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The ∂-equation associated with the Corona problem for several complex variables is examined and the relation of that equation with BMO functions on the boundary is brought to light. A new characterisation, closely related with the H^1 duality, for BMO functions is obtained.

0. **Introduction.** This paper came out of an unsuccessful attempt to prove the Corona theorem for n-dimensions.

If we try to generalise L. Carleson's 1-dimensional proof (with the modifications introduced by L. Hörmander) (cf. [1], [2], [9]), we come up against the following problem:

Solve the $\bar{\partial}$-equation

$$\bar{\partial}u = \mu$$

in, say, the complex n-ball where μ is an arbitrary $\bar{\partial}$-closed differential form that satisfies an appropriate Carleson condition and where we require the solution u to have L^∞ boundary values (also in an appropriate sense, cf. [9]).

We shall show in Part 3 of this paper that it is not always possible to solve the above equation, and that the best we can obtain in general for the boundary values of the solution is a BMO condition.

However along the way a number of positive results will be obtained. In Part 1 we obtain a new characterisation of BMO functions which is closely related with the BMO, H^1 duality. This characterisation, grosso motto, runs as follows: $f \in L^1(\boldsymbol{R}^n)$ is a BMO function in $\boldsymbol{R}^n$ if and only if it is the boundary value of some function F defined in the upper half space $\boldsymbol{R}^{n+1}_+$ such that

$$|\nabla F| = \left(\sum_{i=1}^{n}\left|\frac{\partial F}{\partial x_i}\right| + \left|\frac{\partial F}{\partial y}\right|\right)d(\mathrm{Vol})$$

is a Carleson measure. Exact statements will be given later. The extension F of f in the upper half space is *not*, in general, the harmonic extension and it is not easy to describe it explicitly.

In Part 2 the above results are generalised to the complex ball and to general strictly pseudoconvex domains. This generalisation is tedious but essentially routine.

In Part 3 the real "raison d'etre" of this characterisation appears and it is used to study the $\bar{\partial}$-equation and the Corona problem. It should be remarked however that for the proof of Theorem 3.1.2, which is directly related to the Corona problem, the rather lengthy and tedious Theorem 2.1.1 is not essentially (cf. the remark that follows the proof of Theorem 3.1.2). All that one needs is the much easier Theorem 1.1.1.

Part 1 uses entirely real variable methods. Part 2 uses some of the geometry of pseudo-convex domains. Part 3 finally uses genuine several complex variables methods and in particular the Henkin Integral formulas.

Finally a few words about the style.

As it happens this paper is far too long. To avoid making it longer still I have resorted to a few standard "tricks." Often theorems that are stated in full generality are only proved under some special restrictive condition. This is especially applicable to strictly pseudo-convex domains where all the explicit calculations are carried out only for the complex ball and often only in C^2. But hopefull the reader who possesses some technique will be able to see without too much difficulty how the proofs can be made to work in full generality.

Part 1. The real variable theory.

1.1. Statement of the results. In this paper we shall adopt the notations of [15].

Let us recall that BMO ($\boldsymbol{R}^n$) (or simply BMO) is the space of measurable functions f on $\boldsymbol{R}^n$ that satisfy the following condition,

$$\sup_I \frac{1}{|I|}\int_I |f - f_I|\,dx < +\infty$$

where I runs through all possible cubes of $\boldsymbol{R}^n$ with sides parallel to the axes, $|I|$ denotes the volume of I and $f_I = |I|^{-1}\int_I f dx$, the average of f over I. The key reference for BMO is [6]. Let us also recall that a measure μ on $\mathring{\boldsymbol{R}}^{n+1}_+$ (the interior of $\boldsymbol{R}^{n+1}_+$) is said to satisfy the Carleson condition, or simply to be a Carleson measure, if:

$$\sup_I \frac{|\mu|(\tilde{I})}{|I|} < +\infty$$

where I is as before, $|\mu|$ is the absolute value of μ and

$$\tilde{I} = \{(x, y) \in \mathring{\boldsymbol{R}}^{n+1}_+;\ x \in I,\ y \in (0, h)\}$$

where h is the length of the side of I (cf. [3], [1], [15]).

The main result in this paragraph is summarised in the following theorem.

THEOREM 1.1.1. *Let* $f \in \mathrm{BMO}(\boldsymbol{R}^n)$ *and let us suppose that* f *has compact support. Then there exists an infinitely differentiable function* $F \in C^\infty(\mathring{\boldsymbol{R}}_+^{n+1})$ *that satisfies the following conditions.*

(i) $\lim_{y\to 0} F(x, y) - f(x) \in L^\infty(\boldsymbol{R}^n)$.

(ii) *The measure*

$$|\nabla F|\, dx\, dy = \Big(\sum_{i=1}^{n} \Big|\frac{\partial F}{\partial x_i}\Big| + \Big|\frac{\partial F}{\partial y}\Big|\Big) dx\, dy$$

satisfies the Carleson condition.

(iii) *There exists some* $g(x) \in L^1(\boldsymbol{R}^n)$ *such that*

$$\sup_{y>0} |F(x, y)| \leqq g(x) .$$

(iv) $|\nabla F| = 0(1/y)$.

The above theorem has a number of converses that can be summarised in the following theorems:

THEOREM 1.1.2. *Let* $F(x, y) \in C^1(\mathring{\boldsymbol{R}}_+^{n+1})$ *be a once continuously differentiable function such that* $|\nabla F| dxdy$ *is a Carleson measure and such that the limit*

$$\lim_{y\to 0} F(x, y) = f(x)$$

exists for almost all $x \in \boldsymbol{R}^n$. *Then the following assertions hold.*

(i) *If* $n = 1$ *then* $f \in \mathrm{BMO}(\boldsymbol{R})$.

(ii) *If* $n \geqq 2$ *is arbitrary but in addition we suppose that:* $|\nabla F| = 0(1/y)$ *then* $f \in \mathrm{BMO}(\boldsymbol{R}^n)$.

It should be remarked here, once and for all, that the condition $|\nabla F| = 0(1/y)$, both in Theorem 1.1.1 and Theorem 1.1.2 is purely technical and not very important in our context.

Let us denote by $D = \{z \in \boldsymbol{C};\ |z| \leqq 1\}$ the complex disc.

THEOREM 1.1.3. *Let* $F \in C^1(\mathring{D})$ *be a once continuously differentiable function in* $\mathring{D}$ (the interior of the complex disc) *and let us denote*

$$F_r(e^{i\theta}) = F(re^{i\theta}),\ e^{i\theta} \in \partial D = \boldsymbol{T} \quad 0 \leqq r < 1 .$$

Let us suppose that $|\nabla F| dx\, dy$ *is a Carleson measure in* $\mathring{D}$ *and that*

$$F_r \xrightarrow[r\to 1]{} S \in \mathscr{D}'(\partial\mathscr{D})$$

in the weak distribution topology $\sigma(\mathscr{D}', C^\infty)$.

Then we have:

$$|\langle P, S\rangle| \leqq C\|P\|_1$$

for all analytic polynomials $P(z) = \sum_{n=0}^{N} a_n z^n$ *where* C *is some constant that is independent of* P.

The notion of a Carleson measure in $\mathring{D}$ is analogous to that of $\mathring{\boldsymbol{R}}_+^{n+1}$.

$\langle P, S\rangle$ indicates the scalar product $\left(\text{sometimes abusively denoted as } (1/2\pi)\int_0^{2\pi} P(e^{i\theta})dS(\theta)\right)$ between $\mathscr{D}'$ and C^∞, and

$$\|P\|_1 = \frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|\, d\theta$$

denotes the $L^1(\boldsymbol{T})$ norm of P. Let $S \in \mathscr{S}'(\boldsymbol{R}^n)$ be a temperate distribution. We say that S is of analytic type if

$$\operatorname{supp} \hat{S} \subset \{\xi_i \geqq 0 \quad i = 1, 2, \cdots, n\}$$

we have then:

THEOREM 1.1.4. *Let* $F \in C^1(\mathring{\boldsymbol{R}}_+^{n+1})$ *be a once continuously differentiable function in* $\mathring{\boldsymbol{R}}_+^{n+1}$ *and let us suppose that* $|\nabla F|\,dx\,dy$ *is a Carleson measure. Let us suppose that the continuous functions* $F_y(x) = F(x, y)$ *converge to a distribution* $S \in \mathscr{E}'(\boldsymbol{R}^n)$ *with compact support when* $y \longrightarrow 0$ (the convergence takes place in the weak distribution topology). *Then we have*

$$|\langle S, \varphi\rangle| \leqq C\|\varphi\|_1$$

for all infinite differentiable functions $\varphi_t \in \mathscr{S}(\boldsymbol{R}^n)$ *of rapid decrease at infinity and of anylytic type.*

In §1.2 we shall give a direct and elementary proof of Theorem 1.1.1. In §1.3 we shall give an alternative approach to Theorem 1.1.1, less elementary but which has the advantage that it generalises to strictly pseudoconvex domains. In 1.4 we shall prove the converses and examine the relation they bear with the Stein and Fefferman BMO H^1 duality.

1.2. Proof of Theorem 1.1.1. We start with the slightly weaker

PROPOSITION 1.2.1. *Let* $f \in \mathrm{BMO}(\boldsymbol{R}^n)$ *have compact support. Then there exists* $F \in L^\infty_{\mathrm{loc}}(\mathring{\boldsymbol{R}}_+^{n+1})$ *and* $g \in L^1(\boldsymbol{R}^n)$ *such that:*

(a) $\lim_{y\to 0} F(x, y) - f(x) \in L^\infty(\boldsymbol{R}^n)$.

(b) $|F(x, y)| \leqq g(x)$. $\forall (x, y) \in \boldsymbol{R}_+^{n+1}$

(c) $|\nabla F|$, *taken in the sense of distribution theory, is a Carleson* *measure in* $\mathring{\boldsymbol{R}}_{+}^{n+1}$

REMARK. In (a) and (b) what we mean of course is that there exists a representative of F in the class $L_{\mathrm{loc}}^{\infty}$ for which (a) and (b) hold.

For simplicity's sake, we shall give the proof of Proposition 1.2.1 for $n=1$, but the proof readily generalises to arbitrary dimension.

The proof of Proposition 1.2.1 depends on the following two lemmas.

LEMMA 1.2.1. *Let* $f \in \mathrm{BMO}(\boldsymbol{R})$ *and let us suppose that* $\operatorname{supp} f \subset [0,1]$ *and that the average of* f *in* $[0,1]$ *is* 0. *Then there exists a family* Ω *of distinct closed diadic subintervals of* $[0,1]$ *and a corresponding family* $\{\alpha_\omega \in \boldsymbol{C};\ \omega \in \Omega\}$ *of complex numbers such that*

$$|\alpha_\omega| \leqq C \quad \forall \omega \in \Omega \tag{1.2.1}$$

$$\sum_{\omega \subset I} |\omega| \leqq C|I| \quad \textit{for all intervals } I \tag{1.2.2}$$

$$f(x) - \sum_{\omega \in \Omega} \alpha_\omega \chi_\omega(x) \in L^\infty(\boldsymbol{R}) \tag{1.2.3}$$

where χ_E *denotes in general the characteristic function of the set* E, *and* C *is a constant that depends only on the* BMO *norm of* f.

LEMMA 1.2.2. *Let* f *be as in Lemma* 1.1.1 *and let* σ_1, σ_2 *be two adjacent diadic intervals of equal length. We have then*:

$$|f_{\sigma_1} - f_{\sigma_2}| \leqq C \tag{1.2.4}$$

where C *is a constant that depends only on the* BMO *norm of* f *and where we denote as before* $f_\sigma = 1/|\sigma| \int_\sigma f\,dx$.

REMARK. What is of some interest is that the conditions above, (1.2.1)-(1.2.4) characterise B.M.O. functions. This is a consequence of the proof below. (It can also be seen directly.)

Lemma 1.2.1, which, to my knowledge, has been proved for the first time by J. Garnett (unpublished 1974) depends on a Calderon-Zygmund argument, (stopping time) and holds also in the context of diadic martingales. The proof will be omitted. Lemma 1.2.2 is a trivial application of the BMO condition on the interval $\sigma_1 \cup \sigma_2$.

Proof of Proposition 1.2.1. Let $I = [a, a+h]$ be an arbitrary interval and let us denote by

$$\tilde{I} = \{(x, y);\ a \leqq x \leqq a+h\ ,\quad 0 \leqq y \leqq h\} \subset \boldsymbol{R}_{+}^{2}$$

the square in the upper half plane with I as base.

Let now f be as in Proposition 1.2.1 with $n = 1$ and supp $f \subset [0, 1]$. Let Ω and $\{\alpha_\omega; \omega \in \Omega\}$ be as in Lemma 1.2.1 and let us define

$$F(x, y) = \sum_{\omega \in \Omega} \alpha_\omega \chi_{\tilde{\omega}}(x, y) \quad \forall (x, y) \in \boldsymbol{R}^2_+$$

which is a function defined in the upper half space with support in unit square which is easily seen (by condition (1.2.2)) to belong to $L^\infty_{\mathrm{loc}}(\mathring{\boldsymbol{R}}^2_+)$. We shall prove that F satisfies the conditions of Proposition 1.2.1.

Indeed if we define

$$g(x) = \sum_{\omega \in \Omega} |\alpha_\omega| \chi_\omega$$

we have trivially:

$$(1.2.5) \qquad |F(x, y)| \leqq g(x) \quad \forall (x, y) \in \boldsymbol{R}^2_+$$

$$(1.2.6) \qquad \lim_{y \to 0} F(x, y) = \sum_{\omega \in \Omega} \alpha_\omega \chi_\omega \quad p \cdot p \cdot x \in \boldsymbol{R}\,.$$

Now condition (1.2.2) implies easily that $g \in L^1(\boldsymbol{R})$ and from the above we see that conditions (a) and (b) of Proposition 1.2.1 are satisfied.

Let us now denote:

$$\mu = \frac{\partial F}{\partial x} \qquad \nu = \frac{\partial F}{\partial y}$$

in the sense of distribution theory.

It is an easy matter to see that μ and ν are bounded Radon measures in $\boldsymbol{R}^2_+$. Indeed let $I = [a, a + h]$ be an arbitrary interval of $\boldsymbol{R}$, then $\partial \chi_{\tilde{I}}/\partial x$ (in the distribution sense) is the Lebesgue linear measure concentrated on the two vertical segments $\{x = a, a \leqq y \leqq h\}$ and $\{x = a + h, 0 \leqq y \leqq h\}$ (with sign ± 1 in fact) and $\partial \chi_{\tilde{I}}/\partial x$ is the Lebesgue linear measure on the horizontal segment $\{a \leqq x \leqq a+h, y=h\}$ (with sign -1). From this and conditions (1.2.1) and (1.2.2) it follows that the two series

$$\sum_{\omega \in \Omega} \alpha_\omega \frac{\partial \chi_{\tilde{\omega}}}{\partial x} \qquad \sum_{\omega \in \Omega} \alpha \omega \frac{\partial \chi_{\tilde{\omega}}}{\partial y}$$

converge normally in $M(\mathring{\boldsymbol{R}}^2_+)$ (the space of bounded measures in $\mathring{\boldsymbol{R}}^2_+$) and this proves our assertion

To prove condition (c) we must verify separately the following inequalities

$$(1.2.7) \qquad |\nu|(\tilde{I}) \leqq C|I|$$

$$(1.2.8) \qquad |\mu|(\tilde{I}) \leqq C|I|$$

for an arbitrary diadic interval $I \subset \boldsymbol{R}$ with C some constant independent of I.

From what has been said just above, the inequality (1.2.7) is an immediate consequence of conditions (1.2.1) and (1.2.2). The inequality (1.2.8) is harder however (condition (1.2.4) has to be used here). The rest of the proof will be devoted to the proof of (1.2.8).

