Pacific Journal of Mathematics

GENERATORS OF FACTORS OF BERNOULLI SHIFTS

LAIF SWANSON

Vol. 71, No. 1

November 1977

GENERATORS OF FACTORS OF BERNOULLI SHIFTS

LAIF SWANSON

One of the questions of ergodic theory is that of "relative position" of factors of Bernoulli shift. If \mathscr{F}_0 and \mathscr{F}_1 are factor algebras for a Bernoulli shift T, under what conditions is there an isomorphism ϕ commuting with T such that $\phi \mathscr{F}_0 = \mathscr{F}_1$?

In this paper, we give an example of a Bernoulli shift T of a space X and uncountably many partitions $\{Q_{\alpha}: \alpha \in A\}$ of X with the properties:

(1) $(T, Q_{\alpha}) \cong (T, Q_{\beta})$ for $\alpha, \beta \in A$.

(2) $\bigvee_{-\infty}^{\infty} T^i Q_{\alpha}$ is maximal for its entropy whenever $\alpha \in A$.

(3) There is no isomorphism ϕ commuting with T such that $\phi Q_{\alpha} = Q_{\beta}$ unless $\alpha = \beta$.

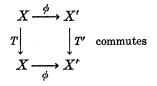
If T is an automorphism of a probability space (X, \mathcal{F}, μ) , a sub-sigma algebra \mathcal{F}_0 of \mathcal{F} is a factor algebra for T if T is an automorphism of (X, \mathcal{F}_0, μ) . That is, \mathcal{F}_0 is a factor algebra for Tmeans $A \in \mathcal{F}_0 \Leftrightarrow TA \in \mathcal{F}_0$. If \mathcal{F}_0 is a factor algebra for T, the automorphism T restricted to (X, \mathcal{F}_0, μ) is called a factor of T, denoted $T | \mathcal{F}_0$.

It is clear that factors of ergodic, weakly mixing, mixing, or Kolmogorov automorphisms are ergodic weakly mixing, mixing, or Kolmogorov respectively. It is known (Ornstein) that factors of Bernoulli shifts are Bernoulli. We investigate the "relative position" of factors of Bernoulli shifts.

This paper is part of a Ph. D. thesis prepared under the supervision of Jacob Feldman. I thank him for many helpful discussions as well as his encouragement. Thanks are also due Donald Ornstein who suggested this problem and listened to my ideas.

Some of the questions one might ask about factors of Bernoulli shifts are:

(1) If T and T' are isomorphic Bernoulli shifts (on spaces X and X') with respective factor algebras \mathscr{F}_0 and \mathscr{F}_0' , under what conditions is there an isomorphism ϕ (defined except on a set of measure zero) such that



(i.e., ϕ is an isomorphism of T and T') and

$$\phi(\mathscr{F}_0) = \mathscr{F}_0$$

(i.e., $A \in \mathscr{F}_0 \Leftrightarrow \phi A \in \mathscr{F}_0$)?

(2) If T and T' are Bernoulli shifts (on spaces X and X' respectively), Q is a partition of X and Q' a partition of X', under what conditions is there an isomorphism ϕ of T and T' so that $\phi Q = Q'$, (i.e. $Q = (Q_1, \dots, Q_n)$, $Q' = (Q_i, \dots, Q'_n)$, and $\phi Q_i = Q'_i$ for each i)?

(3) If T is a Bernoulli shift and \mathscr{F}_0 a factor algebra for T, under what conditions is there another factor algebra \mathscr{F}_1 so that

(a)
$$\mathscr{F}_0 \perp \mathscr{F}_1$$

and

$$(b) \qquad \qquad \mathcal{F}_0 V \mathcal{F}_1 = \mathcal{F}?$$

(We will call such a factor algebra \mathscr{F}_1 a complement of \mathscr{F}_0 .)

We can see a close relationship between questions (1) and (3): If we are given a Bernoulli shift T of X and a factor algebra \mathscr{F}_0 such that $h(T) > h(T | \mathscr{F}_0)$, denote by T_B a Bernoulli shift of a space Y such that $h(T_B) = h(T) - h(T | \mathscr{F}_0)$. Define $T' = T | \mathscr{F}_0 \times T_B$, and denote by \mathscr{F}_0' and \mathscr{F}_1' , respectively, the σ -algebras of the first and second members of the product. Now, the Ornstein isomorphism theorem gives us the fact that T and T' are isomorphic, and an isomorphism ϕ of T and T' taking \mathscr{F}_0 to \mathscr{F}_0' would take some complement of \mathscr{F}_0 to \mathscr{F}_0' , i.e., if ϕ is an isomorphism of T and T' and $\phi(\mathscr{F}_0) = \mathscr{F}_0'$, then $\phi^{-1}(\mathscr{F}_1')$ is a complement for \mathscr{F}_0 .

