}_\lambda(\bar\varphi, H^\infty)$ vanishing on U. Let V be an open subset with $K \subset V$ and the closure of V contained in U. Let $\psi \in C$ be 1 on V and 0 on $T\backslash U$; then $\operatorname{dist}_\lambda(\psi\bar\varphi, H^\infty) \equiv 0$. Now it is known **[13]** that $\operatorname{dist}_\lambda(F, H^\infty) \equiv 0$ is equivalent with $F \in H^\infty + C$. Thus $\psi\bar\varphi \in H^\infty + C$ and there is an $F \in H^\infty$ and a $g \in C$ with $\bar\varphi = F + g$ on V. By Theorem 3.2 there is a function h in A with $h = g$ on K since $m(K) = 0$. By replacing F with $F+h$ and by replacing g with $g-h$ we have $\bar{\varphi}=F+g$ on V where $F\in H^{\infty}$, $g\in C$ and $g\equiv 0$ on K. Once again by Theorem 3.2 there is a function h in A with $h\equiv 1$ on K and $|h|<1$ on $T\backslash K$. Now for positive integers n

$$|h^{n}-\varphi h^{n}F|=|\bar{\varphi}-F||h|^{n}\leqq\begin{cases}|g||h|^{n} & \text{on} \quad V\\ c|h|^{n} & \text{on} \quad T\backslash V\end{cases}$$

where $c=1+\|F\|_{\infty}$. Since g vanishes on K and $|h|<1$ on $T\backslash K$ it follows that $\|h^{n}-\varphi h^{n}F\|_{\infty}\to 0$ as $n\to\infty$, and hence $\|h^{n}+\varphi H^{\infty}\|\to 0$.

By Lemma 3.5

$$\begin{aligned}\limsup_{n,m\to\infty}&\|h^{n}-h^{m}+\varphi H^{\infty}\cap C\|\\ &\leqq\limsup_{n,m\to\infty}\|h^{n}-h^{m}+\varphi H^{\infty}\|+\limsup_{n,m\to\infty}\|h^{n}-h^{m}\|_{Z(\varphi)}\\ &\leqq 0+\limsup_{n,m\to\infty}\|h^{n}-h^{m}\|_{T\backslash V}=0\end{aligned}$$

and so the cosets $\{h^{n}+\varphi H^{\infty}\cap C\}$ form a Cauchy sequence in $C/\varphi H^{\infty}\cap C$. But $C/\varphi H^{\infty}\cap C$ is complete so there exists a continuous function h_0 with $\|h_0-h^{n}+\varphi H^{\infty}\cap C\|\to 0$. By Lemma 3.5 we now have $\|h_0-h^{n}+\varphi H^{\infty}\|\to 0$ and $\|h_0-h^{n}\|_{Z(\varphi)}\to 0$. Since we already know that $\|h^{n}+\varphi H^{\infty}\|\to 0$ it follows that h_0 is in $\varphi H^{\infty}\cap C$. By definition of $Z(\varphi)$ we have $h_0\equiv 0$ on K. But $h^{n}\equiv 1$ on K for all n and h^{n} converges uniformly on K to h_0 so h_0 must be 1 on K. This is a contradiction arising from the assumption that the support of $\operatorname{dist}_{\lambda}(\bar{\varphi}, H^{\infty})$ was properly contained in $Z(\varphi)$ and the proof is complete.

Lemmas 3.6 and 3.7 are our main results which we combine into the following thoerem.

THEOREM 3.8. *Suppose φ is unimodular and $\varphi H^{\infty}\cap C\neq\{0\}$. A necessary and sufficient condition for $\varphi H^{\infty}+C$ to be closed is that there exists an $\epsilon>0$ with the range of the function $\lambda\to\operatorname{dist}_{\lambda}(\bar{\varphi}, H^{\infty})$ contained in the set $\{0\}\cup(\epsilon,\infty)$.*

Proof. The proof is an immediate consequence of Lemmas 3.6 and 3.7.

In **[4]** Douglas and Rudin prove the unimodular functions of the form $\varphi_1\bar{\varphi}_2$, where the functions φ_i are inner, are uniformly dense in the set of all unimodular functions. The first two corollaries are directed at such functions.

