

Pacific Journal of Mathematics

MINIMAL AND MAXIMAL SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS

STEPHEN RICHARD BERNFELD AND JAGDISH CHANDRA

MINIMAL AND MAXIMAL SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS

STEPHEN R. BERNFELD AND JAGDISH CHANDRA

This paper is concerned with the construction of the minimal and the maximal solutions of the nonlinear boundary value problem

$$u'' = f(x, u, u'), \quad 0 < x < 1$$

$$B^i u = \alpha_i u(i) + \beta_i u'(i) = b_i, \quad i = 0, 1$$

under rather mild assumptions on f . In particular, no assumption of monotonicity is made on $f(x, u, u')$ either in u or u' .

1. **Introduction.** This paper is concerned with the construction of the minimal and the maximal solutions of the nonlinear boundary value problem (BVP);

$$(1) \quad u'' = f(x, u, u'), \quad 0 < x < 1$$

$$(2) \quad B^i u \equiv \alpha_i u(i) + \beta_i u'(i) = b_i, \quad i = 0, 1.$$

Obviously, when such boundary value problems are not necessarily uniquely solvable, the existence of the minimal and the maximal solutions plays a useful role in both the quantitative and qualitative theory for these classes of problems. Although considerable literature exists (see, for instance, [9]) about the min-max solutions of initial value problems, very little is known for boundary value problems even in the case of scalar equations (1)-(2). The results in the latter direction usually impose some kind of monotonicity assumption on f in its second and third arguments. In this paper, we establish the minimal and the maximal solutions of BVP (1)-(2) under rather mild assumptions on f . In particular, no assumption of monotonicity is made on $f(x, u, u')$ either in u or u' . The approach taken is essentially an extension of the ideas in [4] where a monotone method was developed for the quasilinear case when f depends on u' linearly. In this paper, we extend the results of [4] in two ways. First, we relax the restriction of linearity of f in u' . Secondly, while in [4] a linear iteration scheme was employed to generate a monotone sequence, here we require a nonlinear iteration scheme. This necessitates our proving existence and uniqueness of solutions of the nonlinear iteration scheme, whereas in the linear case one immediately has existence and uniqueness of the iterative procedure.

The main result can be stated as follows: Suppose there exists a lower and an upper solution for BVP (1)-(2) such that the upper solution dominates the lower solution on the interval of interest. Further, suppose f is continuous and continuously differentiable in its second and third argument, and satisfies a Nagumo condition with respect to these lower and upper solutions. Then there exists maximal and minimal solutions for BVP (1)-(2). Moreover, these are obtained as limits of monotone sequences. Since these sequences converge monotonically, they also provide upper and lower bounds which can be improved by iteration. Thus, if BVP (1)-(2) possesses a unique solution, then this method provides an approximation scheme in which the difference between the upper and lower iterates serves as a good error estimate.

One of the basic motivations in [4] was an extension of the methods in [1], [7] and [11] to a one dimensional quasilinear model of a fluid mechanical problem. The main result of this paper, however, may be considered as an important step in developing a comparison principle for boundary value problems since, for example, the minimal and maximal solutions of a scalar (BVP) may naturally serve as upper and lower bounds for the norm of solutions of higher order systems of differential equations satisfying appropriate boundary conditions. This will be explored elsewhere.

2. Notation and hypotheses. Let $R = (-\infty, \infty)$, $I = [0, 1]$, and $\|u\| = \sup_I |u(x)|$. For any pair of functions $u(x)$ and $v(x)$ with $u(x) \leq v(x)$, $x \in I$, we define the conical segment

$$\langle u, v \rangle = \{w(x) \mid u(x) \leq w(x) \leq v(x), x \in I\}.$$

Let prime denote derivative with respect to x and let subscripts denote derivatives with respect to variables other than x , for example, $f_u = \partial f / \partial u(x, u, u')$. We make the following hypotheses:

(H₁) The real constants α_i, β_i in (2) satisfy, $\alpha_0, \alpha_1, \beta_1 \geq 0, \beta_0 \leq 0$ and $\alpha_i^2 + \beta_i^2 > 0$ for $i = 0, 1$.