Let

$$I = I_0 = \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right] \quad n \geqq 0 \quad p \geqq 0$$

be arbitrary and let us denote:

$$I_1 = \left[\frac{p-1}{2^n}, \frac{p}{2^n}\right], \quad I_2 = \left[\frac{p+1}{2^n}, \frac{p+2}{2^n}\right].$$

the two adjacent intervals. Let us denote:

$$\mu_i = \sum_{\substack{\omega \in \mathcal{A} \\ \omega \subset I_i}} \alpha_\omega \frac{\partial \chi_{\tilde{\omega}}}{\partial x} \quad i = 0, 1, 2 \,.$$

$$\lambda_1 = \sum \alpha_\omega \frac{\partial \chi_{\tilde{\omega}}}{\partial x} \quad \omega \in \Omega \quad \omega = \left[\frac{p}{2^n}, b\right] \quad \text{for some} \quad b > \frac{p+1}{2^n}$$

$$\lambda_2 = \sum \alpha_\omega \frac{\partial \chi_{\tilde{\omega}}}{\partial x} \quad \omega \in \Omega \quad \omega = \left[a, \frac{p+1}{2^n}\right] \quad \text{for some} \quad a < \frac{p}{2^n} \,.$$

(The intervals ω in λ_1 and λ_2 contain I, have a common end point with I and are among the ones that have not already been counted in μ_0.)

$$\rho_1 = \sum \alpha_\omega \frac{\partial \chi_{\tilde{\omega}}}{\partial x} \quad \omega \in \Omega \quad \omega = \left[a, \frac{p}{2^n}\right] \quad \text{for some} \quad a < \frac{p-1}{2^n}$$

$$\rho_2 = \sum \alpha_\omega \frac{\partial \chi_{\tilde{\omega}}}{\partial x} \quad \omega \in \Omega \quad \omega = \left[\frac{p+1}{2^n}, b\right] \quad \text{for some} \quad b > \frac{p+2}{2^n} \,.$$

(The intervals ω in ρ_1, ρ_2 have a common end point with I, have empty intersection with $\mathring{\omega}$ and are among the ones that have not already been counted in the μ_i, $i = 1, 2$.)

It is quite clear that to prove (1.2.8) it suffices to prove the following inequalities.

(1.2.9) $$|\mu_i|(\tilde{I}) \leqq C|I| \quad i = 0, 1, 2$$

(1.2.10) $$|\lambda_1 + \rho_1|(\tilde{I}) \leqq C|I|$$

(1.2.11) $$|\lambda_2 + \rho_2|(\tilde{I}) \leqq C|I| \,.$$

(1.2.9) is an immediate consequence of conditions (1.2.1) and (1.2.2). Let us prove (1.2.10) the proof of (1.2.11) is identical. Let us consider the measure

$$\theta_1 = |\lambda_1 + \rho_1|\chi_{\tilde{I}} .$$

Then θ_1 can be computed explicitly. It is equal to the Lebesgue linear measure on the vertical segment $\{x = p/2^n, 0 \leqq y \leqq 2^{-n}\}$ multiplied by the constant $k_1 = |l_1 - r_1|$ where

$$l_1 = \sum \alpha_\omega \quad \omega \in \Omega \quad \omega = \left[\frac{p}{2^n}, b\right] \quad \text{for some} \quad b > \frac{p+1}{2^n}$$

$$r_1 = \sum \alpha_\omega \quad \omega \in \Omega \quad \omega = \left[a, \frac{p}{2^n}\right] \quad \text{for some} \quad a < \frac{p-1}{2^n} .$$

We must prove therefore that

$$k_1 \leqq C . \tag{1.2.12}$$

Towards that let us denote

$$m_i = \sum_{\substack{\omega \in \Omega \\ \omega \subset I_i}} \alpha_\omega |\omega| \quad i = 0, 1, 2$$

$$s = \sum_{\substack{p/2^n \in \mathring{\omega} \\ \omega \in \Omega}} \alpha_\omega \quad (\mathring{\omega} \text{ is the interior of } \omega) .$$

We have then:

$$f_I = s + l_1 + \frac{m_0}{|I|} + h_I$$

$$f_{I_1} = s + r_1 + \frac{m_1}{|I|} + h_{I_1}$$

where $h \in L^\infty(\boldsymbol{R})$ is the remainder term in (1.2.3). We conclude therefore that:

$$k_1 = |l_1 - r_1| \leqq |f_I - f_{I_1}| + \frac{|m_0| + |m_1|}{|I|} + C$$

and our assertion (1.2.12) follows then from (1.2.2) and (1.2.4).

We almost have a proof of Theorem 1.1.1 now. Indeed the function F constructed above can be easily smoothed out in $\mathring{\boldsymbol{R}}^2_+$ to be made C^∞ and satisfy conditions (i), (ii), and (iii). The only thing that can give a little trouble is condition (iv). It can be shown that provided that the smoothing out above is done with care we can achieve condition (iv) also. We shall not do that however for two reasons, firstly because condition (iv) is totally unessential and secondly because we will be able to get condition (iv) for free (so to speak) in our alternative approach in the next paragraph.

1.3. Alternative approach to Theorem 1.1.1. We shall give here an alternative approach to Theorem 1.1.1. We shall treat the compact case $D = \{z \in \boldsymbol{C};\ |z| \leqq 1\}$ of the complex unit disc. We do that to avoid unnecessary complications at infinity and also because this method is designed to generalise to bounded pseudoconvex domains, and the disc is an (essentially the only) example of a pseudoconvex domain in $\boldsymbol{C}$. The modifications needed to deal with $\boldsymbol{R}_+^{n+1}$ are rather easy.

The starting point of our approach is the following.

PROPOSITION 1.3.1. *Let $f \in \mathrm{BMO}(\partial D) = \mathrm{BMO}(\boldsymbol{T})$* (i.e., a 2π-periodic BMO function on $\boldsymbol{R}$), *then there exists a Carleson measure in $\mathring{D}$ such that*

$$f(\zeta) - \int_{z \in \mathring{D}} P_z(\zeta) d\mu(z) \in L^\infty(\partial D) . \tag{1.3.1}$$

Conversely any function f that satisfies (1.3.1) *is a* BMO *function on the circle ∂D. Here we denote*:

$$P_z(\zeta) = c\frac{1 - |z|^2}{|1 - \bar{\zeta} z|^2} = c\frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} = P_z(\theta)$$

where $z = re^{i\varphi} \in \mathring{D}$ and $\zeta = e^{i\theta} \in \partial D$, for the Poisson kernel of the circle (c is the normalisation constant).

This Proposition is an immediate consequence of the BMO, H^1 duality. A direct proof of this proposition (i.e., one that does not depend on the duality) has also been given by L. Carleson in [4].

Let us define a new function.

$$\tilde{P}_z(u) = P_z(\zeta)\chi_{(r,1)}(\rho)$$
$$\forall z = re^{i\varphi} \in \mathring{D}\,, \quad u = \rho\zeta \in \mathring{D}\,, \quad 0 < r, \rho < 1 \quad \zeta \in \partial D\,,$$

where $\chi_{(r,1)}$ denotes the characteristic function of the interval $(r, 1)$.

Let now $f \in \mathrm{BMO}(\partial D)$ be some BMO function on the circle and let μ be some Carleson measure that satisfies (1.3.1). Let us then define

$$F(u) = \int_{z \in \mathring{D}} \tilde{P}_z(u) d\mu(z) \quad u \in \mathring{D}$$
$$g(\zeta) = \int_{z \in \mathring{D}} P_z(\zeta) d|\mu|(z) \quad \zeta \in \partial D\,.$$

It is perfectly clear then that $F \in L^\infty_{\mathrm{loc}}(\mathring{D})$, that $g \in \mathrm{BMO}(\partial D)$, and that:

$$|F(u)| \leqq g(\zeta) \quad \forall u = \rho\zeta \quad 0 < \rho < 1 \quad \zeta \in \partial D\,. \tag{1.3.2}$$

Also an easy passage to the limit under the integral sign implies that

$$\lim_{\rho \to 1} F(\rho\zeta) = \int_{z \in \mathring{D}} P_z(\zeta) d\mu(z) \tag{1.3.3}$$

for all $\zeta \in \partial D$ such that $g(\zeta) < +\infty$. We shall prove the following

LEMMA 1.3.1.
(i) *$\partial F/\partial\rho$ is a Carleson measure,*
(ii) *$\partial F/\partial\theta$ is a Carleson measure,*
where of course $u = \rho e^{i\theta}$ and the derivatives are taken in the sense of the distribution theory of $\mathring{D}$.

It is quite clear that (1.3.2), (1.3.3), and the above lemma provide us with an alternative proof of Proposition 1.2.1. The proof of Lemma 1.3.1 will be broken up in a number of separate steps.

Let us fix $z \in \mathring{D}$ and let us denote by

$$\nu_z = \frac{\partial \tilde{P}_z(u)}{\partial\rho} \quad u = \rho e^{i\theta} \quad 0 < \rho < 1 .$$

We have then:

LEMMA 1.3.2. *ν_z is for every fixed $z \in \mathring{D}$ a measure that satisfies:*
(i) $\partial F/\partial\rho = \int_{z \in \mathring{D}} \nu_z d\mu(z)$,
(ii) $\|\nu_z\| \leqq 1$.

Proof. ν_z is of course just the Lebesgue linear measure on the circle $u = |z|e^{i\theta}(0 \leqq \theta \leqq 2\pi)$ multiplied by the Poisson kernel $P_z(\theta)$. From this and the fact that $1/2\pi \int_0^{2\pi} P_z(\theta)d\theta = 1$ (ii) follows. (i), on the other hand, is immediate by the definition of F.

Let now $\zeta_0 \in \partial D$ be fixed and let us denote

(1.3.4) $$\tilde{I}_h = \{z \in D; |z - \zeta_0| < h\} .$$

It is quite clear from the above that we have:

(1.3.5) $$|\nu_z|(\tilde{I}_h) \leqq ch\frac{1-r}{|z-\zeta_0|^2} \quad \forall z = re^{i\varphi} \quad 0 \leqq r < 1 \text{ s.t. } |z-\zeta_0| > 1000\,h$$

(1.3.6) $$|\nu_z|(\tilde{I}_h) = 0 \quad \forall z = re^{i\theta} \quad 1 - r > h$$

and we deduce from (1.3.5) and (1.3.6) that

(1.3.7) $$|\nu_z|(\tilde{I}_h) \leqq \frac{ch^2}{|z-\zeta_0|^2} \quad \forall z = re^{i\varphi} \quad 0 \leqq r < 1 \text{ s.t. } |z-\zeta_0| > 1000\,h .$$

We can now give the proof of Lemma 1.3.1 (i).

Proof. Let $\tilde{I}_h$ be as in (1.3.4) and let

$$\nu_1 = \int_{|z-\zeta_0| \leqq 1000h} \nu_z d|\mu|(z); \; \nu_2 = \int_{|z-\zeta_0| > 1000h} \nu_z d|\mu|(z) .$$

It suffices to verify separately the following two inequalities.

$$\nu_1(\tilde{I}_h) \leqq ch \tag{1.3.8}$$

$$\nu_2(\tilde{I}_h) \leqq ch , \tag{1.3.9}$$

where c is some constant independent of h. For indeed we always have $|\partial F/\partial \rho| \leqq \nu_1 + \nu_2$ and our lemma follows.

We have now:

$$\nu_1(\tilde{I}_h) \leqq \int_{|z-\zeta_0|\leqq 1000h} ||\nu_z|| d|\mu|(z) \leqq |\mu|\{z \in \mathring{D}; |z - \zeta_0| \leqq 1000\, h\} \leqq ch$$

by Lemma 1.3.2 (ii) and the hypothesis on μ. This proves (1.3.8). On the other hand using (1.3.7) we see that

$$\nu_2(\tilde{I}_h) \leqq ch^2 \int_{1000h}^{\infty} \frac{dF(t)}{t^2} \tag{1.3.10}$$

where

$$F(t) = |\mu|\{z \in \mathring{D} \mid |z - \zeta_0| \leqq t\} .$$

Our hypothesis on μ implies that

$$F(t) \leqq ct$$

and an easy integration by parts in (1.3.10) then proves the required inequality (1.3.9), and completes the proof of the first part of Lemma 1.3.1.

For every fixed $z \in \mathring{D}$ let us define

$$\rho_z = \frac{\partial F}{\partial \theta} \quad u = \rho e^{i\theta}$$

the derivative being taken in the distribution sense. We have then:

LEMMA 1.3.3. *ρ_z is for every fixed $z \in \mathring{D}$ a measure that satisfies:*
(i) $\partial F/\partial \theta = \int_{z \in \mathring{D}} \rho_z d\mu(z)$
(ii) $||\rho_z|| \leqq C$,
where C is some numerical constant.

Proof. That ρ_z is a measure and that (i) holds is obvious. To prove (ii) we just have to observe that for all fixed $z = re^{i\varphi}$ we have

$$\begin{aligned} ||\rho_z|| = \int_{z \in \mathring{D}} d|\rho_z| &\leqq c\int_r^1 d\rho \int |dP_z(\theta)| \\ &\leqq c(1 - r)\, \mathrm{Max}_\theta |P_z(\theta)| \leqq c \end{aligned}$$

since $P_z(\theta)$ is a function that is monotone in two pieces as θ varies in

$[0, 2\pi]$. This proves the lemma. Let now $\zeta_0 \in \partial D$ be fixed and let $\tilde{I}_h$ be as in (1.3.4). We have then

$$(1.3.11) \qquad |\rho_z|(\tilde{I}_h) \leqq \frac{ch^2}{|z-\zeta_0|^2} \quad \forall z \in \mathring{D} \text{ s.t. } |z-\zeta_0| > 1000\,h\,.$$

To prove this we just have to use the estimate

$$\left|\frac{\partial P_z(\theta)}{\partial \theta}\right| = c\frac{(1-r)|\sin(\theta-\varphi)|}{|z-e^{i\theta}|^4} \leqq \frac{c}{|z-e^{i\theta}|^2}$$

valid for all $z = re^{2\varphi}$ and all $\theta \in [0, 2\pi]$.

From estimate (1.3.11) and Lemma 1.3.3 (ii) we can give the proof of Lemma 1.3.1 just as before. This concludes the proof of Lemma 1.3.1.

Now to give a proof of Theorem 1.1.1 with this method we have to modify the definitions of $\tilde{P}_z$ and of the function F so as to obtain a C^∞ function in $\mathring{B}$. But this is easy. It suffices to truncate P_z with a smooth function rather than the characteristic function and define:

$$(1.3.12) \qquad \tilde{P}_z(u) = P_z(\zeta)\varphi\left(\frac{1-|u|}{1-|z|}\right) \quad \forall u = |u|\zeta\,, \quad \zeta \in \partial D$$

where $\varphi(t)$ $0 \leqq t$ is some positive C^∞ function chosen once and for all that satisfies

$$\varphi \equiv 0 \quad t > 2 \quad \varphi \equiv 1 \quad t \in [0, 1]\,.$$

If we define $F(u)$ then as before we obtain a function that satisfies all the conditions of Theorem 1.1.1. The condition (iv) is the only new thing that has to be verified but it is easy and will be left as an exercise for the dedicated reader.

1.4. The converse of Theorem 1.1.1 and the use of Stoke's formula.

Proof of Theorem 1.1.2 (i). Let F be as in the theorem and let us denote by:

$$\tilde{I}_h = \{(x, y) \in \mathring{R}^2_+;\ x \in (a, a+h),\ y \in (0, h)\}\,.$$

Then by our hypothesis

$$(1.4.1) \qquad \int_{\tilde{I}_h} |\nabla F|\,dx\,dy \leqq ch$$

where c is of course independent of h and a. Then by Fubini's theorem there exists some $h_0 \in (0, h)$ such that

$$\int_a^{a+h} |\nabla F|(x, h_0)|\, dx \leqq c$$

and this of course implies that

$$|F(x, h_0) - F(a, h_0)| \leqq c \quad \forall x \in (a, a+h) . \tag{1.4.2}$$

On the other hand we always have

$$\left|\lim_{y\to 0} F(x, y) - F(x, h_0)\right| \leqq \int_0^{h_0} |\nabla F(x, y)|\, dy \tag{1.4.3}$$

for every fixed $x \in (a, a+h)$. From (1.4.2) and (1.4.3) we conclude therefore that

$$|f(x) - F(a, h_0)| \leqq c + \int_0^h |\nabla F(x, y)|\, dy$$

for all $x \in (a, a+h)$ and integrating the above inequality in $x \in (a, a+h)$ and using (1.4.1) we get

$$\int_a^{a+h} |f(x) - F(a, h_0)|\, dx \leqq ch ,$$

which proves the required result.

The proof of part (ii) is identical only simpler; for, by our additional hypothesis, we do not need to use Fubini to get the preliminary inequality (1.4.2).