COROLLARY 3.9. *Suppose* φ_1, φ_2 *are inner functions with disjoint supports. Let* $\psi = \varphi_1\bar{\varphi}_2$ *then* $\psi H^\infty + C$ *is closed if and only if* $m(\operatorname{supp}\varphi_1) = 0$ *or* 1.

Proof. By the discussion preceding Lemma 3.7 and the fact that φ_1 is continuous on $T\backslash\operatorname{supp}\varphi_i$ it follows that $\operatorname{dist}_\lambda(\bar{\varphi}_1\varphi_2, H^\infty)$ is equal to the characteristic function of $\operatorname{supp}\varphi_1$. If $m(\operatorname{supp}\varphi_1) > 0$ then $\psi H^\infty \cap C = \{0\}$. For suppose $\varphi_1\bar{\varphi}_2 F = g$ for some $F \in H^\infty$ and $g \in C$ then $\varphi_1 F = \varphi_2 g$. Since $\varphi_2 g$ is continuous in a neighborhood of $\operatorname{supp}\varphi_1$ its harmonic extension is also. But the harmonic extension of $\varphi_1 F$ is just $\varphi_1(z)F(z)$ which tends to zero at each point of $\operatorname{supp}\varphi_1$. Thus $\varphi_2 g$ vanishes on $\operatorname{supp}\varphi_1$ and hence $\log|F| = \log|\varphi_2 g| \notin L^1$ since $m(\operatorname{supp}\varphi_1) = 0$. Thus $\psi H^\infty \cap C = \{0\}$ in this case. By Theorem 3.1 we conclude that $\psi H^\infty + C$ is closed if and only if $m(\operatorname{supp}\varphi_1) = 1$.

If $m(\operatorname{supp}\varphi_1) = 0$ then there is a nontrivial $h \in A$ with $h \equiv 0$ on $\operatorname{supp}\varphi_1$ and hence $\psi\varphi_2 h = \varphi_1 h \in C$. So $\psi H^\infty \cap C \neq \{0\}$, $\operatorname{dist}_\lambda(\bar{\psi}, H^\infty)$ is a characteristic function, and by Theorem 3.8 we know that $\psi H^\infty + C$ is closed. This completes the proof.

COROLLARY 3.10. *Suppose* φ_1, φ_2 *are inner functions with* $\operatorname{supp}\varphi_1$ *being a finite set. Then* $\varphi_1\bar{\varphi}_2 H^\infty$ *is closed. More generally, if* ψ *is unimodular,* $\psi H^\infty \cap C \neq \{0\}$, *and the support of* $\operatorname{dist}_\lambda(\bar{\psi}, H^\infty)$ *is a finite set then* $\psi H^\infty + C$ *is closed.*

Proof. If $\psi = \varphi_1\bar{\varphi}_2$ with φ_1, φ_2 as above then the proof of Corollary 3.9 shows that $\psi H^\infty \cap C \neq \{0\}$ and the support of $\operatorname{dist}_\lambda(\bar{\psi}, H^\infty)$ have measure zero. This seems to be related to the structure of functions in $H^\infty + C$. Certainly the simplest example would be $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) \equiv 0$ in which case $\bar{\varphi}$ is in $H^\infty + C$.

One question that I have been unable to answer is whether the hypothesis "$\varphi H^\infty \cap C \neq \{0\}$" in Theorem 3.8 can be replaced by the apparently weaker hypothesis that the support of the function $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty)$ have measure zero. This seems to be related to the structure of functions in $H^\infty + C$. Certainly the simplest example would be $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) \equiv 0$ in which case $\bar{\varphi}$ is in $H^\infty + C$. We have just shown that $\varphi \in H^\infty$ implies that $\bar{\varphi}H^\infty + C$ is closed and it seems reasonable to conjecture that the same thing is true with $\varphi \in H^\infty + C$. This situation is discussed further in §4.

We remark that the local distance function $\operatorname{dist}_\lambda(\cdot, H^\infty)$ has been useful in the study of Toeplitz operators, see Sarason **[13]**.

We now give two examples where $\varphi H^\infty + C$ is not closed, $\varphi H^\infty \cap C \neq \{0\}$, and $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty)$ is supported on a convergent sequence. By Corollary 3.10, we must have $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty)$ supported on an infinite set in order for $\varphi H^\infty + C$ not to be closed.