On the other hand, if $T \cong T'$, and \mathscr{F}_0 and \mathscr{F}_0' are respective factor algebras satisfying

$$h(T | \mathscr{F}_0) = h(T | \mathscr{F}_0')$$

 \mathscr{F}_0 is complemented by \mathscr{F}_1
 \mathscr{F}_0' is complemented by \mathscr{F}_0'

then the Ornstein isomorphism theorem gives isomorphisms

 ϕ_0 from $T|\mathcal{F}_0$ to $T'|\mathcal{F}_0'$

and

$$\phi_1$$
 from $T|\mathcal{F}_1$ to $T'|\mathcal{F}_1'$;

and $\phi = \phi_0 \times \phi_1$ is an isomorphism of T and T' carrying \mathscr{F}_0 to \mathscr{F}_0' . Thus questions (1) and (2) are related in this meanor. If \mathscr{F} and

Thus questions (1) and (3) are related in this manner: If \mathscr{F}_0 and \mathscr{F}_0' are both complemented, there is an isomorphism from T to T'

taking \mathcal{F}_0 to \mathcal{F}_0' . If either is complemented and such an isomorphism exists, they are both complemented.

We can also see that (2) is in a natural way a stronger question than (1): If T and T' are isomorphic Bernoulli shifts, $\bigvee_{-\infty}^{\infty} T^i Q = \mathscr{F}_0$ and $\bigvee_{-\infty}^{\infty} T'^i Q' = \mathscr{F}_0'$, then an isomorphism of T and T' taking Q to Q' will certainly take \mathscr{F}_0 to \mathscr{F}_0' .

Some results about these questions are known. Thouvenot [7] has discovered conditions similar to the condition "finitely determined" (see Ornstein, [4], for this definition) which are equivalent to a factor of a Bernoulli shift being complemented, and has shown that for every Bernoulli shift T and factor algebra \mathscr{F}_0 , there is a factor algebra \mathscr{F}_1 satisfying the weaker conditions

(a)
$$\mathcal{F}_0 \perp \mathcal{F}_1$$

(b) $h(T | \mathscr{F}_0 V \mathscr{F}_1) = h(T)$.

On the other hand, Ornstein and Weiss [6] have shown that if T is a Bernoulli shift, (in fact, any K-automorphism) and \mathscr{F}_0 , \mathscr{F}_1 , and \mathscr{F}_2 factor algebras for T satisfying

$${\mathscr F}_{\scriptscriptstyle 0}\!\subset {\mathscr F}_{\scriptscriptstyle 1}$$
 $h(T\perp {\mathscr F}_{\scriptscriptstyle 0})=h(T\perp {\mathscr F}_{\scriptscriptstyle 1})$

and

then

FiliF

 $\mathcal{F}_0 \perp \mathcal{F}_2$

Thus a factor algebra \mathcal{F}_0 for a Bernoulli shift T can be complemented only if

$${\mathscr F}_{\scriptscriptstyle 1} \supseteq {\mathscr F}_{\scriptscriptstyle 0} \longrightarrow [h(T \,|\, {\mathscr F}_{\scriptscriptstyle 1}) > h(T \,|\, {\mathscr F}_{\scriptscriptstyle 0})] \;.$$

a condition which we will denote by " \mathscr{F}_0 is maximal for its entropy" since it means that if we expand \mathscr{F}_0 we increase the entropy of the factor.

So an immediate question might be "Is it true that if T is a Bernoulli shift and \mathscr{F}_0 a factor algebra which is maximal with respect to its entropy, then \mathscr{F}_0 if complemented?" This is, as we noted earlier, the same question as "If T and T' are isomorphic Bernoulli shifts and \mathscr{F}_0 and \mathscr{F}_0' are respective factor algebras, each maximal

with respect to its entropy, with $h(T | \mathscr{F}_0) = h(T' | \mathscr{F}_0')$, is there an isomorphism ϕ of T and T' so that $\phi(\mathscr{F}_0) = \mathscr{F}_0'?$ The answer to these questions is no—a counterexample has been given by Ornstein [3]. This gives partial answers to questions one and two.