Proof of Theorem 1.1.3. Let F and P be as in the theorem. Let $r \in (0, 1)$ and let us apply Stokes's formula to $D_r = \{z \in \boldsymbol{C};\ |z| \leqq r\}$ we get then

$$\int_{Dr} \frac{\partial F}{\partial \bar{z}} P(z) d\bar{z} \wedge dz = c \int_{\partial D_n} F(z) P(z) dz .$$

From this we conclude, letting $r \longrightarrow 1$, that

$$|\langle S, P\rangle| \leqq c \int_{\mathring{D}} |\nabla F|\, |P|\, dx\, dy$$

(where $z = x + iy$) and this together with our hypothesis on $|\nabla F|$ proves our theorem.

Proof of Theorem 1.1.4. Let us consider the poly-half space

$$\boldsymbol{P}^n = \{z = (z_1, z_2, \cdots) \in \boldsymbol{C}^n;\ \operatorname{Im} z_j \geqq 0\} ,$$

the distinguished boundary of $\boldsymbol{P}^n$ can then be identified with $\boldsymbol{R}^n$ and any function $f \in \mathscr{S}(\boldsymbol{R}^n)$ of analytic type admits a unique extension $\tilde{f}$ to an analytic function in $\mathring{\boldsymbol{P}}^n$.

We shall also identify $\boldsymbol{R}_+^{n+1}$ with a closed subspace of $\boldsymbol{P}^n$ by the correspondence

$$\boldsymbol{R}_+^{n+1} \ni (x_1, x_2, \cdots, x_n, y) \longrightarrow (x_1 + iy, x_2 + iy, \cdots) \in \boldsymbol{P}^n ,$$

and we shall choose $\tilde{F} \in C^1(\mathring{\boldsymbol{P}}^n)$ an extension of F (F is a function as in the statement of Theorem 1.1.4) that satisfies $|\nabla \tilde{F}| \leqq C|\nabla F|$ at every point of $\mathring{\boldsymbol{R}}_+^{n+1}$ (this is always possible) and we shall, as we may, suppose that the supports of both F and $\tilde{F}$ are bounded.

An application of Stokes' formula gives then

$$\int_{A_\varepsilon} \tilde{f}(z)\tilde{F}(z)dz_1 \wedge dz_2 \wedge \cdots = \int_{B_\varepsilon} \tilde{f}(z)\bar{\partial}\tilde{F}(z) \wedge dz_1 \wedge dz_2 \wedge \cdots \tag{1.4.4}$$

where:

$$A_\varepsilon = \{(z_1, \cdots, z_n) \in \boldsymbol{P}^n; \operatorname{Im} z_i = \varepsilon; i = 1, 2, \cdots\}$$
$$B_\varepsilon = \{(z_1, \cdots, z_n) \in \boldsymbol{P}^n; \operatorname{Im} z_i = \operatorname{Im} z_k > \varepsilon; i, k = 1, 2, \cdots\}$$

for all $\varepsilon > 0$. If we let $\varepsilon \to 0$ in (1.4.4) we obtain that

$$|\langle S, f\rangle| \leqq \int_{\mathring{\boldsymbol{R}}_+^{n+1}} |\nabla F|\, |\tilde{f}(x_1 + iy, x_2 + iy, \cdots)|\, dx\, dy$$

and $|\nabla F|\, dx\, dy$ being a Carleson measure, our theorem follows.

I would like to finish this paragraph with some comments on Theorems 1.1.3 and 1.1.4.

Theorem 1.1.3 exhibits another aspect of the well known duality between B. M. O. and H^1 for the disc D. It can be used of course to prove that duality, or if we take the duality for granted, it can be thought of as a converse of Theorem 1.1.1.

To be able to do the same for higher dimensions we must combine Theorem 1.1.4 with the following theorem of L. Carleson [4].

THEOREM (L. Carleson). *Let* $f \in H^1(\boldsymbol{R}^n)$ (the Stein and Weiss H^1 space). *Then there exist finitely many functions* $f_i \in L^1(\boldsymbol{R}^n)$ ($i = 1, 2, \cdots, N$) *where* N *depends only on* n *such that*

$$f = \sum_{i=1}^{N} f_i \quad \|f_i\|_1 \leqq C\|f\|_{H^1}$$

and such that each f_i *becomes of analytic type after an appropriate rotation of the axes* (i.e., $\rho_i(f_i)(x) = f_i(\rho_i(x))$ is of analytic type for an appropriate $\rho_i \in \mathrm{SO}(\boldsymbol{R}^n)$ $i = 1, 2, \cdots, N$).

Indeed if Theorem 1.1.4 is combined with the above result the duality between BMO and H^1 is again obtained. (The only trouble of course here is that Carleson's theorem depends on the fact that

Calderon-Zygmund operators operate on H^1, a fact which itself is best proved via the H^1, BMO duality!)

Part 2. Extension of the results to strictly pseudoconvex domains.

2.1. Statements of the theorems. Let $\mathscr{D} = \{\rho < 0\} \subset \boldsymbol{C}^n$ be a bounded strictly pseudoconvex domain in $\boldsymbol{C}^n$ where ρ is sufficiently differentiable and $d\rho \neq 0$ in some Nhd of $\partial\mathscr{D}$ ($\rho \in C^4$ will do for most purposes) cf. [8]. Our first goal will be to define the notions of a BMO function on $\partial\mathscr{D}$ and of a Carleson measure in $\mathscr{D}$.

Let us define a normalized (i.e., of Euclidean length equal to 1 everywhere) vector field ν_0 in some Nhd of $\partial\mathscr{D}$ that is normal and directed inwards to $\partial\mathscr{D}$ at every point $\zeta \in \partial\mathscr{D}$. Let us denote by $\mu_0 = J\nu_0$ the vector field obtained from ν_0 by applying J, the almost complex structure underlying $\boldsymbol{C}^n$, cf. [12] ($J$is the operator on the tangent space which is obtained by "multiplication by i," that is why one sometimes sees the notation $i\nu_0$ instead of $J\nu_0$ cf. [16], I prefer to use the notation $J\nu_0$ to avoid possible confusion when the tangent space is complexified). μ_0 is then a normalized field in some Nhd of $\partial\mathscr{D}$ that is tangential to $\partial\mathscr{D}$ at every point $\zeta_0 \in \partial\mathscr{D}$.

Let us now complete the orthonormal basis by constructing fields

$$\mu_1, \mu_2, \cdots, \mu_{2n-2}$$

such that at every point $\nu_0, \mu_0, \mu_1, \cdots, \mu_{2n-2}$ form an orthonormal basis of the tangent space. This can be done at least locally; i.e., for every $\zeta_0 \in \partial\mathscr{D}$ there exists Ω some Nhd of ζ_0 in $\boldsymbol{C}^n$ in which $\mu_1, \cdots, \mu_{2n-2}$ can be constructed. It is also clear the fields $\nu_0, \mu_0, \mu_1, \cdots, \mu_{2n-2}$ can be made to have the same degree of smoothness as $\partial\mathscr{D}$. For every $\zeta_0 \in \partial\mathscr{D}$ we shall now define $B_t(\zeta_0)$ the "ball" centered at ζ_0 of radius $t > 0$. In the tangent space $T_{\zeta_0}(\partial\mathscr{D})$ of $\partial\mathscr{D}$ at the point ζ_0 let $B_t^*(\zeta_0) \subset T_{\zeta_0}(\partial\mathscr{D})$ be the paralleliped centered at 0 of side t in the μ_0 direction and side $\sqrt{t}$ in the directions $\mu_1, \cdots, \mu_{2n-2}$. We shall define then $B_t(\zeta_0)$ as the image of $B_t^*(\zeta_0)$ by the exponential mapping $T_{\zeta_0}(\partial\mathscr{D}) \to \partial\mathscr{D}$ which is well defined provided that t is small enough. (There is nothing essential here, of course, about the exponential mapping; in fact any other "ball" of the same "shape" and dimensions as $B_t(\zeta_0)$ could be used in its place.) Let us also define

$$\tilde{B}_t(\zeta_0) = \{B_t(\zeta_0) + \lambda\nu_0(\zeta_0);\ \lambda \in (0, t)\}$$

which is a box inside $\mathscr{D}$ with base $B_t(\zeta_0)$ and height t along the normal at ζ_0. It is the analogue of $\tilde{I}$ in § 1.1. We shall say now that f a measurable function on $\partial\mathscr{D}$ is a BMO function $f =$

BMO $(\partial\mathscr{D})$ if:

$$\sup_B \frac{1}{|B|}\int_B |f - f_B|\, d\sigma < +\infty$$

where B runs through the collection of all balls $B = B_t(\zeta_0)$ ($\zeta_0 \in \partial\mathscr{D}$ $0 < t < t_0$) and where we denote as before

$$f_B = \frac{1}{|B|}\int_B f\, d\sigma$$

the average of f on B; $d\sigma$ is of course the Euclidean $2n-1$ area element on $\partial\mathscr{D}$, and $|B|$ denotes the measure of B for that area. Note that we have (cf. [16], [11])

$$c_2 t^n \leqq |B_t(\zeta_0)| \leqq c_1 t^n \quad \forall \zeta_0 \in \partial\mathscr{D} \quad 0 < t < t_0$$

for two positive constants c_1, c_2. We shall say that $\mu \in M(\mathring{\mathscr{D}})$ a measure in $\mathring{\mathscr{D}}$ (the interior of $\mathscr{D}$) is a Carleson measure if:

$$\sup_{\zeta_0 \in \partial\mathscr{D}\; 0<t<t_0} \frac{|\mu|(\tilde{B}_t(\zeta_0))}{|B_t(\zeta_0)|} < +\infty .$$

Note that the above two notions of a BMO function and of a Carleson measure are independent of the particular choice of the vector fields $\nu_0, \mu_0, \mu_1, \cdots, \mu_{2n-2}$ that we have taken. (The $B_t(\zeta_0)$ do depend on the choice of the fields but in a very inessential way.)

Before we can state our main theorem we shall have to introduce one more notion, the notion of the nonisotropic gradiant near the boundary of $\mathscr{D}$.

Let $F \in C^1(\mathring{\mathscr{D}})$. We shall then define

$$|DF| = |\nu_0(F)| + |\mu_0(F)| + |\rho|^{-1/2} \sum_{j=1}^{2n-2} |\mu_j(F)| .$$

(Let us recall that the μ_j's are vector fields and therefore act on functions; $\mu_j(F)$ is the differentiation in the direction μ_j.) $|DF|$ is then well defined in every Nhd Ω of every point $\zeta_0 \in \partial\mathscr{D}$ in which the fields $\nu_0, \mu_0, \cdots, \mu_{2n-2}$ have been defined. It does depend on the choice of these fields but not in an essential way. In fact if $\nu_0', \mu_0', \cdots, \mu_{2n-2}'$ is a different choice of fields in Ω and if we define $|D'F|$ as above with these new fields we then have

$$c_1 |D'F| \leqq |DF| \leqq c_2 |D'F|$$

at every point $\omega \in \Omega$ where c_1, c_2 are two positive constants independent of ω. We have then the following.

THEOREM 2.1.1. *Let $f \in$ BMO $(\partial\mathscr{D})$. Then there exists $F \in C^\infty(\mathring{\mathscr{D}})$*

such that

(i) $\lim_{\lambda\to 0} F(\zeta + \lambda\nu_0(\zeta)) - f(\zeta) \in L^\infty(\partial\mathscr{D})$.

(ii) $|DF|dV$ *is a Carleson measure in* $\mathring{\mathscr{D}}$ (dV is the volume element in $\mathring{\mathscr{D}}$).

(iii) *There exists some* $g \in L^1(\partial\mathscr{D})$ *and some* $\lambda_0 > 0$ *such that*

$$\sup_{0<\lambda<\lambda_0} |F(\zeta_0 + \lambda\nu_0(\zeta_0))| \leqq g(\zeta_0) \quad \forall\zeta_0 \in \partial\mathscr{D}$$

(iv) $|DE| = 0(1/|\rho|)$.

The above theorem has a number of converses which can be summarised in the following theorems.

THEOREM 2.1.2. *Let* $F \in C^1(\mathring{\mathscr{D}})$ *be a once continuously differentiable function in* $\mathring{\mathscr{D}}$ *and let us suppose that* $|DF|$ *is a Carleson measure and that it satisfies the condition* $|DF| = 0(1/|\rho|)$. *Let us further suppose that*

$$\lim_{\lambda\to 0} F(\zeta + \lambda\nu_0(\zeta_0)) = f(\zeta) \quad \zeta \in \partial\mathscr{D}$$

exists for almost every point $\zeta \in \partial\mathscr{D}$. *Then* f *is a* BMO *function of* $\partial\mathscr{D}$.

To simplify notations we shall state our next theorem for the complex ball

$$B = \{z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n;\ |z_1|^2 + \cdots + |z_n|^2 \leqq 1\}$$

let $S \in \mathscr{D}'(\partial B)$ be a distribution on the boundary of the ball. We say that S is of analytic type if there exists $F \in A(\mathring{B})$, some analytic function defined in the interior $\mathring{B}$ of B, such that

$$F_\rho \xrightarrow[\rho\to 1]{} S \quad \text{in} \quad \sigma(\mathscr{D}'; C^\infty)$$

where $F_\rho \in C^\infty(\partial\mathscr{D})$ is defined by:

$$F_\rho(\zeta) = F(\rho\zeta) \quad 0 < \rho < 1 \quad \zeta \in \partial B\,.$$

Analogous definitions exist of course for general domains. We have then the following.

THEOREM 2.1.3. *Let* $F \in C^1(\mathring{B})$ *be a once continuously differentiable function in the interior of the ball* B, *and let us suppose there exists some distribution* $S \in \mathscr{D}'(\partial B)$ *such that*

$$F_\rho \xrightarrow[\rho\to 1]{} S \quad in \quad \sigma(\mathscr{D}'; C^\infty)\,.$$

Then the following assertions hold:

(i) *If we suppose that* $|DF|dV$ *is a Carleson measure then* $S \in \mathrm{BMO}(\partial B)$.

(ii) *If we suppose that* $(|F| + |\nabla F|)dV$ *is a Carleson measure and in addition that* S *is of analytic type then* $S \in \mathrm{BMO}(\partial B)$. dV *denotes of course the volume element in* B *and* ∇F *is the Euclidean gradiant* (cf., § 1.1).

The above theorem holds of course also for general strictly pseudoconvex domains. We only state it here for the ball for simplicity's sake. In fact right through this paragraph we shall have to negotiate generality of general pseudoconvex domains $\mathscr{D}$ against the simplicity of the notations of the complex ball B. All of the explicit calculations will be carried out for the ball but they all generalise easily to strictly pseudoconvex domains. A dedicated reader can do it for himself. In fact in what follows we shall push the simplification one step further; we shall suppose that the dimension of $\boldsymbol{C}^n$ is $n = 2$, this case is perfectly typical.