(i) If φ is unimodular and continuous except for a jump discontinuity at λ of magnitude $\epsilon > 0$ then by approximating $\bar{\varphi}$ at λ by constants we see that $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) < \epsilon$. Since $\operatorname{dist}_z(\bar{\varphi}, H^\infty) = 0$ for all $z \neq \lambda$ we must have $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) > 0$. Otherwise, $\varphi \in H^\infty + C$ and this is impossible since φ has a jump discontinuity.

Now let $\{\lambda_n\}$ be a sequence in T converging to 1. Let φ be continuous except for jump discontinuities at λ_n of magnitude $\epsilon_n > 0$, where $\epsilon_n \to 0$. Then $0 < \operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) < \epsilon_n$ and so by Theorem 3.8, $\varphi H^\infty + C$ is not closed.

(ii) Suppose $\{a_n\}$ is a sequence with $|a_n| < 1$, $a_n \to \lambda \in T$, and $\Sigma'(1 - |a_n|) < \infty$. Let B denote the infinite Blaschke product determined by this sequence. Let $\epsilon > 0$, then by a theorem of Frostman [**5**, pg. 111] $\alpha \in C$ can be chosen, with $0 < |\alpha| < 1$, such that the function

$$B_1(z) = \frac{B(z) - \alpha}{1 - \bar{\alpha}B(z)} \qquad (|z| < 1)$$

is also an infinite Blaschke product and $\|B - B_1\|_\infty < \epsilon$. We assume that $0 < \epsilon < 1$ and then the support of B_1 is the set $\{\lambda\}$.

Let $\varphi = B\bar{B}_1$ then $\operatorname{dist}_z(\bar{\varphi}, H^\infty) = 0$ for $z \notin \lambda$, since B, B_1 are continuous on $T\backslash\{\lambda\}$, and $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) \leqq \operatorname{dist}_\lambda(\bar{\varphi}, 1) \leqq \|B - B_1\|_\infty < \epsilon$. Now suppose $F \in H^\infty$ then,

$$\begin{aligned}\operatorname{dist}_\lambda(\bar{\varphi}, F) &= \limsup_{\substack{z\to\lambda \\ |z|<1}} |B_1(z) - B(z)F(z)| \\ &\geqq \limsup_n |B_1(a_n) - B(a_n)F(a_n)| \\ &= \limsup_n |B_1(a_n)| = |\alpha|\end{aligned}$$

and we conclude that $\operatorname{dist}_\lambda(\bar{\varphi}, H^\infty) > 0$. Notice that B can be factored into the product $B'B''$ where B'' is the finite Blaschke product formed from an initial segment of the sequence $\{a_n\}$ and B' is an infinite Blaschke product. Since B'' is continuous and $|B''| \equiv 1$ on T we see that $\operatorname{dist}_z(B_1\bar{B}', H^\infty) = \operatorname{dist}_z(\bar{\varphi}, H^\infty)$ for all z in T. The point of this modification is that we may assume that $\Sigma(1 - |a_n|)$ is arbitrarily small. We may perform a similar modification on B_1 and thus we have proved the following:

PROPOSITION. *Suppose $\lambda \in T$ and two positive numbers ϵ and δ are given. Then there are two Blaschke products B and B_1 with zeros $\{a_n\}$ and $\{b_n\}$ such that*

(i) *Both sequence* $\{a_n\}$ *and* $\{b_n\}$ *converge to* λ.
(ii) $0 < \text{dist}_\lambda (B_1\bar{B}, H^\infty) < \epsilon$.
(iii) $\Sigma(1 - |a_n|) + \Sigma(1 - |b_n|) < \delta$.

Now let $\{\lambda_n\}$ be a sequence in T converging to 1. Let $\{\epsilon_n\}$ and $\{\delta_n\}$ be two sequences of positive numbers such that $\epsilon_n \to 0$ and $\Sigma\delta_n < \infty$. By the proposition we have Blaschke products $\{B_n\}$ and $\{B_{n,1}\}$ satisfying (i)–(iii) for ϵ_n and δ_n. Now by (iii) and the fact that $\Sigma\delta_n < \infty$ we can define new Blaschke products $B = \Pi B_n$ and $B_1 = \Pi B_{n,1}$. By letting $\varphi = B_1\bar{B}$ it is now clear that $\text{dist}_\lambda (\bar{\varphi}, H^\infty) < \epsilon_n$. Since $\epsilon_n \to 0$ we have that $\varphi H^\infty + C$ is not closed, by Theorem 3.8.