(H₂) There exist continuously differentiable functions u_0, v_0 which satisfy

$$(3) \quad u_0(x) \leq v_0(x), \quad x \in I;$$

furthermore, u_0 satisfies the inequalities

$$(4) \quad \begin{aligned} u_0'' &\geq f(x, u_0, u_0') \\ B^i u_0 &\leq b_i, \quad i = 0, 1. \end{aligned}$$

and v_0 satisfies (4) with inequalities reversed. Recall that this con-

dition says that u_0 and v_0 are lower and upper solutions of (1)-(2) respectively.

(H₃) f is continuous on $I \times R \times R \rightarrow R$ and satisfies a Nagumo condition with respect to u_0, v_0 ; that is, for $x \in I, u \in \langle u_0, v_0 \rangle, u' \in R,$

$$(5) \quad |f(x, u, u')| \leq j(|u'|)$$

where $j(s)$ is a positive and continuous function on $[0, \infty)$ such that there exists a positive constant N , for which

$$(6) \quad \int_{\lambda}^N \frac{s ds}{j(s)} \max_{x \in I} v_0(x) - \min_{x \in I} u_0(x) \stackrel{\text{def}}{=} \Delta,$$

where

$$(7) \quad \lambda = \max(|u_0(0) - v_0(1)|, |u_0(1) - v_0(0)|).$$

(H₄) $f(x, u, u')$ is continuously differentiable in u and u' on $I \times R \times R$.

REMARK 1. As a consequence of the Nagumo condition in (H₃), there exists a positive number N such that $|u'(x)| \leq N$ for $x \in I$, where N is defined in (6) and $u'' = f(x, u, u')$ [8]. Notice that N depends only on u_0, v_0 and j .

REMARK 2. In view of (H₃) and (H₄), there exist positive numbers $N, \gamma(N), \gamma'(N)$ such that $|f_u| \leq \gamma, |f_{u'}| \leq \gamma'$ for $x \in I, u \in \langle u_0, v_0 \rangle$ and $|u'(x)| \leq N$.

REMARK 3. The assumption that $f(x, u, u')$ is continuously differentiable in u, u' on $I \times R \times R$ may be relaxed by requiring only that $f_u, f_{u'}$ exist and are bounded for $x \in I, u \in \langle u_0, v_0 \rangle$ and $|u'(x)| \leq N$.

3. Basic lemmas. For $x \in I, z \in \langle u_0, v_0 \rangle$, define $F(x, u, u'; z) = f(x, z(x), u') + \gamma u - \gamma z$, where γ is defined in Remark 2 of §2. For simplicity, we will always write $F(x, u, u'; z) = F(x, u, u')$. Clearly, F is continuous on $I \times R \times R \rightarrow R, F_u = \gamma > 0$, and $F_{u'} = f_{u'}$.

LEMMA 1. Let (H₁), (H₂), and (H₄) hold. Then u_0 and v_0 are respectively lower and upper solutions for the BVP

$$(8) \quad u'' = F(x, u, u'), \quad 0 < x < 1$$

$$(9) \quad B^i u = b_i, \quad i = 0, 1.$$

Proof. Consider the case of a lower solution. We need only to show that $f(x, u_0, u_0') \geq F(x, u_0, u_0')$ for $x \in I$. To see this, note that

$$\begin{aligned}
 f(x, u_0, u'_0) - F(x, u_0, u'_0) \\
 &= f(x, u_0, u'_0) - f(x, z(x), u'_0) - \gamma(u_0 - z) \\
 &= [f_u(x, \tilde{u}_0, u'_0) - \gamma](u_0 - z) \geq 0
 \end{aligned}$$

where $\tilde{u}_0 \in \langle u_0, z \rangle$ and we pick N large enough so that $|u'_0(x)|, |v'_0(x)| \leq N$ for $x \in I$. Thus the above inequality holds since $f_u \leq \gamma$ and $u_0 \leq z$. Similarly, v_0 can be seen to be an upper solution.