2.2. The geometry of $B \subset \boldsymbol{C}^2$. Let B be the unit ball in $\boldsymbol{C}^2$ and let $\mathbf{1} = (1, 0)$ be its north pole. We shall introduce then local coordinates in N, some Nhd of $\mathbf{1}$ in ∂B, by setting

$$\zeta = (1 - \alpha_1 + i\beta_1, \alpha_2 + i\beta_2) \in N \subset \partial B . \tag{2.2.1}$$

$(\beta_1, \alpha_2, \beta_2)$ become then local coordinated of N as they run through a Nhd of zero in $\boldsymbol{R}^3$ and α_1 satisfies:

$$2\alpha_1 = \alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 . \tag{2.2.2}$$

We can choose our fields μ_1, μ_2 so that they satisfy:

$$\mu_0 = \frac{\partial}{\partial \beta_1} \quad \mu_1 = \frac{\partial}{\partial \alpha_2} \quad \mu_2 = \frac{\partial}{\partial \beta_2}$$

at the point $\mathbf{1}$. The one parameter family of balls $B_t(\mathbf{1})$ $(0 \leqq t \leqq t_0)$ is then equivalent to the family

$$C_t(\mathbf{1}) = \{|\beta_1| \leqq t;\ |\alpha_2|, |\beta_2| \leqq \sqrt{t}\} \quad (0 \leqq t \leqq t_0) ,$$

where we say that two one parameter families of sets $(A_t)_{t \geqq 0}$ $(B_t)_{t \geqq 0}$ are equivalent if there exist two positive constants $c_1, c_2 > 0$ s.t.

$$B_{c_2 t} \subset A_t \subset B_{c_1 t} \quad \forall t .$$

Using these local coordinates it is easy to show that the family $C_t(\mathbf{1})$ is also equivalent to the family

$$\{\zeta \in \partial B;\ |1-\zeta_1| < t\}$$

and since the above relations are rotation invariant we deduce that for every $\zeta_0 \in \partial B$ the two families

$$B_t(\zeta_0) 0 \leqq t \leqq t_0;\ \{\zeta \in \partial B;\ |1-\bar{\zeta}_0 \cdot \zeta| < t\} \quad 0 \leqq t \leqq t_0$$

are equivalent and also the two families

$$B_t^{\sim}(\zeta_0)(0 \leqq t \leqq t_0);\ \{z \in B;\ |1-\bar{\zeta}_0 \cdot z| < t\} \quad 0 \leqq t \leqq t_0$$

are equivalent where we denote:

$$\bar{z} \cdot u = \bar{z}_1 u_1 + \bar{z}_2 u_2 \quad \forall z, u \in \boldsymbol{C}^2 .$$

We shall denote now by

$$P_z(\zeta) = c \frac{(1-|z|^2)^2}{|1-\bar{\zeta} \cdot z|^4} \quad z \in \mathring{B}, \quad \zeta \in \partial B$$

the Poisson-Szegö kernel of the ball (cf. [16]), which for $\zeta \in N$ becomes in terms of our local coordinates $P_z(\beta_1, \alpha_2, \beta_2)$ for $\zeta = (\beta_1, \alpha_2, \beta_2)$.

We have then the following.

LEMMA 2.2.1. *There exists two positive constants C and c such that*

(i) $$\left| \frac{\partial P_z(\zeta)}{\partial \beta_1} \right| \leqq \frac{C}{|1-\mathbf{1} \cdot z|^3} = \frac{C}{|1-z_1|^3}$$

(ii) $$\left| \frac{\partial P_z(\zeta)}{\partial \alpha_2} \right|, \left| \frac{\partial P_z(\zeta)}{\partial \beta_2} \right| \leqq \frac{C}{|1-\mathbf{1} \cdot z|^{5/2}},$$

for all $\zeta \in B_t(1)$ and all $z \in B$ that satisfy $|1-\mathbf{1} \cdot z| > ct$ $(0 < t < t_0)$

Proof. (i) We have of course

$$\left| \frac{\partial P_z}{\partial \beta_1} \right| \leqq C \frac{(1-|z|^2)^2 |1-\bar{z} \cdot \zeta| (1 + |\partial \alpha_1 / \partial \beta_1|)}{|1-\bar{z} \cdot \zeta|^6}$$

using (2.2.2) we get

$$\frac{\partial \alpha_1}{\partial \beta_1} = \frac{\beta_1}{1-\alpha_1} .$$

On the other hand provided that c is large enough we have

(2.2.3) $$|1-\bar{z} \cdot \zeta| \geqq C|1-\mathbf{1} \cdot z| \geqq C(1-|z|)$$

for (z, ζ) in the required range. From this (i) follows.

(ii) If we set

$$z = (x_1 + iy_1, x_2 + iy_2) .$$

The kernel $P_z(\zeta)$ becomes

$$P_z(\zeta) = c\frac{(1-|z|^2)^2}{[(1-x_1+x_1\alpha_1-y_1\beta_1-x_2\alpha_2-y_2\beta_2)^2+(x_1\beta_1-y_1+y_1\alpha_1+x_2\beta_2-y_2\alpha_2)^2]^2}$$

and we deduce that

$$\left|\frac{\partial P_z(\zeta)}{\partial \alpha_2}\right| \leqq C\frac{(1-|z|^2)^2|1-\bar{z}\cdot\zeta|}{|1-\bar{z}\zeta|^6}\{|x_2|+|y_2|+(|x_1|+|y_1|)|\partial\alpha_1/\partial\alpha_2|\}$$

but we have:

$$|x_2|, |y_2| \leqq |z_2| \leqq c(1-|z_1|)^{1/2} \leqq c|1-1\cdot z|^{1/2}$$

and also using (2.2.2)

$$\left|\frac{\partial\alpha_1}{\partial\alpha_2}\right| = \left|\frac{\alpha_2}{1-\alpha_1}\right| \leqq C\sqrt{t} \quad \zeta \in B_t(1) .$$

From this and (2.2.3) it follows that for z, ζ in the required range we wave

$$\left|\frac{\partial P_z}{\partial\alpha_2}\right| \leqq \frac{C}{|1-1\cdot z|^{5/2}} + \frac{C\sqrt{t}}{|1-1\cdot z|^3} \leqq \frac{C}{|1-1\cdot z|^{5/2}}$$

and this proves the lemma.

2.3. Proof of Theorem 2.1.1. Our construction of the function F of Theorem 2.1.1 from the function $f \in \mathrm{BMO}(\partial B)$ is based on the following:

PROPOSITION 2.3.1. *Let* $f \in \mathrm{BMO}(\partial B)$ *then there exists* μ *some Carleson measure in* $\mathring{B}$ *such that*

$$f(\zeta) - \int_{z\in\mathring{B}} P_z(\zeta)d\mu(z) \in L^\infty(B) . \tag{2.3.1}$$

Conversely if μ *is a Carleson measure and* f *satisfies* (2.3.1) *then* $f \in \mathrm{BMO}(\partial B)$

A very easy proof of the above proposition can be given if we use the BMO, H^1 duality (cf. [5]). Alternatively, Carleson's argument can be adapted in this setting to give a direct proof. (This was done by Y. Meyer, unpublished.) At any rate we intend to take it for granted.

Let us define (in analogy with §1.3)

$$\tilde{P}_z(u) = P_z\Big(\frac{u}{|u|}\Big)\chi_{(|z|,1)}(|u|) \quad z,\, u \in \mathring{B}$$

where $|z|^2 = |z_1|^2 + |z_1|^2$, of course, and where $\chi_{(|z|,1)}$ is the characteristic function of the interval $(|z|, 1)$.

Let now $f \in \mathrm{BMO}(\partial B)$ be given and let μ be some Carleson measure that satisfies (2.3.1) and let us define

$$F(u) = \int_{z\in\mathring{B}} \tilde{P}_z(u)d\mu(z),\ u \in \mathring{B};\ g(\zeta) = \int_{z\in\mathring{B}} P_z(\zeta)d|\mu|(z)\ , \quad \zeta \in \partial B\,.$$

It is an easy matter to verify then just as in §1.3 that $F \in L^\infty_{\mathrm{loc}}(\mathring{B})$ that

$$(2.3.2) \qquad |F(u)| \leqq g\Big(\frac{u}{|u|}\Big) \quad \forall u \in \mathring{B}$$

and that

$$(2.3.3) \qquad \lim_{\rho\to 1} F(\rho\zeta) = \int_{z\in\mathring{B}} P_z(\zeta)d\mu(z)$$

for all $\zeta \in \partial B$ such that $g(\zeta) < +\infty$. We have then:

LEMMA 2.3.1. *Let F be defined as above, then* $|DF|$ (interpreted in the sense of the distribution theory of $\mathring{B}$) *is a Carleson measure of* $\mathring{B}$.

Observe that by the remarks made in §2.1 the conclusion of the lemma is independent of the particular choice of vector fields ν_0, μ_0, μ_1, μ_2 that we take. The proof of the lemma will be given in several distinct steps.

The first thing we do is to observe that there exists some Nhd. Ω of 1 in C^2 which $(\beta_1, \alpha_2, \beta_2)$, the local coordinats of $\Omega \cap \partial B$ and $\rho = |u|$ form a set of local coordinates of $u = \rho\zeta \in \Omega$, $\zeta = (\beta_1, \alpha_2, \beta_2) \in \partial B$. We shall use these local coordinates to take partial derivatives in Ω. Let now $\tilde{P}_z(u)$ be as above and let us define for each fixed $z \in \mathring{B}$ the partial derivatives with respect to $\partial/\partial\rho$, $\partial/\partial\beta_1, \cdots$ of $\tilde{P}_z(u)$ (considered as a function of $u = (\rho, \beta_1, \alpha_2, \beta_2)$) in the sense of distribution theory in $\Omega \cap \mathring{B}$. We have then:

LEMMA 2.3.2. *For each fixed $z \in \mathring{B}$ the following distributions*

$$\sigma_1^{(z)} = \frac{\partial\tilde{P}_z(u)}{\partial\rho};\ \sigma_2^{(z)} = \frac{\partial\tilde{P}_z(u)}{\partial\beta_1}\ ;$$

$$\sigma_3^{(z)} = (1-\rho)^{-1/2}\frac{\partial\tilde{P}_z(u)}{\partial\alpha_2};\ \sigma_4^{(z)} = (1-\rho)^{-1/2}\frac{\partial\tilde{P}_z(u)}{\partial\beta_2}\ ;$$

$$\sigma_5^{(z)} = (1-\rho)^{-1/2}(\beta_1^2+\alpha_2^2+\beta_2^2)^{1/2}\frac{\partial \tilde{P}_z(u)}{\partial \beta_1}$$

are measures in $\Omega \cap \mathring{B}$. Furthermore there exist two positive constants C and c such that

(2.3.4) $|\sigma_i^{(z)}|(\tilde{B}_h(1)) \leqq C \quad 1 \leqq i \leqq 5,\ z=(r,0) \quad 0 \leqq r \leqq 1 \quad \forall h>0$

and

(2.3.5) $$|\sigma_i^{(z)}|(\tilde{B}_h(1)) \leqq C\frac{h^{5/2}}{|1-1\cdot z|^{5/2}}\,, \quad 1 \leqq i \leqq 5$$

for all $z \in B$ and $h>0$ that satisfy

(2.3.6) $$|1-1\cdot z| \geqq ch\,.$$

The proof of the above lemma which consists of eight distinct parts will be deferred until the next paragraph.

Let Ω be some Nhd of $\mathbf{1}$ in $\boldsymbol{C}^2$ in which we have our local coordinates $(\rho, \beta_1, \alpha_2, \beta_2)$. If Ω is small enough we can take our normal field ν_0 (cf. §2.1)

$$\nu_0 = -\frac{\partial}{\partial \rho} \quad \text{in} \quad \Omega$$

and we can also choose our fields μ_0, μ_1, μ_2 such that

(2.3.7) $$\mu_0 = -\frac{\partial}{\partial \beta_1}\,, \quad \mu_1 = \frac{\partial}{\partial \alpha_2}\,, \quad \mu_2 = \frac{\partial}{\partial \beta_2}$$

at the point $\mathbf{1}$.

Let us denote by:

$$\mu_0' = -\frac{\partial}{\partial \beta_1}\,, \quad \mu_1' = \frac{\partial}{\partial \alpha_2}\,, \quad \mu_2' = \frac{\partial}{\partial \beta_2}$$

which are vector fields in Ω. (2.3.7) implies then that:

(2.3.8) $$\mu_i = \mu_i' + \sum_{j=1}^{3} a_{ij}\mu_j' \quad i = 0, 1, 2\,.$$

where a_{ij} are functions in Ω that satisfy

(2.3.9) $$a_{ij}(u) = 0(|1-u|) \quad u \in \Omega$$

where $|\ |$ indicates of course the Euclidean distance in $\boldsymbol{C}^2$. For every fixed $z \in \mathring{B}$ we can now take the derivatives $\mu_i(\tilde{P}_z(u))$ in the sense of distribution theory of $\tilde{P}_z(u)$ (considered as a function of $u \in \mathring{B}$) along the fields $\mu_i (i = 0, 1, 2)$. The distributions $\mu_i(\tilde{P}_z(u))$ are in fact measures. Let us denote

$$\tau_1^{(z)} = |\nu_0(\tilde{P}_z(u))|;\ \tau_2^{(z)} = |\mu_0(\tilde{P}_z(u))|$$
$$\tau_i^{(z)} = (1-\rho)^{-1/2}|\mu_{i-2}(\tilde{P}_z(u))| \quad i = 3, 4 .$$

We have then:

LEMMA 2.3.3. *The measures* $\tau_i^{(z)}(1 \leqq i \leqq 4)$ *satisfy the following relations*:

$$\tau_i^{(z)}(\tilde{B}_h(1)) \leqq C \quad z = (r, 0) \quad r > 0 \quad h > 0 \tag{2.3.10}$$

$$\tau_i^{(z)}(\tilde{B}_h(1)) \leqq \frac{Ch^{5/2}}{1 - \mathbf{1}\cdot z|^{5/2}} \tag{2.3.11}$$

Proof. For $i = 1, 2$ both the above inequalities are immediate consequences of Lemma 2.3.2 a for $i = 3, 4$ we have to be but little more careful because of the factor $(1-\rho)^{-1/2}$. We shall give the proof for $i = 3$, the proof for $i = 4$ is identical. Using (2.3.8) and (2.3.9) we see that

$$\tau_3^{(z)}(\tilde{B}_h(1)) \leqq C[\sigma_3^{(z)}(\tilde{B}_h(1)) + \sigma_4^{(z)}(\tilde{B}_h(1)) + \sigma_5^{(z)}(\tilde{B}_h(1))] .$$

(This is the only point where $\sigma_5^{(z)}$ is used and the extra factor $(\beta_1^2 + \alpha_2^2 + \beta_2^2)^{1/2}$ is supplied by (2.3.9).) (2.3.10) follows then again by Lemma 2.3.2.

Using again (2.3.8) and (2.3.9) we see that

$$\mu_1(P_z(\zeta)) \leqq C[|\mu_1'(P_z(\zeta))| + |\mu_2'(P_z(\zeta))| + \sqrt{h}|\mu_0'(P_z(\zeta))]$$

for all $\zeta \in B_h(1)$. (Observe that $\zeta \in B_h(1)$ we have $|1-\zeta| \leqq \sqrt{h}$.) From this and Lemma 2.2.1 we deduce that

$$|\mu_1(P_z(\zeta))| \leqq \frac{C}{|1 - \mathbf{1}\cdot z|^{5/2}} \tag{2.3.12}$$

for all $\zeta \in B_h(1)$ and all $z \in \mathring{B}$ that satisfies $|1 - \mathbf{1}\cdot z| > ch$, where c is as in Lemma 2.2.1. Inequality (2.3.11) follows then from (2.3.12) exactly as in §2.4 (proof of 2.3.5 for $\sigma_3^{(z)}$. This completes the proof of Lemma 2.3.3.

Let us finally remark that once we have passed to the fields μ_0, μ_1, μ_2 it is no longer necessary to keep z of the form $(r, 0)$ $(0 \leqq r < 1)$ in (2.3.10). Indeed, the situation is invariant by rotation and therefore we deduce that

$$\tau_i^{(z)}(\tilde{B}_h(1)) \leqq C;\ 1 \leqq i \leqq 4 , \quad z \in \mathring{B} , \quad h > 0 . \tag{2.3.13}$$

The only provision being that h should be small enough for $\tilde{B}_h(1)$ to stay in some set where the fields μ_1, μ_2 can be defined.

It is clear now from the definition of $F(u)$ that if we take the derivatives of F in the distribution sense in $\Omega \cap \mathring{B}$ we obtain:

$$\Sigma_1 = \nu_0(F) = \int_{z \in \mathring{B}} \tau_1^{(z)} d\mu(z)$$

$$\Sigma_2 = \mu_0(F) = \int_{z \in \mathring{B}} \tau_2^{(z)} d\mu(z)$$

$$\Sigma_3 = (1 - \rho)^{-1/2}\mu_1(F) = \int_{z \in \mathring{B}} \tau_3^{(z)} d\mu(z)$$

$$\Sigma_4 = (1 - \rho)^{-1/2}\mu_2(F) = \int_{z \in \mathring{B}} \tau_4^{(z)} d\mu(z)$$

It is also clear from Lemma 2.3.2 that $\Sigma_i (1 \leqq i \leqq 4)$ are measures in $\Omega \cap \mathring{B}$ we have then

LEMMA 2.3.4. *There exists a constant c that depends only on μ such that*

$$|\Sigma_i|(\tilde{B_h}(1)) \leqq ch^2 \quad \forall h > 0 .$$

Proof of Lemma 2.3.4. Let $h > 0$ be arbitrary but fixed and let us define the following measures

$$P_i = \int_{|1-\mathbf{1}\cdot z| \leqq ch} |\tau_i^{(z)}| d|\mu|(z) \quad 1 \leqq i \leqq 4$$

$$Q_i = \int_{|1-\mathbf{1}\cdot z| > ch} |\tau_i^{(z)}| d|\mu|(z) \quad 1 \leqq i \leqq 4$$

where c is as in (2.3.6).