In both of these constructions we forced $\text{dist}_\lambda (\bar{\varphi}, H^\infty)$ to be arbitrarily small but positive on the sequence $\{\lambda_n\}$ and identically zero everywhere else except possibly at $\lambda = 1$. No conclusion was drawn concerning the value of $\text{dist}_\lambda (\bar{\varphi}, H^\infty)$ at $\lambda = 1$. Suppose the following had occurred: $\text{dist}_\lambda (\bar{\varphi}, H^\infty)$ is bounded away from zero on the sequence $\{\lambda_n\}$ and identically zero everywhere else with the possible exception of $\lambda = 1$. Assuming that $\varphi H^\infty \cap C \neq \{0\}$ in this case we would have $\varphi H^\infty + C$ is closed if and only if $\text{dist}_1 (\bar{\varphi}, H^\infty) > 0$. However, we observe that the function $\text{dist}_\lambda (\bar{\varphi}, H^\infty)$ is upper-semicontinuous and hence its value at 1 would have to be positive and $\varphi H^\infty + C$ is closed. Thus we have the following extension of Theorem 3.8:

COROLLARY 3.11. *Suppose* φ *is unimodular,* $\varphi H^\infty \cap C \neq \{0\}$, *and* $\text{dist}_\lambda (\bar{\varphi}, H^\infty)$ *is bounded away from zero on a dense subset of its support. Then* $\varphi H^\infty + C$ *is closed.*

4. Unimodular functions in $H^\infty + C$. Suppose φ is unimodular and $Z(\varphi)$ is a finite set, then the support of $\text{dist}_\lambda (\bar{\varphi}, H^\infty)$ is a finite set and by Corollary 3.10 we know that $\varphi H^\infty + C$ is closed. In particular this is true if $Z(\varphi)$ is the empty set. If this is the case, then $\text{dist}_\lambda (\bar{\varphi}, H^\infty) \equiv 0$ and so $\bar{\varphi}$ is in $H^\infty + C$. Conversely, suppose $\bar{\varphi} \in H^\infty + C$ and $\varphi H^\infty \cap C \neq \{0\}$ then $\text{dist}_\lambda (\bar{\varphi}, H^\infty) \equiv 0$ so $Z(\varphi)$ is the empty set by Lemma 3.7. Thus $Z(\varphi)$ is empty if and only if $\bar{\varphi}$ is unimodular in $H^\infty + C$ and $\varphi H^\infty \cap C \neq \{0\}$.

The simplest way to exhibit such a function is to take $F \in H^\infty$, $g \in C$ with $|Fg| = 1$, and let $\bar{\varphi} = Fg$. Clearly, $\varphi \in H^\infty + C$ since $H^\infty + C$ is an algebra and $Z(\varphi)$ is empty since $\varphi F = g^{-1}$ where g has no zeros. Our next result shows that this is always the case (replace $\bar{\varphi}$ with φ).

THEOREM 4.1. *Suppose* φ *is a unimodular function in* $H^\infty + C$ *with* $\bar{\varphi} H^\infty \cap C \neq \{0\}$. *Then* $\varphi = Fg$ *where* F *is in* H^∞ *and* g *is a continuous function.*

Proof. By Lemma 3.7 we have $Z(\bar{\varphi})$ is the null set. Since T is compact and $Z(\bar{\varphi})$ is empty we can find functions $F_1, \cdots, F_n$ in H^∞ and $g_1, \cdots, g_n$ in C with $\varphi g_i = F_i$ for $i = 1, \cdots, n$ and $\Sigma |g_i|^2 > 0$. Since the g_i's are continuous there is a positive integer m and functions $h_1, \cdots, h_n$ in A with

$$\Sigma \| \bar{z}^m h_i - \bar{g}_i \|_\infty \| g_i \|_\infty \leqq \tfrac{1}{2} \min (\Sigma | g_i |^2).$$

Now the function $\Sigma h_i g_i$ is continuous and nonvanishing since

$$\begin{aligned} |\Sigma h_i g_i| &= |\Sigma |g_i|^2 + \Sigma(\bar{z}^m h_i - \bar{g}_i) g_i | \\ &\geqq \Sigma |g_i|^2 - \Sigma \| \bar{z}^m h_i - \bar{g}_i \|_\infty \| g_i \|_\infty \\ &\geqq \tfrac{1}{2} \Sigma |g_i|^2 > 0. \end{aligned}$$

Let $g = (\Sigma h_i g_i)^{-1} \in C$ and $F = \Sigma h_i F_i \in H^\infty$ then

$$\varphi g^{-1} = \varphi \Sigma h_i g_i = \Sigma h_i F_i = F$$

or $\varphi = Fg$ and the proof is complete.