Since Ornstein's methods involve a skew product with a member of the uncountable family of non-Bernoulli K-automorphisms (Ornstein-Shields, [5]), an immediate question is "Is there a Bernoulli shift with an uncountable number if factor algebras $\{\mathscr{F}_{\alpha}: \alpha \in A\}$ with the properties:

(i) \mathscr{F}_{α} is maximal with respect to its entropy, for every $\alpha \in A$ (ii) $h(T|\mathscr{F}_{\alpha}) = h(T|\mathscr{F}_{\beta})$ for every $\alpha, \beta \in A$

(iii) If α , $\beta \in A$, there is no isomorphism ϕ commuting with T such that $\phi(\mathcal{F}_{\alpha}) = \mathcal{F}_{\beta}$ unless $\alpha = \beta$?

We deal with a weaker question, like question (2): Is there an uncountable family $\{T_{\alpha}: \alpha \in A\}$ of isomorphic Bernoulli shifts of spaces X_{α} and partitions Q_{α} of X_{α} such that

(i) $\bigvee_{-\infty}^{\infty} T^i_{\alpha} Q_{\alpha}$ is maximal with respect to its entropy for every $\alpha \in A$

(ii) $(T_{\alpha}, Q_{\alpha}) \cong (T_{\beta}, Q_{\beta})$ for every $\alpha, \beta \in A$

(iii) If α , $\beta \in A$, there is no isomorphism ϕ of T_{α} and T_{β} such that $\phi(Q_{\alpha}) = Q_{\beta}$ unless $\alpha = \beta$?

The family of processes constructed by Ornstein [3] answers this question. Each automorphism is the skew product of a certain Bernoulli shift with one of the non-Bernoulli K-automorphisms and the identity. These automorphisms are proved Bernoulli by Ornstein [3]. We assume a familiarity with the uncountable family of K-automorphisms.

The construction. We first construct a Bernoulli shift T_B ; we will then skew this with a non-Bernoulli K-automorphism and the identity.

 T_B is the common extension of the gadget transformations for a sequence G(n) of gadgets which are defined inductively. It is generated by a partition $R = (R_0, R_1, R_e, R_f)$ which also denotes the partition associated with each gadget. The gadgets depend on certain increasing sequences of integers f(n), k(n), and s(n). G(1) is (A_1, \dots, A_p) and $G^*(1) \subset R_0$.

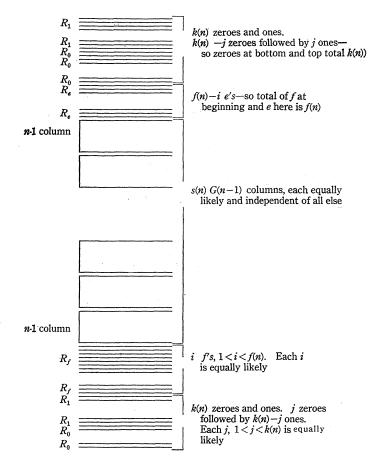
We will describe T_{B} inductively by assuming that G(n) has already been constructed and constructing G(n + 1):

Divide G(n) into s(n) isomorphic gadgets $G(n, 1), \dots, G(n, s(n))$.

Let $\bar{G}(n + 1) = G(n, 1) * \cdots * G(n, s(n)).$

Divide each column C of $\overline{G}(n+1)$ into (f(n)-1)k(n) columns $C(i, j), 1 \leq i \leq f(n), 1 \leq j \leq k(n)$. Precede C(i, j) by j intervals of R_0 , k(n) - j intervals of R_1 , and i intervals R_f , and follow it by f(n) - i intervals of R_i , k(n) - j intervals of R_0 , and j intervals of R_i . These columns are the columns of G(n+1).