We have then clearly:

$$|\Sigma_i|(\tilde{B_h}(1)) \leqq P_i(\tilde{B_h}(1)) + Q_i(\tilde{B_h}(1)) , \quad 1 \leqq i \leqq 4 , \quad h > 0 . \tag{2.3.14}$$

We also have by (2.3.13)

$$\begin{aligned} P_i(\tilde{B_h}(1)) &\leqq \int_{|1-\mathbf{1}\cdot z| \leqq ch} |\tau_i^{(z)}|(\tilde{B_h}(1)) d|\mu|(z) \\ &\leqq C|\mu|\{z \in \mathring{B};\, |1 - \mathbf{1}\cdot z| \leqq ch\} , \quad 1 \leqq i \leqq 4 . \end{aligned}$$

From this and the hypothesis on μ it follows that

$$P_i(\tilde{B_h}(1)) \leqq ch^2 . \tag{2.3.15}$$

We have similarly

$$\begin{aligned} Q_i(\tilde{B_h}(1) &\leqq \int_{|1-\mathbf{1}\cdot z| > ch} |\tau_i^{(z)}|(\tilde{B_h}(1)) d|\mu|(z) \\ &\leqq Ch^{5/2} \int_{ch}^{\infty} \frac{dF(t)}{t^{5/2}} \end{aligned} \tag{2.3.16}$$

where we denote by:

$$F(t) = |\mu|\{z \in B;\ |1 - \mathbf{1}\cdot z| \leqq t\}\ .$$

But by our hypothesis we have

$$F(t) \leqq ct^2\ .$$

An integration by parts in the last member of (2.3.16) gives then at once that

$$Q_i(B_h^{\sim}(1)) \leqq ch^2\ . \tag{2.3.17}$$

And if we combine (2.3.14), (2.3.15) and (2.3.17) we obtain our lemma. We can now give the

Proof of Lemma 2.3.1. It follows from Lemma 2.3.4 that:

$$|DF|(B_h^{\sim}(1)) \leqq Ch^2\ , \tag{2.3.18}$$

where C is a constant that only depends on μ and in fact only on the Carleson constant of μ which in turn only depends on the BMO norm of the original function f. Since that norm is rotation invariant we see that we can rotate (2.3.18) and finally obtain the Carleson condition

$$|DF|(B_h^{\sim}(\zeta_0)) \leqq Ch^2 \quad \forall\zeta_0 \in \partial B \quad h > 0\ .$$

This proves the lemma.

Proof of Theorem 2.1.1. To satisfy the conditons (i), (ii), and (iii) of Theorem 2.1.1 we only have to modify the definition of $\tilde{P}_z$ in a manner analogous to the one in (2.3.12). This makes the function F infinitely differentiable. Conditions (i), (ii), and (iii) follow then from (2.3.2), (2.3.3), and Lemma 2.3.1. The verification of (iv) has to be done separately but for the same reasons as before it will be omitted.

2.4. Proof of Lemma 2.3.2.

The proof for $\sigma_1^{(z)}$. For every fixed $z \in \mathring{B}$, $\sigma_1^{(z)}$ is seen at once to be the 3-dimensional Lebesgue on the sphere $\partial B_{|z|} = \{u = |z|\zeta;\ \zeta \in \partial B\}$ multiplied by the function $P_z(\zeta)$. (2.3.4) follows therefore from the fact that $\|P_z\|_{L^1(\partial B)} = 1$. On the other hand:

$$\begin{aligned}|\sigma_1^{(z)}|(B_h^{\sim}(1)) &\leqq |B_h(1)|\cdot \sup_{\zeta\in B_h(1)} |P_z(\zeta)| \\ &\leqq h^2 \sup_{\zeta\in B_h(1)} \frac{(1-|z|^2)^2}{|1-\bar{\zeta}\cdot z|^4}\end{aligned}$$

But for $1 - |z| > h$ we also have $\sigma_1^{(z)}(\widetilde{B}_h(1)) = 0$ and for $z \in \mathring{B}$ satisfying (2.3.6) with $c > 0$ large enough we have

$$|1 - \bar{\zeta}\cdot z| \geqq C|1 - \mathbf{1}\cdot z| \geqq C(1 - |z|) \quad \forall \zeta \in B_h^{\sim}(1)$$

and from this, (2.3.5) follows.

Proof of (2.3.5) *for* $\sigma_2^{(z)}, \sigma_3^{(z)}, \sigma_4^{(z)}$, *and* $\sigma_5^{(z)}$. That $\sigma_2^{(z)}, \sigma_3^{(z)}, \sigma_4^{(z)}$, and $\sigma_5^{(z)}$ are all measures is evident. We have further

$$|\sigma_2^{(z)}|(B_h^{\sim}(1)) \leqq |B_h^{\sim}(1)| \sup_{\zeta \in B_h(\mathbf{1})} \left|\frac{\partial P_z(\zeta)}{\partial \beta_1}\right| \leqq \frac{ch^3}{|1 - \mathbf{1}\cdot z|^3} \leqq \frac{ch^{5/2}}{|1 - \mathbf{1}\cdot z|^{5/2}}$$

in the required range by Lemma 2.2.1 (i), provided that c is large enough in (2.3.6) which is the required result for $\sigma_2^{(z)}$. We also have for $i = 3, 4$

$$\begin{aligned} |\sigma_i^{(z)}|(\widetilde{B}_h(1)) &\leqq |B_h(1)| \left(\int_{1-h}^1 (1 - \rho)^{-1/2} d\rho\right) \sup_{\zeta \in B_h(\mathbf{1})} \left\{\left|\frac{\partial P_z(\zeta)}{\partial \alpha_2}\right| + \left|\frac{\partial P_z(\zeta)}{\partial \beta_2}\right|\right\} \\ &\leqq h^{5/2} \frac{C}{|1 - \mathbf{1}\cdot z|^{5/2}} \leqq \frac{Ch^{5/2}}{|1 - \mathbf{1}\cdot z|^{5/2}} \end{aligned}$$

in the required range by Lemma 2.2.1(ii). We finally have

$$\begin{aligned} |\sigma_5^{(z)}|(\widetilde{B}_h(1)) &\leqq |B_h(1)| \left(\int_{|z|}^1 (1 - \rho)^{-1/2} d\rho\right) \sup_{\zeta \in B_h(\mathbf{1})} \left[(|\beta_1| + |\alpha_2| + |\beta_2|)\left[\frac{\partial P_z}{\partial \beta_1}\right]\right. \\ &\leqq h^2(1 - |z|)^{1/2} \frac{C\sqrt{h}}{|1 - \mathbf{1}\cdot z|^3} \leqq \frac{ch^{5/2}}{|1 - \mathbf{1}\cdot z|^{5/2}} \end{aligned}$$

again by Lemma 2.2.1 (observe that the $\beta_1, \alpha_2, \beta_2$, which are the local coordinates of ζ in $B_h(1)$, are bounded by $\sqrt{h}$). To complete the proof it suffices to prove (2.3.4) for $i = 2, 3, 4, 5$. Towards that we shall show that:

$$(2.4.1) \qquad |\sigma_i^{(z)}|(\Omega \cap \mathring{B}) \leqq C; \; 2 \leqq i \leqq 5 \quad z = (r, 0) \quad 0 \leqq r \leqq 1\,,$$

where Ω is some fixed Nhd of 1 in C^2 (in which we have our local coordinates).

Since $z = (r, 0)$ we have:

$$(2.4.2) \qquad \begin{aligned} P = P_z(\zeta) &= C\frac{(1 - r^2)^2}{[(1 - r + r\alpha_1)^2 + r^2\beta_1^2]^2} \\ &\leqq C\frac{(1 - r^2)^2}{[(1 - r + r(\beta_1^2 + \alpha_2^2 + \beta_2^2))^2 + r^2\beta_1^2]^2} \end{aligned}$$

Proof of (2.4.1) *for* $i = 2$. Using Fubini's theorem we obtain

(2.4.3)
$$
\begin{aligned}
|\sigma_i^{(z)}|(\Omega \cap \overset{\circ}{B}) &\leqq C(1-r)\iiint\left|\frac{\partial P}{\partial \beta_1}\right| d\beta_1 d\alpha_2 d\beta_2 \\
&= C(1-r)\iint d\alpha_2 \alpha\beta_2 \int\left|\frac{\partial P}{\partial \beta_1}(\beta_1, \alpha_2, \beta_2)\right| d\beta_1
\end{aligned}
$$

where the integration in $\beta_1, \alpha_2, \beta_2$ ranges through some fixed small cube (depending only on Ω).

Now it is clear that for every fixed α_2, β_2 in the range of integration we can split the range of integration in β_1 into N intervals on each of which $P(\beta_1, \alpha_2, \beta_2)$ is a monotone function of β_1. N, the number of intervals, is bounded by some numerical constant.

From this we deduce that for every fixed α_2, β_2 in the range of integration we have

$$
\int\left|\frac{\partial P}{\partial \beta_1}(\beta_1, \alpha_2, \beta_2)\right| d\beta_1 \leqq C \max_{\beta_1} |P(\beta_1, \alpha_2, \beta_2)| .
$$

Using (2.4.2) to get the above max and substituting its value in (2.4.3) We finally obtain

$$
|\sigma_2^{(z)}|(\Omega \cap \overset{\circ}{B}) \leqq C(1-r)^3 \iint \frac{d\alpha_2 d\beta_2}{[(1-r)+\alpha_2^2+\beta_2^2]^4} \leqq C\,, \quad 0 \leqq r \leqq 1
$$

which gives the required result.

Proof of (2.4.1) *for* $i = 3$. The argument runs as before. A use of Fubini gives that

$$
\begin{aligned}
|\sigma_3^{(z)}|(\Omega \cap \overset{\circ}{B}) &\leqq C\int_r^1 (1-\rho)^{-1/2} d\rho \iiint\left|\frac{\partial P_z}{\partial \alpha_2}\right| d\beta_1 d\alpha_2 d\beta_2 \\
&\leqq C(1-r)^{1/2} \iint \operatorname*{Max}_{\alpha_2} |P_z(\beta_1, \alpha_2, \beta_2)|\, d\beta_1 d\beta_2
\end{aligned}
$$

because here again P_z as a function of α_2 for fixed β_1 and β_2 is monotone in "finitely many pieces." If again we obtain the maximum from (2.4.2) and substitute it in the above integral we obtain

$$
\begin{aligned}
|\sigma_3^{(z)}|(\Omega \cap \overset{\circ}{B}) &\leqq C(1-r)^{5/2} \iint \frac{d\beta_1 d\beta_2}{[((1-r)+\beta_1^2+\beta_2^2)^2+\beta_1^2]^2} \\
&\leqq ct^{5/2} \int_0^\infty dR \int_0^{2\pi} \frac{R d\varphi}{[(t+R^2)^2+R^2\cos^2\varphi]^2}
\end{aligned}
$$

where we have set $1 - r = t$. But

$$
\begin{aligned}
\int_0^{2\pi} \frac{R\,d\varphi}{[(t+R^2)^2+R^2\cos^2\varphi]^2} &\leqq 2\int_{2|\cos\varphi|^2 \leqq 1} \frac{R\,d\varphi}{[(t+R^2)^2+R^2\cos^2\varphi]^2} \\
\leqq C\int_0^{2\pi} \frac{R\sin\varphi\,d\varphi}{[(t+R^2)^2+R^2\cos^2\varphi]^2} &\leqq C\int_{-\infty}^{+\infty} \frac{d\sigma}{[(t+R^2)^2+\sigma^2]^2}
\end{aligned}
$$

as we obtain by the substitution $R\cos\varphi = \sigma$. The substitution $\sigma = (t + R^2)S$ in the last integral gives therefore that:

$$|\sigma_3^{(z)}|(\Omega \cap \mathring{B}) \leqq ct^{5/2}\int_0^\infty \frac{dR}{(t + R^2)^3}\int_{-\infty}^{+\infty}\frac{dS}{(1 + S^2)^2}\,.$$

A final substtituion $R = \sqrt{t}\,R^*$ implies therefore the required result.

$$|\sigma_3^{(z)}|(\Omega \cap \mathring{B}) \leqq C\,.$$

The proof of (2.4.1) for $i = 4$ is identical. It remains therefore to give the

Proof of (2.4.1) *for* $i = 5$. A use of Fubini gives as before:

$$|\sigma_5^{(z)}|(\Omega \cap \mathring{B}) \leqq C(1 - r)^{1/2}\iiint(\beta_1^2 + \alpha_2^2 + \beta_2^2)^{1/2}\left|\frac{\partial P_z}{\partial \beta_1}\right| d\beta_1 d\beta_2 d\beta_2$$

$$\text{(2.4.4)} \qquad \leqq C(1 - r)^{1/2}\Big[\iint\alpha_2^2 + \beta_2^2)^{1/2} d\alpha_2 d\beta_2 \int\left|\frac{\partial P_z}{\partial \beta_1}\right| d\beta_1$$

$$+ \iint d\alpha_2 d\beta_2 \int\left|\beta_1\frac{\partial P_z}{\partial \beta_1}\right| d\beta_1\Big]\,.$$

We shall estimate the two integrals separately. The first one can be estimates as in the proof of (2.4.1) for $i = 2$ by:

$$C(1 - r)^{5/2}\iint\frac{(\alpha_2^2 + \beta_2^2)^{1/2} d\alpha_2 d\beta_2}{[(1 - r) + \alpha_2^2 + \beta_2^2]^4} \leqq C\,.$$

To estimate

$$\int\left|\beta_1\frac{\partial P_z}{\partial \beta_1}\right| d\beta_1$$

we have to use once more the fact for fixed α_2 and β_2 the function P_z is monotone in each of N disjointed intervals ($N \leqq C$) and to integrate by parts on each interval. We obtain therefore

$$\int\left|\beta_1\left|\frac{\partial P_z}{\partial \beta_1}\right|\right| d\beta_1 \leqq C\Big[\operatorname*{Max}_{\beta_1}|\beta_1 P_z(\beta_1, \alpha_2, \beta_2)| + \int P_z d\beta_1\Big]\,.$$

But we have as before:

$$(1 - r)^{1/2}\iint\operatorname*{Max}_{\beta_1}|\beta_1 P_z(\beta_1, \alpha_2, \beta_2)|\, d\alpha_2 d\beta_2 \leqq C\iint\frac{(1 - r)^{5/2} d\alpha_2 d\beta_2}{(1 - r + \alpha_2^2 + \beta_2^2)^3} \leqq C$$

and also

$$\iiint P_z(\beta_1, \alpha_2, \beta_2) d\beta_1 d\alpha_2 d\beta_2 \leqq C\,.$$

If we substitute all these estimates back in (2.4.4) we obtain the required result.

2.5. Proof of the converse of Theorem 2.1.1. The proof of Theorem 2.1.2 is identical to the proof of Theorem 1.1.2(ii), and will be therefore omitted. Before we give the proof of Theorem 2.1.3 we shall prove the following geometric

LEMMA 2.5.1. *Let $\mathscr{D} = \{\rho < 0\}$ be a strictly pseudoconvex domain and let us suppose that $F \in C^1(\mathring{\mathscr{D}})$ is such that $|DF|dV$ is a Carleson measure (dV denotes the volume elemeot in $\mathring{\mathscr{D}}$). Let us consider the following two forms:*

$$\bar{\partial}F = \sum_i a_i d\bar{z}_i$$

$$\frac{1}{\sqrt{|\rho|}}\bar{\partial}\rho \wedge \bar{\partial}F = \sum_{i<j} a_{ij} d\bar{z}_i \wedge d\bar{z}_j\ .$$

Then the measures

$$\mu = \left[\sum_i |a_i|\right]dV$$

$$\nu = \left[\sum_{i<j} |a_{ij}|\right]dV$$

are both Carleson measures in $\mathring{\mathscr{D}}$.

Proof. That μ is a Carleson measure is immediate. To prove that ν is a Carleson measure we have to choose a special set of coordinates on the cotangent space.