LEMMA 2. *Let the assumptions of Lemma 1 hold. Further, suppose (H_3) is satisfied. Then F satisfies a Nagumo condition with respect to u and v provided*

$$(10) \quad \frac{j(s)}{s^2} \text{ is finite for } s \longrightarrow \infty,$$

and

$$(11) \quad \frac{\gamma(N)}{N^2} \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

Proof. Define $l(N) = \gamma(N) \|u_0 - v_0\|$. Then, clearly,

$$|F(x, u, u')| \leq j(|u'|) + l(N)$$

for $x \in I$ and $u \in \langle u_0, v_0 \rangle$. We want to pick N so large that

$$\int_{\lambda}^N \frac{s ds}{j(s) + l(N)} \geq \Delta,$$

where λ is given by (7). From (10) there exists a $\tau > 0$, $\rho > 0$ such that $j(s) \leq \rho s^2$ for $s > \tau$ and from (11) there exists a function $K(m)$ such that $K(m) \rightarrow 0$ as $m \rightarrow \infty$ and $l(N) \leq K(m)N^2$ whenever $N \geq m$. For some arbitrary (for the moment) positive number t pick $N \geq tm$. Then, since we can assume $m > \lambda$ and $m > \tau$

$$\begin{aligned}
 \int_{\lambda}^N \frac{s ds}{j(s) + l(N)} &> \int_m^N \frac{s ds}{j(s) + l(N)} \geq \int_m^N \frac{s ds}{(\rho s^2 + K(m)N^2)} \\
 &= \frac{1}{2} \ln \left[\frac{\rho N^2 + K(m)N^2}{\rho m^2 + K(m)N^2} \right] \\
 &\geq \frac{1}{2} \ln t^2 + \frac{1}{2} \ln \left[\frac{\rho + K(m)}{\rho + t^2 K(m)} \right] \\
 &\geq \Delta,
 \end{aligned}$$

provided we choose $t = e^{2\Delta}$ and m such that $K(m) = \rho e^{-2\Delta}$. Now Lemma 2 is established by picking $N \geq e^{2\Delta} m$.

REMARK 4. As a consequence of Lemma 2 and Nagumo's lemma,

we have that if $u \in \langle u_0, v_0 \rangle$ satisfies (8), then $|u'(x)| \leq N$, for $x \in I$, where

$$(12) \quad N \geq e^{2\lambda} K^{-1}(\rho e^{-2\lambda}).$$

Here $K(\cdot)$ and ρ are defined as above. We can assume without loss of generality that $K(\cdot)$ is a decreasing function.

REMARK 5. The conditions (10) and (11) cannot be weakened much for the following reasons: If we allow $\gamma(N) = O(N^2)$, then since

$$\int_{\lambda}^N \frac{s ds}{j(s) + N^2} \leq N \int_{\lambda}^N \frac{ds}{j(s) + N^2} \leq \frac{1}{N} \int_{\lambda}^N ds < 1,$$

F may not satisfy a Nagumo condition unless λ happens to be sufficiently small.

On the other hand if we assume $j(s)$ only satisfies (6) and do not require (10), then by defining $j(s) = sh(s)$, where $h(s)$ is given in Example 2.3 in [3] and assuming $\gamma(N) = O(N)$, we have

$$\int_{\lambda}^N \frac{s ds}{j(s) + N} < \int_{\lambda}^N \frac{s ds}{j(s) + s} = \int_{\lambda}^N \frac{ds}{h(s) + 1} < \int_{\lambda}^{\infty} \frac{ds}{h(s) + 1} < \infty.$$

Thus, in general a compatibility condition between $j(s)$ and $\gamma(N)$ is needed to insure that F satisfies a Nagumo condition. Clearly, (10) and (11) are satisfied in [4], because both j and γ are linear there.