Suppose $\varphi = Fg$ as in the theorem. From the inner-outer factorization of H^∞ functions (see Hoffman's book **[9]**) we know that $F = \psi \mathcal{O}$ where ψ is an inner function and $\mathcal{O}$ is an outer function, i.e., $\mathcal{O}$ is described for $|z| < 1$ by

$$\mathcal{O}(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |\mathcal{O}(e^{it})| \, dt.$$

Then $|\mathcal{O}| = |g|^{-1}$ and clearly $\mathcal{O}^{-1}$ is also in H^∞. In other words, the function $u = \mathcal{O} g$ is an invertible function in the algebra $H^\infty + C$ and $\mathcal{O} = \psi u$.

In [**12**, pg. 293] Sarason has asked whether every unimodular function in $H^\infty + C$ can be described in this way, i.e., as the product of an inner function and an invertible function in $H^\infty + C$. Sarason shows that every invertible in H^∞ and g is a nonvanishing continuous function. Thus the desired factorization is equivalent to showing that every unimodular function in $H^\infty + C$ can be written as the product of an H^∞ function with a continuous function. By Theorem 4.1 it suffices to show that $\bar{\varphi} H^\infty \cap C \neq \{0\}$ for all unimodular functions φ in $H^\infty + C$. Our main result shows that this is also equivalent to showing that $\bar{\varphi} H^\infty + C$ is always closed.

We have been unable to answer Sarason's question in general but we do have an affirmative answer in the following special case.

We start by defining the support of a unimodular function φ in

$H^\infty + C$ in the same manner as we did for inner functions, i.e., $\operatorname{supp} \varphi$ consists of those points $\lambda \in T$ for which there is a sequence $\{z_n\}$ with $|z_n| < 1$ on which the harmonic extension of φ tends to zero. Clearly $\operatorname{supp} \varphi$ is a closed subset. Now Douglas **[3]** has shown that a function in $H^\infty + C$ is invertible if and only if its harmonic extension is bounded away from 0 in some annulus $r < |z| < 1$. Thus a unimodular function φ in $H^\infty + C$ is invertible if and only if $\operatorname{supp} \varphi$ is empty.

LEMMA 4.2. *Suppose φ is unimodular in $H^\infty + C$ and $\lambda \in T \backslash \operatorname{supp} \varphi$. If $\{z_n\}$ is a sequence with $|z_n| < 1$ and $z_n \to \lambda$ then $|\varphi(z_n)| \to 1$.*

Proof. Write $\varphi = \psi \mathcal{O} + g$ where ψ is an inner function, $\mathcal{O}$ is an outer function, g is continuous, and $g(\lambda) = 0$. Now $\lambda \notin \operatorname{supp} \psi$ since $g(\lambda) = 0$ and hence $|\psi(z_n)| \to 1$. Let $\epsilon > 0$ and $V = \{|g| < \epsilon\}$ then V is an open neighborhood of λ and

$$\begin{aligned} ||\mathcal{O}| - 1| &= ||\psi\mathcal{O}| - |\varphi|| \\ &\leqq |g| < \epsilon \end{aligned}$$

almost everywhere in V. For $|z| < 1$, $\log|\mathcal{O}(z)|$ is equal to the Poisson integral of the boundary function $\log|\mathcal{O}|$. Since $|\mathcal{O}| \geqq 1 - \epsilon$ in V it follows from well-known properties of the Poisson integral that

$$\liminf_n |\mathcal{O}(z_n)| \geqq 1 - \epsilon$$

and hence

$$\begin{aligned} \liminf_n |\varphi(z_n)| &= \liminf_n |\psi(z_n)\mathcal{O}(z_n) + g(z_n)| \\ &= \liminf_n |\mathcal{O}(z_n)| \geqq 1 - \epsilon. \end{aligned}$$

Thus, $\liminf |\varphi(z_n)| \geqq 1$ since ϵ is arbitrary and the proof is complete since $\|\varphi\|_\infty = 1$.