So a typical column of G(n) looks like:



We use X to denote $\bigcup_n G^*(n)$, and so T_B is an automorphism of X. Let \overline{T}_{α} be any of the non-Bernoulli K-automorphisms defined by Ornstein and Shields, and let Y_{α} be the space on which \overline{T}_{α} is defined. We define an automorphism, which we call T_{α} , of the space $X \times Y_{\alpha}$ (with the product σ -algebra and measure) by

$$T_{\scriptscriptstyle lpha}(x,\,y) = egin{cases} T_{\scriptscriptstyle B}x,\,ar{T}_{\scriptscriptstyle lpha}y) & ext{ if } x \in R_{\scriptscriptstyle 0} \ T_{\scriptscriptstyle B}x,\,y) & ext{ if } x
otin R_{\scriptscriptstyle 0} \ . \end{cases}$$

Ornstein shows that this automorphism is a Bernoulli shift of $X \times Y_{\alpha}$ if sequences f, s, and k satisfy certain conditions which will cause, among other things, the number of zeroes in an *n*-block for T_{B} to be large compared to the length of an *n*-block for T_{α} and small compared to the length of an (n + 1)-block for T_{α} .

We define a partition $Q^{\alpha} = (Q_0^{\alpha}, Q_1^{\alpha}, Q_e^{\alpha}, Q_f^{\alpha})$ of $X \times Y_{\alpha}$ by

$$egin{aligned} Q^lpha_0 &= \{(x,\ y)\in X imes \ Y_lpha; x\in R_0\};\ Q^lpha_1 &= \{(x,\ y)\in X imes \ Y_lpha; x\in R_1\}\ Q^lpha_e &= \{(x,\ y)\in X imes \ Y_lpha; x\in R_e\};\ Q^lpha_f &= \{(x,\ y)\in X imes \ Y_lpha; x\in R_f\} \end{aligned}$$

Ornstein shows that $\bigvee_{-\infty}^{\infty} T^i Q^{\alpha}$ is maximal with respect to its entropy, and is not complemented, i.e., there is no factor algebra \mathscr{F}_1 for T_{α} so that $\bigvee_{-\infty}^{\infty} T^i_{\alpha} Q^{\alpha} \perp \mathscr{F}_1$ and $\bigvee_{-\infty}^{\infty} T^i_{\alpha} Q^{\alpha} V \mathscr{F}_1$ is the product σ -algebra for $X \times Y_{\alpha}$.

We will show that if a "family" of K-automorphisms, depending on sequences f and s of integers, is fixed, and \overline{T}_{α} and \overline{T}_{β} are nonisomorphic members of this family, then there is no isomorphism ϕ of \overline{T}_{α} and \overline{T}_{β} such that $\phi(Q^{\alpha}) = Q^{\beta}$. (Note that since \overline{T}_{α} and \overline{T}_{β} are from the same family, $h(\overline{T}_{\alpha}) = h(\overline{T}_{\beta})$, and so $h(T_{\alpha}) = h(T_{\beta})$, and so, by the Ornstein isomorphism theorem, $T_{\alpha} \cong T_{\beta}$.) We will show the nonexistence of ϕ by assuming that there is such a ϕ and proving that $\overline{T}_{\alpha} \cong \overline{T}_{\beta}$, using the following facts about the family of Kautomorphisms.

(1) $\bar{T}_{\alpha} = \bar{T}_{\beta}$ if and only if $g_{\alpha}(n)$ for all but finitely many n.

(2) Let $R^{\beta,k}$ be the partition of Y_{β} defined by $R^{\beta,k} = (R_{0}^{\beta,k}, R_{1}^{\beta,k})$ where $R_{1}^{\beta,k} = G^{*}(k)$. There is a certain number $\bar{\varepsilon}$ with the property that if R^{k} is any partition of Y_{α} , $y_{\alpha} \in Y_{\alpha}$ and $y_{\beta} \in Y_{\beta}$ and the R^{k} name of y_{α} agrees with the $R^{\beta,k}$ -name of y_{β} in all but $\bar{\varepsilon}$ places, then $g_{\alpha}(k) = g_{\beta}(k)$.

Let p_1 and p_2 be the projections of $X \times Y_{\alpha}$ onto X and Y_{α} respectively. Since $T_{\beta}\phi = \phi T_{\alpha}$, $\phi Q^{\alpha} = Q^{\beta}$, and $\bigvee_{-\infty}^{\infty} T_{\alpha}^{i}Q^{\alpha} = \phi(p_1)$, we see that $p_1(\phi(x, y)) = x$.