We shall now use the maximum principle to assert that there is at most one solution of the BVP (8)-(9) contained in $\langle u_0, v_0 \rangle$. Since we will be making much use of the maximum principle [10], we state it here for completeness as:

LEMMA A. Let $q(x), r(x)$ be real-valued functions on I with $r(x) \geq 0, x \in I$. Suppose (H_1) holds and $\phi \in C'(I)$ satisfies

$$(13) \quad \phi'' + q(x)\phi' - r(x)\phi \leq 0$$

$$(14) \quad \alpha_i \phi(i) + \beta_i \phi'(i) \geq 0, \quad i = 0, 1.$$

Then $\phi(x) \geq 0$ for $x \in I$. If the inequalities in (13) and (14) are reversed then $\phi(x) \leq 0$.

LEMMA 3. Let the assumptions of Lemma 2 hold. In addition, assume N satisfies (12) and

$$(15) \quad N \geq \max(\|u'_0\|, \|v'_0\|).$$

Then the BVP (8)-(9) has at most one solution in $\langle u_0, v_0 \rangle$.

Proof. Suppose u_1 and u_2 are two solutions of BVP (8)-(9) in $\langle u_0, v_0 \rangle$. Then from Lemma 2 and Remark 4, we conclude that for $x \in I$, $|u'_i(x)| \leq N$, $i = 1, 2$. Set $\phi = u_1 - u_2$. Then using the mean-value theorem, we obtain

$$\begin{aligned}\phi'' &= f(x, z, u'_1) - f(x, z, u'_2) + \gamma(u_1 - u_2) \\ &= f_u(x, z, \psi(x))\phi' + \gamma\phi \\ B'\phi &= 0\end{aligned}$$

where $|\psi(x)| \leq N$ for $x \in I$. An application of Lemma A then concludes the proof of Lemma 3.

We are now in a position to use a result in [5] to obtain the existence of a solution of the BVP (8)-(9) in $\langle u_0, v_0 \rangle$.

LEMMA 4. *Assume (H₁)-(H₄), (10) and (11) hold and let N satisfy (12) and (15). Then there exists a solution of $u(x)$ of the BVP (8)-(9) such that $u \in \langle u_0, v_0 \rangle$ and $|u'(x)| \leq N$ for $x \in I$.*

Proof. First observe that from Lemma 1, u_0, v_0 are lower and upper solutions respectively of the BVP (8)-(9), and from Lemma 2, F satisfies a Nagumo condition with respect to u_0 and v_0 . Since $\beta_0 \leq 0$ and $\beta_1 \geq 0$, the result in [5] together with Remark 4, establishes Lemma 4. We should remark that although in [5], it is assumed that the strict inequalities $u_0(0) < v_0(0)$ and $u_0(1) < v_0(1)$ are satisfied, these can be relaxed. For instance, using well known approximation arguments [2, 6] the result in [5] is valid for $u_0(0) \leq v_0(0)$ and $u_0(1) \leq v_0(1)$.

Thus, from Lemmas 3 and 4, we conclude that the BVP (8)-(9) is uniquely solvable in $\langle u_0, v_0 \rangle$.

4. Minimal and maximal solution. For each function $z(x) \in C'(I) \cap \langle u_0, v_0 \rangle$, define the image $w(x)$ of the mapping A to be the solution of the nonlinear BVP (8)-(9), that is, $w = Az$ if and only if $w(x)$ satisfies (8) and (9). From the previous section, $w(x)$ is uniquely defined for each $z(x) \in C'(I) \cap \langle u_0, v_0 \rangle$, is contained in $C'(I) \cap \langle u_0, v_0 \rangle$ and satisfies $|w'(x)| \leq N$ for $x \in I$.

LEMMA 5. *Assume (H₁)-(H₄), (10) and (11) hold and let N satisfy (12) and (15). Then,*

- (i) $Au_0 \geq u_0, Av_0 \leq v_0$
- (ii) A is monotone on $\langle u_0, v_0 \rangle$, that is, if $z_1, z_2 \in C'(I) \cap \langle u_0, v_0 \rangle$, and $z_1 \leq z_2$ then $Az_1 \leq Az_2$.