THEOREM 4.3. *Suppose φ is a unimodular function in $H^\infty + C$ with $m(\operatorname{supp} \varphi) = 0$. Then there is an inner function ψ and a function u invertible in $H^\infty + C$ with $\varphi = \psi u$.*

Proof. We have that $\varphi = \psi \mathcal{O} + g$ with ψ an inner function, $\mathcal{O}$ an outer function, g continuous, and by Theorem 3.2 we can assume g

variables on $\operatorname{supp}\varphi$. Let $V = \{|g| < \frac{1}{2}\}$ then V is an open neighborhood of $\operatorname{supp}\varphi$. Let $\lambda \in V \backslash \operatorname{supp}\varphi$ and $|z_n| < 1$ with $z_n \to \lambda$. Then by Lemma 4.2,

$$\liminf_n |\psi(z_n)\mathcal{O}(z_n)| = \liminf_n |\varphi(z_n) - g(z_n)| \geqq \tfrac{1}{2} > 0$$

and in particular λ is not $\operatorname{supp}\psi$. Thus $\operatorname{supp}\psi \cap V \subset \operatorname{supp}\varphi$ and we can factor ψ into the product $\psi_1\psi_2$ where ψ_1, ψ_2 are both inner functions, $\operatorname{supp}\psi_1 = \operatorname{supp}\psi \cap V$, and $\operatorname{supp}\psi_2$ is disjoint from $\bar{V}$.

Now g vanishes on $\operatorname{supp}\varphi$ so $\bar{\psi}_1 g \in C$ and since

$$\bar{\psi}_1\varphi = \psi_2\mathcal{O} + \bar{\psi}_1 g$$

we have $\bar{\psi}_1\varphi$ is a unimodular function in $H^\infty + C$. But $\bar{\psi}_1\varphi$ has no support in $T \backslash \operatorname{supp}\varphi$ since ψ_1 is continuous on this set. If $\lambda \in \operatorname{supp}\varphi$ and $z_n \to \lambda$ ($|z_n| < 1$) then $|\psi_2(z_n)| \to 1$ and the harmonic extension of $\bar{\psi}_1 g$ is continuous and vanishes at λ. If the harmonic extension of $\bar{\psi}_1\varphi$ tends to zero on the sequence $\{z_n\}$ we must therefore have $|\mathcal{O}(z_n)| \to 0$. But the proof of Lemma 4.2 shows that this is impossible since $g(\lambda) = 0$. Thus the support of $\bar{\psi}_1\varphi$ is invertible in $H^\infty + C$. Hence $\varphi = \psi_1 u$ is the desired factorization and the proof is complete.

In closing we remark that the factorization in Theorem 4.3 is not unique since it is possible to have two relatively prime inner functions (i.e., no common divisors in H^∞) whose quotient is invertible in $H^\infty + C$; in fact, their quotient can be continuous. This seems to be one of the major difficulties with this problem.

References

1. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math., **81** (1949), 239–255.

2. A. Devinatz, *An extension of a limit theorem of G. Szegö*, J. Math. Anal. Appl., **14** (1966), 499–510.

3. R. A. Douglas, *Toeplitz and Wiener–Hopf operators in $H^\infty + C$*, Bull. Amer. Math. Soc., **74** (1968), 895–899.

4. R. A. Douglas and W. Rudin, *Approximations by inner functions*, Pacific J. Math., **31** (1969), 313–320.

5. O. Frostman, *Potential d'équilibre et capacité des ensembles*, Lundus Univ. Mat. Sem., **3** (1935).

6. H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

7. H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math., **99** (1958), 165–202.

8. H. Helson and D. Sarason, *Past and future*, Math. Scand., **21** (1967), 5–16.

9. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

10. W. Rudin and E. L. Stout, *Boundary properties of functions of several complex variables*, J. Math. Mech., **14** (1965), 991–1000.

11. D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc., **127** (1967), 179–203.

12. ———, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc., **79** (1973), 286–299.

13. ———, *On products of Toeplitz operators*, Acta Sci. Math., (Szezed) **35** (1973), 7–12.

14. L. Zalcman, *Bounded analytic functions on domains of infinite connectivity*, Trans. Amer. Math. Soc., **144** (1969), 241–269.

Received September 20, 1976 and in revised form March 23, 1977. The author was supported in part by NSF Grant GP-36418X1.

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Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics

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