Let ν_0 and μ_0 be as in §2.1 and let us choose $\mu_1, \mu_2, \cdots, \mu_{n-1}$ smooth vector fields in some Nhd Ω of some point $\zeta_0 \in \partial\mathscr{D}$ such that the fields

$$\nu_0,\ J\nu_0,\ \mu_1,\ J\mu_1,\ \mu_2,\ J\mu_2,\ \cdots,\ \mu_{n-1},\ J\mu_{n-1} \tag{2.5.1}$$

form an orthonormal basis of the tangent space at every point of Ω. Such a choice is clearly possible. Let us then choose differential forms $\omega_0, \omega_1, \cdots, \omega_{n-1}$ in Ω such that the forms

$$\omega_0,\ -J\omega_0,\ \omega_1,\ -J\omega_1,\ \cdots$$

form a basis of the cotangent space that is dual to the basis (2.5.1) (we denote of course $J\omega(X) = \omega(JX)$). The complex differential forms

$$\omega_0 \pm iJ\omega_0,\ \omega_1 \pm iJ\omega_1,\ \cdots,\ \omega_{n-1} \pm iJ\omega_{n-1} \tag{2.5.2}$$

form then a basis of the complexified cotangent space and since $d\rho$

is proportional to ω_0 it follows that

$$(2.5.3) \qquad \omega_0 + iJ\omega_0 = k\bar{\partial}\rho;\ \omega_0 - iJ\omega_0 = l\partial\rho\ ,$$

where k and l are smooth functions.

Let us now express dF and $\bar{\partial}F$ in terms of the basis (2.5.2). We obtain then

$$dF = \sum_{j=0}^{n-1} \alpha_j(\omega_j + iJ\omega_j) + \sum_{j=0}^{n-1} \beta_j(\omega_j - iJ\omega_j)$$
$$\bar{\partial}F = \sum_{j=0}^{n-1} \alpha_j(\omega_j + iJ\omega_j)\ .$$

To prove that ν is a Carleson measure it suffices therefore to prove that

$$\nu' = \frac{1}{\sqrt{|\rho|}}\left[\sum_{j=1}^{n-1}|\alpha_j|\right]dV$$

is a Carleson measure. But we have

$$2\alpha_j = dF(\mu_j + iJ\mu_j) = \mu_j(F) + iJ\mu_j(F) \quad (j \geqq 1)$$

(i.e., α_j involves only complex tangential derivatives of F) and this of course together with the hypothesis on F completes the proof of the lemma.

We shall also need the following two facts about functions and distributions of analytic type on ∂B the boundary of the complex ball $B \subset \boldsymbol{C}^n$.

PROPOSITION 2.5.1. (i) *Let $f \in L^1(\partial B)$ and let us suppose that:*

$$\int_{\partial B} f \wedge \bar{\partial}\psi = 0$$

for all ψ smooth in some Nhd *of B and of type $(n, n-2)$. Then f is of analytic type.*

(ii) *Let us denote by P the orthogonal projection of $L^2(\partial B)$ on the subspace*

$$H^2(\partial B) = \{f \in L^2(\partial B);\ \text{of analytic type}\}\ ,$$

Then $P[\mathrm{BMO}(\partial B)] \subset \mathrm{BMO}(\partial B)$.

Part (i) is standard and well known (cf. [10], [18]). Part (ii) holds because P is given by the Szegö kernel that is a singular integral operator (cf. [17]). Both parts (i) and (ii) hold for general strictly pseudoconvex domains, although part (ii) is much harder to show in general.

Proof of Theorem 2.1.3(i). We can now give the proof of Theorem 2.1.3(i). Towards that we shall suppose in addition that $S \in C^\infty(\partial B)$ and we shall prove the a priori estimate

$$\|S\|_{\mathrm{BMO}} \leqq CK(|DF|\,dV) \tag{2.5.4}$$

where $K(|DF|\,dV)$ denotes the constant involved in the definition of the Carleson measure. Once the estimate (2.5.4) is known, Theorem 2.1.3(i) follows at once by an easy regularisation process.

Let then F be as in the theorem using then our hypothesis on F, Lemma 2.5.1 and Theorem 3.1.1(i). It follows that there exists some $\varphi \in \mathrm{BMO}(\partial B)$ such that:

$$\|\varphi\|_{\mathrm{BMO}} \leqq CK(|DF|\,dV)$$

and such that:

$$\int_{\partial B} \varphi \wedge \psi = \int_{\mathring{B}} \bar\partial F \wedge \psi$$

for all form ψ, smooth and $\bar\partial$-closed, in some Nhd of B of type $(n, n-1)$.

On the other hand an easy use of Stokes's theorem in the ball $B_\rho = \{z \in C^n;\ |z| \leqq \rho\}$ and a passage to the limit as $\rho \to 1$ gives us

$$\int_{\partial B} S \wedge \psi = \int_{\mathring{B}} \bar\partial F \wedge \psi$$

for the same class of ψ's as above.

We conclude therefore that:

$$\int_{\partial B} (S - \varphi) \wedge \psi = 0$$

again for the same ψ's, and from that and Proposition 2.5.1(i) we conclude that there exists some function α of analytic type such that

$$S = \varphi + \alpha\,. \tag{2.5.5}$$

But the hypothesis of our theorem are clearly stable by complex conjugation. We conclude therefore that there exist $\tilde\varphi$ and $\tilde\alpha$ such that $\tilde\alpha$ is of analytic type and such that:

$$\bar S = \tilde\varphi + \tilde\alpha \quad \|\tilde\varphi\|_{\mathrm{BMO}} \leqq CK(|DF|\,dV)\,. \tag{2.5.6}$$

From (2.5.5) and (2.5.6) we conclude therefore that:

$$\begin{aligned} S &= (I - P)(\varphi + \alpha) + P(\bar{\tilde\varphi} + \bar{\tilde\alpha}) \\ &= (I - P)\varphi + P(\bar{\tilde\varphi}) + P(\bar{\tilde\alpha}) \end{aligned}$$

but $\bar{\tilde{a}}$ being the complex conjugate of an analytic function, it follows that $P(\bar{\tilde{a}})$ is a constant; and since constants do not affect the BMO norm ($\|\ \|_{\mathrm{BMO}}$. It is not a norm, it is only a norm on functions modulo constants.). We obtain

$$\|S\|_{\mathrm{BMO}} \leqq \|(I-P)\varphi\|_{\mathrm{BMO}} + \|P(\bar{\tilde{\varphi}})\|_{\mathrm{BMO}}$$

and (2.5.4) follows then from Proposition 2.5.1(ii).

For the proof of the second part of the theorem we shall need the following.

LEMMA 2.5.2. *The exists some $\omega \in C^\infty_{n,n-1}(C^n)$ such that*

$$\int_{\partial B} \varphi\omega = \int_{\zeta \in \partial B} \varphi(\zeta) d\sigma(\zeta) \quad \forall \varphi \in C^\infty(\partial B)$$

where $d\sigma$ is the normalised Lebesgue measure on ∂B.

Proof. Let $\pi = \bar{z}_n dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}$. It is clear then by Stokes's theorem that

$$\int_{\partial B} \pi \neq 0 .$$

π determines then some $\lambda \in M(\partial B)$ by the equation

$$\int_{\partial B} \varphi\pi = \int_{\zeta \in \partial B} \varphi(\zeta) d\lambda(\zeta) \quad \forall \varphi \in C^\infty(\partial B) . \tag{2.5.7}$$

Let now $\alpha \in SU(n)$ be some complex rotation on C^n and denote by $\pi_\alpha \in C^\infty_{n,n-1}(C^n)$ the form $\pi_\alpha = \alpha^*(\pi)$ where:

$$\alpha\colon z \longrightarrow \alpha z .$$

Let us also denote by λ_α the measure that is determined by ∂B from π_α by the analogue of (2.5.7) and let us define

$$\omega = \int_{\alpha \in SU(n)} \pi_\alpha d\alpha \in C^\infty_{n,n-1}(C^n);\ \lambda = \int_{\alpha \in SU(n)} \lambda_\alpha d\alpha \in M(\partial B)$$

where $d\alpha$ denotes the normalized Haar measure on $SU(n)$. $d\mu$ is rotation invariant and it satisfies

$$\int_{\partial B} \varphi\omega = \int_{\zeta \in \partial B} \varphi(\zeta) d\mu(\zeta) \quad \forall \varphi \in C^\infty(\partial B) \tag{2.5.8}$$
$$\int_{\partial B} d\mu = \int_{\partial B} d\lambda = \int_{\partial B} \pi \neq 0$$

$d\mu$ is therefore proportional to the normalised uniform measure on ∂B $d\sigma$ and (2.5.8) proves the lemma.

Proof of Theorem 2.1.3(i). Let us suppose that F and S are as in the theorem and let us suppose in addition the $S \in C^\infty(\partial B)$.

Let $f(z)$ be an analytic polynomial in $z \in C^n$. We see then by an easy application of Stokes's theorem to $B_\rho = \{z \in C^n; |z| \leqq \rho\}$ and a passage to the limit as $\rho \to 1$ that

$$\int_{\zeta \in \partial B} f\bar{S} d\sigma(\zeta) = \int_{\partial B} f\bar{S}\omega = \int_{\mathring{B}} d(f\bar{F}\omega)$$

and therefore:

$$\left|\int_{\zeta \in \partial B} f\bar{S} d\sigma(\zeta)\right| \leqq \left|\int_{\mathring{B}} f\partial\bar{F} \wedge \omega\right| + \left|\int_{\mathring{B}} f\bar{F} \wedge \bar{\partial}\omega\right| .$$

By our hypothesis on F it follows therefore that

$$\left|\int_{\zeta \in \partial B} f\bar{S} d\sigma(\zeta)\right| \leqq CK\|f\|_{L^1(\partial B)} \tag{2.5.9}$$

where C is a numerical constant and K is the constant associated to the Carleson measure $(|\nabla F|+|F|)dV$. From (2.5.9) it follows therefore that there exists $\varphi, \theta \in L^\infty(\partial B)$ such that

$$S = \varphi + \theta; \ \|\varphi\|_\infty \leqq CK$$

and θ is orthogonal to every analytic polynomial in $L^2(\partial B)$. Let P now by the projection of Proposition 2.5.1(ii). We have then $P\theta = 0$ and therefore

$$S = PS = P\varphi$$

by our hypothesis on S. And this implies by Proposition 2.5.1 (ii) that:

$$\|S\|_{\mathrm{BMO}} \leqq CK$$

and with this a priori estimate we can complete the proof of Theorem 2.1.3 at once.

Part 3. The $\bar{\partial}$-equation and the Corona problem.

3.1. Statement of the results. One novelty in this paragraph will be the systematic use of differential forms. Let $\Omega \subset C^n$ be an open subset. We shall then denote by $C^\infty_{p,q}(\Omega)$ and $L^s_{p,q}(\Omega)$ the differential forms of type (p, q) in Ω with coefficients in C^∞ and L^s respectively. We shall also denote by $M_{p,q}(\Omega)$ the "differential forms" in Ω with coefficients in $M(\Omega)$, the space of bounded measures in Ω. Strictly speaking $M_{p,q}(\Omega)$ is not a space of forms but a space of currents, but we shall ignore this complication here and will not use the (irrelevant for our purposes) formalism of currents. (We shall tacidly identify currents of degree 0 with distributions.)

The only thing that has to be correctly understood is the meaning of the form $\bar{\partial}\mu$ for:

$$\mu = \sum_{I,J} \mu_{I,J} dz_I \wedge d\bar{z}_J \in M_{p,q}(\Omega);\ \mu_{I,J} \in M(\Omega) \quad \forall I, J\,.$$

We shall simply *define*:

$$\bar{\partial}\mu = \sum_{I,J,ii} \frac{\partial \mu_{I,J}}{\partial \bar{z}_i} d\bar{z}_i \wedge dz_I \wedge d\bar{z}_J \in \mathscr{D}'_{p,q+1}(\Omega)$$

where the derivatives $\partial\mu_{I,J}/\partial\bar{z}_i$ are taken in the sense of distribution of Ω $\mathscr{D}'(\Omega)$. We can verify that $\bar{\partial}\mu$ satisfies all the obvious formal properties.

We shall use the notation $C^\infty_{p,q}(K)$ where K is a closed subset of C^n to indicate the space of (p, q) forms defined and smooth in some Nhd of K.

Together with the above spaces of forms we shall need to consider spaces of forms on $\partial\mathscr{D}$, the boundary of a strongly pseudoconvex domain $\mathscr{D}$. e.g., we shall need to consider the spaces $C^\infty_{p,q}(\partial\mathscr{D})$ (resp. $\mathrm{BMO}_{p,q}(\partial B)$, $L^s_{p,q}(\partial B)$). These are *not* spaces of forms on the differential manifold $\partial\mathscr{D}$; they are spaces of forms in C^n (this is why we are allowed to talk about the type (p, q) which comes about from the ∂, $\bar{\partial}$ decomposition), but the coefficients are only defined on $\partial\mathscr{D}$ and are $C^\infty(\partial B)$ (resp. $\mathrm{BMO}(\partial B)$, $L^s(\partial B)$) functions on ∂B, using the editoriolizing language of the topologists $C^\infty_{p,q}(\partial B)$ is a C^∞ section of $\Lambda_{p,q}T^*(C^n)$ over $\partial\mathscr{D}$.

Before we can state our main theorem we shall need to introduce the following definition. Let

$$\mu = \sum_{I,J} \mu_{I,J} dz_I \wedge d\bar{z}_J \in M_{p,q}(\mathring{\mathscr{D}})$$

where $\mathscr{D} = \{\rho < 0\}$ is as in §1.1. We shall say that μ satisfies the Carleson condition if the measure $\mu = \sum_{I,J} |\mu_{I,J}|$ and the measure $\nu = \sum_{I,J} |\nu_{I,J}|$ are Carleson measures in $\mathring{\mathscr{D}}$, where we denote

$$\nu = \sum_{I,J} \nu_{I,J} dz_I \wedge d\bar{z}_J = |\rho|^{-1/2}\mu \wedge \bar{\partial}\rho \in M_{p,q+1}(\mathring{\mathscr{D}})\,.$$

We have then:

THEOREM 3.1.1.

(i) *Let $\mu \in M_{p,q}(\mathring{\mathscr{D}})$ $(q \geqq 1)$ satisfy the Carleson condition and be such that $\bar{\partial}\mu = 0$ in $\mathring{\mathscr{D}}$. Then there exists some $g \in \mathrm{BMO}_{p,q-1}(\partial\mathscr{D})$ such that*

$$\int_{\partial\mathscr{D}} g \wedge \psi = \int_{\mathring{\mathscr{D}}} \mu \wedge \psi \tag{3.1.1}$$

for all $\psi \in C^\infty_{n-p,n-q}(\overline{\mathscr{D}})$ *that satisfy* $\bar\partial\psi = 0$ *in some* Nhd *of* $\mathscr{D}$

(ii) *Conversely let us suppose that* $g \in \mathrm{BMO}_{p,q-1}(\partial\mathscr{D})$. *Then there exists* $\mu \in M_{p,q}(\mathring{\mathscr{D}})$ *a form that satisfies the Carleson condition in* $\mathring{\mathscr{D}}$ *and* $f \in L^\infty(\partial\mathscr{D})$ *such that:*

$$\int_{\partial\mathscr{D}} g \wedge \psi = \int_{\mathring{\mathscr{D}}} \mu \wedge \psi + \int_{\partial\mathscr{D}} f \wedge \psi$$

for all $\psi \in C^\infty_{n-p,n-q}(\overline{\mathscr{D}})$ *that satisfies* $\bar\partial\psi = 0$ *in some* Nhd *of* $\overline{\mathscr{D}}$.

Equation (3.1.1) is, of course, a global formulation of the so-called $\bar\partial_b$-problem.

When the dimension of the space $n = 1$, Theorem 3.1.1 (i) can be improved and we can choose $g \in L^\infty(\partial\mathscr{D})$ cf. [2], [9], and this is crucial for the proof of the Corona theorem.