Proof. (i) Suppose $Au_0 = w$. Set $\phi = w - u_0$. Then exactly as in the proof of Lemma 3 with $z = u_0$

$$\phi'' - f'_u(x, u_0, \tilde{u}'_0)\phi' - \gamma\phi \leq 0$$

and

$$B^i\phi \geq 0, \quad i = 0, 1$$

where $|\tilde{u}'_0(x)| \leq N$ for $x \in I$. Therefore, from Lemma A, we conclude that $w \geq u_0$. Similarly we can show that $Av_0 \leq v_0$. This proves (i). (ii) Suppose $z_1, z_2 \in C'(I) \cap \langle u_0, v_0 \rangle$ and $z_1 \leq z_2$. Let $Az_i = w_i, i = 1, 2$. Then setting $\phi = w_2 - w_1$, and using the same techniques as in Lemma 3, ϕ satisfies

$$\phi'' - f'_w(x, z_2, w')\phi' - \gamma\phi = [f'_u(x, \tilde{z}, w'_1) - \gamma](z_2 - z_1) \leq 0$$

where $\tilde{z} \in \langle u_0, v_0 \rangle$ and $|w'(x)| \leq N$. The above inequality follows from the fact that $f'_u \leq \gamma$ for $x \in I, u \in \langle u_0, v_0 \rangle$ and $|u'(x)| \leq N$. Also, $B^i\phi = 0$. Again from the generalized maximum principle we conclude that $w_1 \leq w_2$. This completes the proof.

From Lemma 5, we see that A is monotone on $\langle u_0, v_0 \rangle$ and maps this closed, bounded and convex set into itself. Thus, using the mapping $Az = w$ defined by BVP (8)-(9), we introduce the sequences $\{u_n\}$ and $\{v_n\}$ by means of

$$u_n = Au_{n-1} \text{ where } u_0 \text{ is given in } (H_2),$$

$$v_n = Av_{n-1} \text{ where } v_0 \text{ is given in } (H_2).$$

THEOREM. *Let (H₁)-(H₄), (10) and (11) hold and assume N satisfies (12) and (15). Let $\{u_n\}, \{v_n\}$ be defined as above. Then $\{u_n\}$ and $\{v_n\}$ converge uniformly and monotonically to minimal and maximal solutions u_{\min}, v_{\max} , respectively, of BVP (1)-(2) on $\langle u_0, v_0 \rangle$: that is, if w is any solution of BVP (1)-(2) in $\langle u_0, v_0 \rangle$, then*

$$(16) \quad u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq u_{\min} \leq w \leq v_{\max} \dots \leq v_n \leq \dots \leq v_1 \leq v_0.$$

Proof. In view of Lemma 5, the proof follows essentially the same arguments as given in the proof of Theorem 1 in [4]. We only outline it here. By Lemma 5, $u_{n-1} \leq u_n$ for $n = 0, 1, 2, \dots$. If w is any solution of (1)-(2) in $\langle u_0, v_0 \rangle$, then $u_0 \leq w$ and $Au_0 \leq Aw = w$. This implies that $u_n \leq w$. Since $u_0 \leq v_0$, then by Lemma 5, $u_n \leq v_n$, and $v_{n+1} \leq v_n$ by the above arguments. Thus (16) follows where u_{\min} and v_{\max} denote limits of the monotone bounded sequences $\{u_n\}, \{v_n\}$ respectively.

It remains only to show that u_{\min} is a solution of the BVP (1)-(2) (with a similar argument for v_{\max}). If u_{\min} is a solution, then

it is the minimal solution in $\langle u_0, v_0 \rangle$, since $u_n \leq w$ for all n and any solution w of (1) and (2) in $\langle u_0, v_0 \rangle$. It is easy to see that the sequence $\{u_n\}$ is uniformly bounded and equicontinuous and thus converges (the full sequence by monotonicity) on I . By considering the integral equation which is equivalent to the BVP (8)-(9) and using the fact that $\lim u_n = \lim u_{n-1}$, it follows that $\lim u_n = u_{\min}$ is a solution of the BVP (1)-(2).