Recall that Y_{α} is partitioned by $P = (P_s, P_f, P_s)$ which generates for \overline{T}_{α} . We define a partition P^{α} of $X \times Y_{\alpha}$ by

$$(x, y) \in P_e^{\alpha} \longleftrightarrow y \in P_e$$
$$(x, y) \in P_f^{\alpha} \longleftrightarrow y \in P_f$$
$$(x, y) \in P_s^{\alpha} \longleftrightarrow y \in P_s.$$

Let P' be the partition $\phi^{-1}(P^{\beta})$ of $X \times Y_{\alpha}$. We can make new "names" \overline{P} and \overline{P}' for the points of $X \times Y_{\alpha}$ by looking at the P and P' names, respectively, in only those positions for which the Q^{α} name is Q_0^{α} . Of course these "names" do not properly shift with T_{α} . The \overline{P}_{α} name of (x, y) is the P name of y; the \overline{P}' name of (x, y) is the P name of y; the \overline{P}' name of (x, y).

We know, since $Q^{\alpha}VP^{\alpha}$ is a generator for T_{α} , that P' names can be coded from $Q^{\alpha}VP^{\alpha}$ names. That is, for every $\varepsilon > 0$ there is an $n = n(\varepsilon)$ and a partition $P_1 \subset \bigvee_{-n}^n T_{\alpha}(Q^{\alpha}VP^{\alpha})$ such that $|P_1 - P'| < \varepsilon$. Let $R^{k,\varepsilon}$ be the partition $(R_{\varepsilon}^{k,\varepsilon}, R_1^{k,\varepsilon})$ of Y_{α} where

$$R_1^{k,x} = \{y_lpha \in Y_lpha \colon p_2 \phi(x, y_lpha) \in G^*(k)\}$$
 .

Let k be large enough that the length of a (k-1) block for T_B is at least twice $n(\bar{\varepsilon}/4)$; that is, choose k so large that strings in the $Q^{\alpha}VP^{\alpha}$ name which are as long as a (k-1) block for T_B predict P' with accuracy at least $1 - (\bar{\varepsilon}/4)$. We will show that whether these strings predict $R_{1,x}^{k,x} R_{0,x}^{k,x}$ is nearly independent of x.

Let A be a $Q^{\alpha}VP^{\alpha}$ string whose Q^{α} part is two consecutive kblocks and whose P part and P' part are mostly k-blocks in the same (k + 1)-block. (Most strings of this length have their \overline{P} -part and $\overline{P'}$ part made up mostly of k-blocks in the same (k + 1)-block.) So we have chosen A so that it is the set of points (x, y) which for a certain j have $T_{B}^{i}x$ in the base of a particular column of $\mathscr{F}(k)$, and have $T_{B}^{j+h(k)}x$ in the base of a particular column of $\mathscr{F}(k)$, and whose P_{α} and P' names in the positions j through j + 2h(k) are made mostly of certain k-blocks in the same (k + 1)-block.

Let B be another $Q^{\alpha}VP^{\alpha}$ string whose \overline{P}^{α} part is the same as for A and whose Q^{α} part is the same as A's except that the number of zeroes at the beginning of the second k-block is larger. That is, B is the set of (x, y) for which $T_B^j x$ is in the base of the same column for $\mathscr{F}(k)$ as for A, but $T_B^{j+k(k)}x$ is in a column which begins with z more zeroes than the corresponding column for A. Thus, the P^{α} part of B is "bunched up" at the beginning of the second k-block when you compare it to that of A.

Since the first halves of the strings A and B are the same, the positions for k-blocks in the \overline{P}' name of B must be the same as for A (because of the rigidity of the block structure for \overline{T}_B). Thus the "predictions" of $R^{k,x}$ and $R^{k,x'}$, where $x' = T_B^x x$, based on strings of length 2n are the same. So $|R^{k,x} - R^{k,x'}| < \overline{\epsilon}/2$. But if k is large enough, all but $\overline{\epsilon}/2$ of the x's have strings of length 2n which are the same of the string for some $T_B^x \overline{k}$ for a fixed \overline{x} . Thus, for this fixed \overline{x} , $|R^{k,\overline{x}} - R^{k,x}| < \overline{\epsilon}$ for all x's except a family of measure less than $\overline{\epsilon}/2$. This and the ergodic theorem give that the $R^{k,\overline{x}}$ name of most y_{α} agrees with the $R^{\beta,k}$ name of $P_2\phi(x, y_{\beta})$ for most x. Thus $\overline{T}_{\alpha} \cong \overline{T}_{\beta}$.