THEOREM 3.1.2. *Let B be the unit ball in C^2. Then there exists some* $f \in C^\infty_{0,1}(\mathring{B})$ *such that* $\bar\partial f = 0$ *and such that the form f satisfies the Carleson condition and such that whenever* $u \in L^1(\partial B)$ *satisfies the equation*

$$\int_{\partial B} u \wedge \varphi = \int_B f \wedge \varphi$$

for all $\varphi \in C^\infty_{2,1}(B)$ *that are $\bar\partial$-closed in some* Nhd *of B then*

$$\operatorname{ess\,sup} |u| = +\infty .$$

Another way to express the above theorem is to say that the $\bar\partial_b u = \mu$ problem for μ satisfying the Carleson condition is not always solvable in $L^\infty(\partial B)$.

3.2. The Henkin construction. In this paragraph we shall content ourselves in recalling and explicitating some of Henkin's notations and theorems from [7]. They will be basic for the proof of Theorem 3.1.1 (i).

Let $\mathscr{D} \subset C^n$ and ρ be as before and (following Henkin) let us suppose for simplicity that $\mathscr{D}$ is in fact strictly convex.

We shall denote then

$$\varphi(\zeta, z) = \sum_{k=1}^n p_k(\zeta, z)(\zeta_k - z_k);\ p = \{p_1, p_2, \cdots, p_n\}$$

$$\varphi^*(\zeta, z) = \sum_{k=1}^n p^*_k(\zeta, z)(\zeta_k - z_k);\ p^* = \{p^*_1, p^*_2, \cdots, p^*_n\}$$

where

$$p_k(\zeta, z) = \frac{\partial\rho}{\partial\zeta_k}(\zeta);\ p_k^*(\zeta, z) = p_k(z, \zeta)\ .$$

We set $\omega(\zeta) = \bigwedge_{i=1}^n d\zeta_i$ and we orient C^n by

$$\int_{\mathscr{D}} (-i)^n \omega(\bar{\zeta}) \wedge \omega(\zeta) > 0\ .$$

Let $\eta = (\eta_1, \eta_2, \cdots, \eta_n) = \eta(\zeta, z, \lambda)$ be a smooth vector valued function of ζ, $z \in C^n$ and the real parameter λ. We set then

$$\omega'(\eta) = \sum_{k=1}^n (-1)^{k-1}\eta_k \bigwedge_{i\neq k} d\eta_i\ .$$

The differential form $\omega'(\eta) \wedge \omega(\zeta) \wedge \omega(z) = \theta$ may be decomposed then as the sum.

$$\theta = \sum_{q=0}^{n-1} \omega'_q(\eta) \wedge \omega(\zeta) \wedge \omega(z)$$

where $\omega'_q(\eta)$ is a form of degree q with respect to $d\bar{z}$ and, correspondingly of degree $(n - q - 1)$ with respect to $d\bar{\zeta}$ and $d\lambda$. We shall further write

$$\omega(\zeta + z) = \sum_{p=0}^n \omega_p(\zeta + z)$$

where $\omega_p(\zeta + z)$ is a form of type $(p, 0)$ in z and of type $(n - p, 0)$ in ζ. We shall now introduce the following definition. We shall say that $f \in C^\infty_{p,q}(\partial\mathscr{D})$ satisfies $\bar{\partial}_b f = 0$ if f admits some C^∞ extension $\tilde{f}$ in some Nhd of $\mathscr{D}$ such that

$$\bar{\partial}\tilde{f} = \bar{\partial}\rho \wedge h + \rho k \tag{3.2.1}$$

in some Nhd of $\partial\mathscr{D}$ where h and k are forms defined in some Nhd of $\partial\mathscr{D}$ (observe that the above definition is independent of the particular extension $\tilde{f}$ cf. [13]). We shall also introduce the following condition which is stronger than the condition $\bar{\partial}_b f = 0$. Let $f \in L^1_{p,q}(\partial\mathscr{D})$. We shall say that $H_b(f) = 0$ if

$$\int_{\partial\mathscr{D}} f \wedge \varphi = 0 \tag{3.2.2}$$

for all $\varphi \in C^\infty_{n-p,n-q-1}(\partial\mathscr{D})$ such that $\bar{\partial}_b\varphi = 0$. (Notice that we do not attempt to give an intrinsic meaning to $H_b(f)$ or to $\bar{\partial}_b f$, as far as we are concerned $\bar{\partial}_b f = 0$ and $H_b(f) = 0$ are just abbreviations for (3.2.1) and (3.2.2).)

G. M. Henkin has proved in [7] the following basic:

THEOREM (Henkin). *Let $f \in C^\infty_{p,q}(\partial\mathscr{D})$ be such that $H_b f = 0$, and $q < n$. Let us define*

(3.2.3)

$$g = \int_{(\zeta,\lambda)\in\partial\mathscr{D}\times[0,1]} f(\zeta) \wedge \omega'_{q-1}\left[(1-\lambda)\frac{p(\zeta, z)}{\varphi(\zeta, z)} + \lambda\frac{p^*(\zeta, z)}{\varphi^*(\zeta, z)}\right] \wedge \omega_p(\zeta + z)\,.$$

Then $g \in L^1_{p,q-1}(\partial\mathscr{D})$ and g satisfies the equation:

$$\int_{\partial\mathscr{D}} g \wedge \bar\partial\varphi = C\int_{\partial\mathscr{D}} f \wedge \varphi \tag{3.2.4}$$

for all $\varphi \in C_{n-p,n-q-1}(\overline{\mathscr{D}})$, where $C \neq 0$ is a constant that depends only on n, p, q.

We shall specialise now the above formulas to $n = 2$, $p = 2$, $q = 1$. We have then

$$\omega'(\eta) = \eta_1 d\eta_2 - \eta_2 d\eta_1$$

from which it readily follows that

$$\omega'_0\left[(1-\lambda)\frac{p(\zeta, z)}{\varphi(\zeta, z)} + \lambda\frac{p^*(\zeta, z)}{\varphi^*(\zeta, z)}\right] = \frac{p_1 p_2^* - p_2 p_1^*}{\varphi\varphi^*}d\lambda + \chi$$

where χ is a form that involves $d\bar\zeta$. On the other hand $\omega_2(z+\zeta)$ is equal to $dz_1 \wedge dz_2$. If we substitute the above expressions in the formula (3.2.3) of Henkin's theorem we obtain:

$$g = \int_{\partial\mathscr{D}} \frac{p_1(\zeta, z)p_2^*(\zeta, z) - p_2(\zeta, z)p_1^*(\zeta, z)}{\varphi(\zeta, z)\varphi^*(\zeta, z)} f(\zeta)dz_1 \wedge dz_2$$

where of course:

$$f(\zeta) = f_1(\zeta)d\zeta_1 \wedge d\zeta_2 \wedge d\bar\zeta_1 + f_2(\zeta)d\zeta_1 \wedge d\zeta_2 \wedge d\bar\zeta_2\,.$$

We shall now specialise further and suppose that $\mathscr{D} = \mathring{B}$, the interior of the unit ball, and that $\rho(\zeta) = |\zeta_1|^2 + |\zeta_2|^2 - 1$. We have then:

$$p_i(\zeta, z) = \bar\zeta_i \qquad p_i^*(\zeta, z) = \bar z_i \qquad i = 1, 2$$
$$\varphi(\zeta, z) = |\zeta|^2 - \bar\zeta\cdot z \qquad \varphi^*(\zeta, z) = \bar z\cdot\zeta - |z|^2$$

with the usual notation $\bar\zeta\cdot z = \bar\zeta_1\cdot z_1 + \bar\zeta_2\cdot z_2$ and $\bar\zeta\cdot\zeta = |\zeta|^2$. We deduce that:

$$\frac{p_1p_2^* - p_2p_1^*}{\varphi\varphi^*} = \frac{\bar\zeta_1\bar z_2 - \bar\zeta_2\bar z_1}{(|\zeta|^2 - \bar\zeta\cdot z)(\bar z\cdot\zeta - |z|^2)}$$

let us then set:

$$K(z, \zeta) = \frac{\bar{z}_1\bar{\zeta}_2 - \bar{z}_2\bar{\zeta}_1}{|1 - \bar{\zeta}\cdot z|^2};$$

$$K_1(z, \zeta) = \frac{\bar{\zeta}_2 - \bar{z}_2}{|1 - \bar{\zeta}\cdot z|^2}; \quad K_2 = \frac{\bar{\zeta}_1 - \bar{z}_1}{|1 - \bar{\zeta}\cdot z|^2}$$

which are all well defined for $z, \zeta \in B$ except when $z = \zeta \in \partial B$. We have then:

$$\frac{p_1 p_2^* - p_2 p_1^*}{\varphi\varphi^*} = K(z, \zeta) \quad \forall z, \zeta \in \partial B, \qquad z \neq \zeta$$

$$K(z, \zeta) = \bar{z}_1 \frac{\bar{\zeta}_2 - \bar{z}_2}{|1 - \bar{\zeta}\cdot z|^2} - \bar{z}_2 \frac{\bar{\zeta}_1 - \bar{z}_1}{|1 - \bar{\zeta}\cdot z|^2} = \bar{z}_1 K_1(z, \zeta) - \bar{z}_2 K_2(z, \zeta).$$

and by an easy computation we obtain that for $\zeta \in \partial B$:

$$\frac{\partial K_1}{\partial z_1} = \bar{\zeta}_1 \frac{\bar{\zeta}_2 - \bar{z}_2}{|1 - \bar{\zeta}\cdot z|^2(1 - \bar{\zeta}\cdot z)}$$

$$\frac{\partial K_1}{\partial \bar{z}_1} = \zeta_1 \frac{\bar{\zeta}_2 - \bar{z}_2}{|1 - \bar{\zeta}\cdot z|^2(1 - \zeta\cdot\bar{z})}$$

$$\frac{\partial K_1}{\partial z_2} = \bar{\zeta}_2 \frac{\bar{\zeta}_2 - \bar{z}_2}{|1 - \bar{\zeta}\cdot z|^2(1 - \bar{\zeta}\cdot z)}$$

$$\frac{\partial K_1}{\partial \bar{z}_2} = \zeta_1 \frac{\bar{z}_1 - \bar{\zeta}_1}{|1 - \bar{\zeta}\cdot z|^2(1 - \zeta\cdot\bar{z})}$$

and the analogous expressions for K_2. From the above we conclude that

$$L_i = \frac{\partial K_i}{\partial \bar{z}_1}\frac{\partial \rho}{\partial \bar{z}_2} - \frac{\partial K_i}{\partial \bar{z}_2}\frac{\partial \rho}{\partial \bar{z}_1} = (-1)^{i+1}\zeta_i \frac{\bar{\zeta}\cdot z - |z|^2}{|1 - \bar{\zeta}\cdot z|^2(1 - \zeta\cdot\bar{z})}; \quad (i = 1, 2)$$

and

$$(3.2.5) \quad \begin{cases} L = \dfrac{\partial K}{\partial \bar{z}_1}\dfrac{\partial \rho}{\partial \bar{z}_2} - \dfrac{\partial K}{\partial \bar{z}_2}\dfrac{\partial \rho}{\partial \bar{z}_1} = z_2 K_1 + z_1 K_2 + \bar{z}_1 L_1 - \bar{z}_2 L_2 \\[2ex] \quad = z_2 K_1 + z_1 K_2 + \bar{z}\cdot\zeta \dfrac{\bar{\zeta}\cdot z - |z|^2}{|1 - \bar{\zeta}\cdot z|^2(1 - \zeta\cdot\bar{z})} \\[2ex] \quad = z_2 K_1 + z_1 K_2 + \bar{z}\cdot\zeta \dfrac{1 - |z|^2}{|1 - \bar{\zeta}\cdot z|^2(1 - \zeta\cdot\bar{z})} \\[2ex] \qquad + \bar{z}\cdot\zeta \dfrac{\bar{\zeta}\cdot z - 1}{|1 - \bar{\zeta}\cdot z|^2(1 - \zeta\cdot\bar{z})} \\[2ex] \quad = M_1 + M_2 + M_3 \end{cases}$$

where

$$M_1 = z_2 K_1 + z_1 K_2$$

$$M_2 = \bar{z}\cdot\zeta \frac{1 - |z|^2}{|1 - \bar{\zeta}\cdot z|^2(1 - \zeta\cdot\bar{z})}$$

$$M_3 = \frac{-\bar{z}\cdot\zeta}{(\bar{z}\cdot\zeta - 1)^2}\,.$$

Let us now denote by

$$T\colon L^1_{2,1}(\partial B) \longrightarrow C^\infty(\mathring{B}) \tag{3.2.6}$$

the linear mapping:

$$Tf(z) = \frac{-1}{C}\int_{\zeta\in\partial B} K(z, \zeta)f(\zeta)$$

where C is the constant of equation (3.2.4). If we take the derivatives of Tf with respect to z we see that the coordinates of ∇Tf can be expressed as a linear combination (with coefficient functions in $C^\infty(B)$, i.e., C^∞ up to the boundary) of following ten integrals

$$\int_{\partial B} K_i(z, \zeta)f(\zeta)\,;\quad \int_{\partial B}\frac{\partial K_i}{\partial z_j}(z, \zeta)f(\zeta)\,;\quad \int_{\partial B}\frac{\partial K_i}{\partial \bar{z}_j}(z, \zeta)f(\zeta) \tag{3.2.7}$$

with $i, j = 1, 2$.

Let us now denote by $A_k(z, \zeta)$ ($k = 1, 2, 3, 4$) the following four kernels:

$$\frac{(1 - |z|^2)^{1/2}(\bar{z}_j - \bar{\zeta}_j)}{|1 - z\cdot\bar{\zeta}|^2(1 - \bar{z}\cdot\zeta)}\,,\quad \frac{(1 - |z|^2)^{1/2}(\bar{z}_j - \bar{\zeta}_j)}{|1 - z\cdot\bar{\zeta}|\,(1 - z\cdot\bar{\zeta})} \tag{3.2.8}$$

for $j = 1, 2$. It follows then from the expressions of the derivatives obtained above that the last two integrals in (3.2.7) can be expressed as linear combinations with constant coefficients of integral of the form:

$$(1 - |z|^2)^{-1/2}\int_{\partial B} A_k(z, \zeta)\alpha(\zeta)f(\zeta) \qquad k = 1, 2, 3, 4 \tag{3.2.9}$$

where $\alpha(\zeta) \in C^\infty(\partial B)$ (in fact $\alpha(\zeta)$ is any of the four functions ζ_1, ζ_2, $\bar{\zeta}_1$, $\bar{\zeta}_2$ restricted to ∂B). We also have

LEMMA 3.2.1. *Let $f \in L^1_{2,1}(\partial B)$. Let us suppose that $H_b(f) = 0$ and let us define $\Gamma(z) \in C^\infty(\mathring{B})$ by the equation*

$$\bar{\partial}(Tf) \wedge \bar{\partial}\rho = \Gamma(z)d\bar{z}_1 \wedge d\bar{z}_2\,.$$

$\Gamma(z)$ is then a linear combination with coefficients in $C^\infty(B)$ (in fact the coefficients are polynomials in z and $\bar{z}$ and are C^∞ in the whole of $\mathbf{C}^2$) of the following four integrals.

$$\int_{\partial B} K_i(z, \zeta)f(\zeta)\,;\quad \int_{\partial B} B(z, \zeta)\zeta_j f(\zeta)\,;\qquad i, j = 1, 2$$

where:

$$B(z, \zeta) = \frac{1 - |z|^2}{|1 - \bar{\zeta}\cdot z|^2(1 - \zeta\cdot\bar{z})}.$$

Proof. The proof is an immediate consequence of the expression of L in (3.2.5) and the fact that our hypothesis $H_b(f) = 0$ implies that:

$$\int_{\partial B} M_3(z, \zeta)f(\zeta) = 0 \qquad \forall z \in \mathring{B}$$

3.3. Estimates on the kernels. In this paragraph we shall make a number of estimates on the kernels that were introduced in the previous paragraph.