REFERENCES

1. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review, **18** (1976), 620-709.
2. J. Bebernes and R. Wilhelmson, *A remark concerning a boundary value problem*, J. Differential Equations, **10** (1971), 389-391.
3. S. R. Bernfeld, *The extendability of solutions of perturbed scalar differential equations*, Pacific J. Math., **42** (1972), 277-288.
4. J. Chandra and P. W. Davis, *A monotone method for quasilinear boundary value problems*, Arch. Rat. Mech. Anal., **54** (1974), 257-266.
5. L. Erbe, *Nonlinear boundary value problems for second order differential equations*, J. Differential Equations, **7** (1970), 459-472.
6. P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, New York, 1964.
7. H. B. Keller, *Elliptic boundary value problems suggested by nonlinear diffusion processes*, Arch. Rat. Mech. Anal., **35** (1969), 363-381.
8. L. K. Jackson, *Subfunctions and second-order differential inequalities*, Advances in Math., **2** (1968), 307-363.
9. V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. 1., Academic Press, New York, 1969.
10. M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Inc., 1967.
11. D. H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J., **21** (1972), 979-1000.

Received September 14, 1975.

UNIVERSITY OF TEXAS AT ARLINGTON
 AND
 U.S. ARMY RESEARCH OFFICE-NORTH CAROLINA
 P.O. Box 12211
 RESEARCH TRIANGLE PARK, NC 27709

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

C. W. CURTIS

University of Oregon
Eugene, OR 97403

C. C. MOORE

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. FINN AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Charalambos D. Aliprantis and Owen Sidney Burkinshaw, <i>On universally complete Riesz spaces</i>	1
Stephen Richard Bernfeld and Jagdish Chandra, <i>Minimal and maximal solutions of nonlinear boundary value problems</i>	13
John H. E. Cohn, <i>The length of the period of the simple continued fraction of $d^{1/2}$</i>	21
Earl Vern Dudley, <i>Sidon sets associated with a closed subset of a compact abelian group</i>	33
Larry Finkelstein, <i>Finite groups with a standard component of type J_4</i>	41
Louise Hay, Alfred Berry Manaster and Joseph Goeffrey Rosenstein, <i>Concerning partial recursive similarity transformations of linearly ordered sets</i>	57
Richard Michael Kane, <i>On loop spaces without p torsion. II</i>	71
William A. Kirk and Rainald Schoneberg, <i>Some results on pseudo-contractive mappings</i>	89
Philip A. Leonard and Kenneth S. Williams, <i>The quadratic and quartic character of certain quadratic units. I</i>	101
Lawrence Carlton Moore, <i>A comparison of the relative uniform topology and the norm topology in a normed Riesz space</i>	107
Mario Petrich, <i>Maximal submonoids of the translational hull</i>	119
Mark Bernard Ramras, <i>Constructing new R-sequences</i>	133
Dave Riffelmacher, <i>Multiplication alteration and related rigidity properties of algebras</i>	139
Jan Rosiński and Wojbor Woyczynski, <i>Weakly orthogonally additive functionals, white noise integrals and linear Gaussian stochastic processes</i>	159
Ryōtarō Satō, <i>Invariant measures for ergodic semigroups of operators</i>	173
Peter John Slater and William Yslas Vélez, <i>Permutations of the positive integers with restrictions on the sequence of differences</i>	193
Edith Twining Stevenson, <i>Integral representations of algebraic cohomology classes on hypersurfaces</i>	197
Laif Swanson, <i>Generators of factors of Bernoulli shifts</i>	213
Nicholas Th. Varopoulos, <i>BMO functions and the $\bar{\partial}$-equation</i>	221