References

1. D. S. Ornstein, Bernoulli shifts with the same entropy are isomorphic, Advances in Math., 4 (1970), 337-352.

2. D. S. Ornstein, Factors of a Bernoulli shifts are Bernoulli shifts, Advances in Math., 5 (1970), 349-364.

3. —, Factors of a Bernoulli shift, Israel J. Math., 21 (1975), 145-153.

4. ____, Ergodic Theory, Randomness, and Dynamical Systems, Yale University Press, 1974.

5. D. S. Ornstein, and P. C. Shields, An uncountable family of K-automorphisms, Advances in Math., 10 (1973), 63-68.

6. D. S. Ornstein, and B. Weiss, *Finitely determined implies very weak Bernoulli*, Israel J. Math., **17** (1974), 94-104.

7. J.-P. Thouvenot, Quelques propriétés des systèmes dynamiques qui se décomposent en deux systèmes dont l'un est un schéma de Bernoulli, Israel J. Math., **21** (1975), 177-207.

Received May 17, 1976. Supported by NSF grant No. MCS 76-06005.

TEXAS A & M UNIVERSITY College Station, TX 77843

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor) University of California Los Angeles, California 90024

C. W. CURTIS University of Oregon Eugene, OR 97403

C.C. MOORE University of California Berkeley, CA 94720 J. DUGUNDJI

Department of Mathematics University of Southern California Los Angeles, California 90007

R. FINN AND J. MILGRAM Stanford University Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA CALIFORNIA INSTITUTE OF TECHNOLOGY UNIVERSITY OF CALIFORNIA MONTANA STATE UNIVERSITY UNIVERSITY OF NEVADA, RENO NEW MEXICO STATE UNIVERSITY OREGON STATE UNIVERSITY UNIVERSITY OF OREGON OSAKA UNIVERSITY UNIVERSITY OF SOUTHERN CALIFORNIA STANFORD UNIVERSITY UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON

AMERICAN MATHEMATICAL SOCIETY NAVAL WEAPONS CENTER

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of MathematicsVol. 71, No. 1November, 1977

Charalambos D. Aliprantis and Owen Sidney Burkinshaw, <i>On universally</i> <i>complete Riesz spaces</i>	1
Stephen Richard Bernfeld and Jagdish Chandra, <i>Minimal and maximal</i> solutions of nonlinear boundary value problems	13
John H. E. Cohn, <i>The length of the period of the simple continued fraction of</i> $d^{1/2}$	21
Earl Vern Dudley, <i>Sidon sets associated with a closed subset of a compact abelian group</i>	33
Larry Finkelstein, <i>Finite groups with a standard component of type J</i> ₄	41
Louise Hay, Alfred Berry Manaster and Joseph Goeffrey Rosenstein, Concerning partial recursive similarity transformations of linearly ordered sets	57
Richard Michael Kane, <i>On loop spaces without p torsion. II</i>	71
William A. Kirk and Rainald Schoneberg, <i>Some results on</i>	, 1
pseudo-contractive mappings	89
Philip A. Leonard and Kenneth S. Williams, <i>The quadratic and quartic</i>	
character of certain quadratic units. I	101
Lawrence Carlton Moore, A comparison of the relative uniform topology	
and the norm topology in a normed Riesz space	107
Mario Petrich, Maximal submonoids of the translational hull	119
Mark Bernard Ramras, <i>Constructing new R-sequences</i>	133
Dave Riffelmacher, <i>Multiplication alteration and related rigidity properties</i> of algebras	139
Jan Rosiński and Wojbor Woyczynski, Weakly orthogonally additive	
functionals, white noise integrals and linear Gaussian stochastic	
processes	159
Ryōtarō Satō, Invariant measures for ergodic semigroups of operators	173
Peter John Slater and William Yslas Vélez, <i>Permutations of the positive</i>	
integers with restrictions on the sequence of differences	193
Edith Twining Stevenson, <i>Integral representations of algebraic cohomology</i>	
classes on hypersurfaces	197
Laif Swanson, Generators of factors of Bernoulli shifts	213
Nicholas Th. Varopoulos, <i>BMO functions and the</i> $\overline{\partial}$ <i>-equation</i>	221