LEMMA 3.3.1. *The two kernels* K_i $(i = 1, 2)$ *defined in* §3.2 *satisfy the following conditions*:

$$\sup_{\zeta\in\partial B}\int_{\xi\in\partial B}\sup_{0<r<1}|K_i(r\zeta, \xi)|\,d\sigma(\xi) < +\infty$$

$$\sup_{\xi\in\partial B}\int_{\zeta\in\partial B}\sup_{0<r<1}|K_i(r\zeta, \xi)|\,d\sigma(\zeta) < +\infty\,.$$

Proof. The two kernels K_1, K_2, taken together, are rotation invariant in the sense that if $g \in SU(2)$ is a complex rotation of C^2 then we have

$$K_i(gz, g\zeta) = a_g^{(i)}K_1(z, \zeta) + b_g^{(i)}K_2(z, \zeta)$$

where $a_g^{(i)}$, $b_g^{(i)}$ are constants that only depend on g. From this we see that it suffices to show that:

$$\int_{\partial B}\sup_{0<r<1}|K(r1, \zeta)|\,d\sigma(\zeta) < +\infty \tag{3.3.1}$$

$$\int_{\partial B}\sup_{0<r<1}|K(r\zeta, 1)|\,d\sigma(\zeta) < +\infty\,. \tag{3.3.2}$$

Introducing then the coordinates of §2.1 $\zeta = (1 - \alpha_1 + i\beta_1, \alpha_2 + i\beta_2)$ in some Nhd of 1. We see that (3.3.1) and (3.3.2) are both consequences of the following two inequalities.

$$I = \iiint\sup_r\left|\frac{1 - r}{(1 - r + \beta_1^2 + \alpha_2^2 + \beta_2^2)^2 + \beta_1^2}\right|d\beta_1 d\alpha_2 d\beta_2 < +\infty$$

$$J = \iiint\sup_r\left|\frac{(\beta_1^2 + \alpha_2^2 + \beta_2^2)^{1/2}}{(1 - r + \beta_1^2 + \alpha_2^2 + \beta_2^2)^2 + \beta_1^2}\right|d\beta_1 d\alpha_2 d\beta_2 < +\infty$$

where the integration is taken in some fixed cube of $(\beta_1, \alpha_2, \beta_2)$

centered at the origin. Now we have:

$$J = \iiint \frac{(\beta_1^2 + \alpha_2^2 + \beta_2^2)^{1/2}}{(\beta_1^2 + \alpha_2^2 + \beta_2^2)^2 + \beta_1^2} d\beta_1 d\alpha_2 d\beta_2 < +\infty$$

as an easy calculation readily shows. The change of variables $\beta_1^2 + \alpha_2^2 + \beta_2^2 = R^2$, $\beta_1 = R\cos\varphi$ and then the change of variables $R\cos\varphi = \sigma$ shows that:

$$\begin{aligned} I &\leqq \int_0^{R_0} dR \int_{-R_0}^{R_0} \sup_{0<t<1} \left| \frac{(t+R^2)R}{(t+R^2)^2 + \sigma^2} \right| d\sigma \\ &= \int_0^{R_0} R\,dR \int_{-R_0}^{R_0} \sup_{\alpha > R^2} \left\{ \frac{\alpha}{\alpha^2 + \sigma^2} \right\} d\sigma \\ &\leqq C \int_0^{R_0} R\,dR \left\{ \int_0^R \frac{R\,d\sigma}{R^2 + \sigma^2} + \int_R^{R_0} \frac{d\sigma}{\sigma} \right\} < +\infty \end{aligned}$$

and this completes the proof of the lemma.

As an immediate corollary of the above lemma we obtain the following:

LEMMA 3.3.2. *Let $f \in C_{2,1}^{\infty}(\partial B)$. Then there exists some constant C such that:*

$$|Tf(z)| \leqq C \qquad \forall z \in \mathring{B} \tag{3.3.3}$$

we also have:

$$Tf(r\zeta) \xrightarrow[r \to 1]{} Tf(\zeta) \qquad \forall \zeta \in \partial B \tag{3.3.4}$$

and the convergence in (3.3.4) *is uniform with respect to $\zeta \in \partial B$.*

Proof. (3.3.3) is an immediate consequence of the provious lemma.

The convergence in (3.3.4) follows from the fact that:

$$\| K_i(r\zeta, \xi) - K_i(\zeta, \xi) \|_{L^1(\partial B : d\sigma(\xi))} \xrightarrow[r \to 1]{} 0$$

uniformly in ζ. The uniformity follows once more from the rotation properties of the K_i's that allow us to bring any ζ to the point 1.

In what follows we shall denote by $S(z, \zeta)$ any of the following 12 kernels:

$$K_i(z, \zeta) = \frac{\bar{z}_i - \bar{\zeta}_i}{|1 - \bar{\zeta}\cdot z|^2}, \qquad i = 1, 2 \tag{3.3.5}$$

$$\frac{(1 - |z|^2)^{1/2} \operatorname{Re}\operatorname{Im}\{\zeta_i - z_i\} \operatorname{Re}\operatorname{Im}\{1 - \zeta\cdot\bar{z}\}}{|1 - \bar{\zeta}\cdot z|^4}, \qquad i = 1, 2 \tag{3.3.6}$$

$$\frac{(1-|z|^2)\operatorname{Re}\operatorname{Im}\{1-\zeta\cdot\bar{z}\}}{|1-\bar{\zeta}\cdot z|^4} \tag{3.3.7}$$

where Re Im { } means that either the real or the imaginary part in { } has to be taken. Let us now take as before the local coordinates $\zeta = (1-\alpha_1+i\beta_1, \alpha_2+i\beta_2)$ in some Nhd of 1 on ∂B. $S(z, \zeta)$ becomes then $S(z; \beta_1, \alpha_2, \beta_2)$ a function of z and of the local coordinates $(\beta_1, \alpha_2, \beta_2)$. We have then:

LEMMA 3.3.3. *Let S be as above, then there exists two positive constants C and c such that:*

(i) $\int |S(z, \zeta)|\, d\sigma(\zeta) \leq C \qquad \forall z \in \mathring{B}$

(ii) *For all $h > 0$ we have:*

$$\left|\frac{\partial S}{\partial \beta_1}\right| \leq \frac{C}{|1-\mathbf{1}\cdot z|^3} \tag{3.3.8}$$

$$\left|\frac{\partial S}{\partial \alpha_2}\right|, \left|\frac{\partial S}{\partial \beta_2}\right| \leq \frac{C}{|1-\mathbf{1}\cdot z|^{5/2}} \tag{3.3.9}$$

for all (z, ζ) in the range

$$\zeta \in B_h(1);\ z \in \mathring{B}\ \ |1-\mathbf{1}\cdot z| > ch\ .$$

Proof. (i) Using, as in the proof of Lemma 3.3.1, the rotation properties of the above kernels we see that if we may suppose in (i) that $z = (r, 0)$ $0 < r < 1$. When in (i) S is as in (3.3.5) our assertion is already contained in Lemma 3.3.1. When it is one other kernels an easy culculation shows that the integral in (i) is dominated by a linear combination of the following two integrals:

$$(1-r)\iiint \frac{d\beta_1 d\alpha_2 d\beta_2}{[(1-r+\beta_1^2+\alpha_2^2+\beta_2^2)^2+\beta_1^2]^{3/2}}$$

$$(1-r)^{1/2}\iiint \frac{(\beta_1^2+\alpha_2^2+\beta_2^2)^{1/2} d\beta_1 d\alpha_2 d\beta_2}{[(1-r+\beta_1^2+\alpha_2^2+\beta_2^2)^2+\beta_1^2]^{3/2}}$$

where the integration range is some fixed cube in $(\beta_1, \alpha_2, \beta_2)$. And using the changes of variables $\beta_1^2+\alpha_2^2+\beta_2^2 = R^2$, $\beta_1 = R\cos\varphi$ and then $R\cos\varphi = \sigma$. We can verify readily that the above two integrals are uniformly bounded in $0 < r < 1$ as required by the lemma.

(ii) *Proof of* (3.3.8): For S as in (3.3.5), (3.3.6), and (3.3.7) we have respectively.

$$\left|\frac{\partial S}{\partial \beta_1}\right| \leq C\left[\frac{1}{|1-\bar{\zeta}\cdot z|^2} + \frac{|z-\zeta|}{|1-\bar{\zeta}\cdot z|^3}\right] \tag{3.3.10}$$

$$\left|\frac{\partial S}{\partial \beta_1}\right| \leqq C(1-|z|^2)^{1/2}\left[\frac{1}{|1-\bar\zeta\cdot z|^3}+\frac{|z-\zeta|}{|1-\bar\zeta\cdot z|^4}\right] \tag{3.3.11}$$

$$\left|\frac{\partial S}{\partial \beta_1}\right| \leqq C\frac{1-|z|^2}{|1-\bar\zeta\cdot z|^4} \tag{3.3.12}$$

and if we use the inequality:

$$|\zeta - z| \leqq C|1-\bar\zeta\cdot z|^{1/2} \qquad z\in \mathring{B}\,,\quad \zeta\in\partial B \tag{3.3.13}$$

and the inequality:

$$\begin{aligned}(1-|z|^2) &\leqq C(1-|z_1|^2) \leqq C|1-z_1| \\ &\leqq C|1-\mathbf{1}\cdot z| \leqq C|1-z\cdot\bar\zeta|\end{aligned} \tag{3.3.14}$$

which is valid in our range of ζ and z provided that c is large enough. We obtain the inequality (3.3.8) in all three cases (3.3.10), (3.3.11), and (3.3.12).

Proof of (3.3.9). For S as in (3.3.5), (3.3.6), and (3.3.7), we have respectively:

$$\left|\frac{\partial S}{\partial \alpha_2}\right| \leqq C\left[\frac{1}{|1-\bar\zeta\cdot z|^2}+\frac{|z-\zeta|}{|1-\bar\zeta\cdot z|^3}\right] \tag{3.3.15}$$

$$\left|\frac{\partial S}{\partial \alpha_2}\right| \leqq C(1-|z|^2)^{1/2}\left[\frac{1}{|1-\bar\zeta\cdot z|^3}+\frac{|z-\zeta|}{|1-\bar\zeta\cdot z|^4}\left(|z_2|+\left|\frac{\partial\alpha_1}{\partial\alpha_2}\right|\right)\right] \tag{3.3.16}$$

$$\left|\frac{\partial S}{\partial \alpha_2}\right| \leqq C(1-|z|^2)\left[\frac{|z_2|+|\partial\alpha_1/\partial\alpha_2|}{|1-\bar\zeta\cdot z|^4}\right]. \tag{3.3.17}$$

To obtain the inequalities (3.3.16), and (3.3.17) we use the same calculation as in the proof of Lemma 2.2.1 (ii).

Using then (3.3.13), (3.3.14), and the same inequalities as in the proof of Lemma 2.2.1 (ii), we see that we obtain (3.3.9) in all three cases (3.3.15), (3.3.16), and (3.3.17). $\partial S/\partial\beta_2$ behaves exactly as $\partial S/\partial\alpha_2$. We have finally:

LEMMA 3.3.4. *Let $S(z,\zeta)$ be any one of the* 12 *kernels as in* (3.3.5), (3.3.6) *or* (3.3.7), *and let $\mu\in M(\mathring{B})$ be a bounded measure in $\mathring{B}$, then the integral*

$$f(\zeta)=\int_{z\in\mathring{B}} S(z,\zeta)\,d\mu(z)$$

converges absolutely for almost all $\zeta\in\partial B$ and $f\in L^1(\partial B)$. If in addition μ is a Carleson measure then $f(\zeta)\in$ BMO (∂B).

Proof. Let us denote by

$$\varphi(\zeta) = \int |S(z, \zeta)|\, d\,|\mu|(z)$$

we have then

$$\begin{aligned}\int_{\partial B} \varphi(\zeta) d\sigma(\zeta) &= \int_{\zeta \in \partial B} \int_{z \in \overset{\circ}{B}} |S(z, \zeta)|\, d\,|\mu|\,(z) d\sigma(\zeta) \\ &= \int_{z \in \overset{\circ}{B}} d|\mu|(z) \int_{\zeta \in \partial B} |S(z, \zeta)|\, d\sigma(\zeta) < +\infty\end{aligned}$$

by Lemma 3.3.3 (i). This proves the first part of our lemma.

To prove the second part we first observe that it suffices to verify the BMO condition on the balls $B_h(1)$ centered at 1. Indeed as we have already observed twice before, the kernel $S(z, \zeta)$ transform into linear combinations of themselves under complex rotations, and by an appropriate rotation we can bring any point $\zeta_0 \in \partial B$ to the north pole 1.

Let $h > 0$ be arbitrary but fixed and let us denote by:

$$f_1(\zeta) = \int_{|1-\mathbf{1}\cdot z| \leqq ch} S(z, \zeta) d\mu(z)$$

$$f_2(\zeta) = \int_{|1-\mathbf{1}\cdot z| > ch} S(z, \zeta) d\mu(z)$$

where c is as in the Lemma 3.3.3.

We have then by our hypothesis on μ and by Lemma 3.3.3 (i):

$$\int_{B_h(1)} |f_1(\zeta)|\, d\sigma(\zeta) \leqq C|\mu|\{z \in \overset{\circ}{B};\ |1 - \mathbf{1}\cdot z| \leqq ch\} \leqq Ch^2\,. \tag{3.3.18}$$

By Lemma 3.3.3 (ii) we also have:

$$|S(z, \zeta) - S(z, 1)| \leqq C\left[\frac{h}{|1 - \mathbf{1}\cdot z|^3} + \frac{h^{1/2}}{|1 - \mathbf{1}\cdot z|^{5/2}}\right] \tag{3.3.19}$$

for all

$$\zeta \in B_h(1)\ ; \quad z \in \overset{\circ}{B}\,, \quad |1 - \mathbf{1}\cdot z| > ch\,.$$

We conclude therefore from (3.3.19) that:

$$|f_2(\zeta) - f_2(1)| \leqq Ch \int_{ch}^{\infty} \frac{dF(t)}{t^3} + Ch^{1/2} \int_{ch}^{\infty} \frac{dF(t)}{t^{5/2}} \tag{3.3.20}$$

where, as before, we denote:

$$F(t) = |\mu|\{z \in \overset{\circ}{B};\ |1 - \mathbf{1}\cdot z| < t\}$$

and if we integrate by parts in (3.3.20) and use the fact that $F(t) \leqq ct^2$, which is a consequence of our hypothesis, we conclude that:

$$|f_2(\zeta) - f_2(1)| \leqq C \qquad \forall \zeta \in B_h(1)$$

which together with (3.3.18) implies that

$$\int_{B_h(1)} |f(\zeta) - f_2(1)|\, d\sigma(\zeta) \leqq Ch^2$$

and completes the proof of the lemma.

3.4. Proof of the first half of Theorem 3.1.1. Let $\mathscr{D} = \{\rho < 0\}$ be as before and let us denote by $\sum_{p,q} \subset M_{p,q}(\mathscr{D})$ the space of forms $\mu \in M_{p,q}(\mathscr{D})$ that satisfy

$$|\rho|^{-1/2}\bar\partial\rho \wedge \mu \in M_{p,q}(\mathscr{D}) \, .$$

We can then identify $C_{p,q}$, the space of (p, q)-forms that satisfy the Carleson condition with a subspace of $\sum_{p,q}$. The two spaces $\sum_{p,q}$ and $C_{p,q}$ have a natural norm.

For arbitrary $f \in C^\infty_{2,1}(\partial B)$ we shall denote

$$Qf(z) = \bar\partial(Tf) \wedge dz_1 \wedge dz_2 \in C^\infty_{2,1}(\mathring{B})$$

where T is the mapping defined in (3.2.5). We have then

PROPOSITION 3.4.1. *There exists*

$$\Lambda(z, \zeta) = (\Lambda_1(z, \zeta)d\bar z_1 + \Lambda_2(z, \zeta)d\bar z_2) \wedge dz_1 \wedge dz_2$$

a vector kernel of type (2, 1) *that is defined and smooth for* $z \in \mathring{B}$ *and* $\zeta \in \partial B$ *that has the following properties.*

The integral:

$$\check\mu(\zeta) = \int_{\mathring{B}} \mu(z) \wedge \Lambda(z, \zeta)$$

is absolutely convergent for all $\mu \in \sum_{0,1}$ *and almost all* $\zeta \in \partial B$ *and it satisfies*

(i) $\|\check\mu\|_{L^1(\partial B)} \leqq C\|\mu\|_\Sigma$

(ii) *If* $\mu \in C_{0,1}$ *then* $\check\mu \in \mathrm{BMO}\,(\partial B)$ *and* $\|\check\mu\|_{\mathrm{BMO}(\partial B)} \leqq \|\mu\|_C$

(iii) $\int_{\mathring{B}} \mu \wedge Qf = \int_{\partial B} \check\mu f$

for all $f \in C^\infty_{2,1}(\partial B)$ *that satisfies* $H_b(f) = 0$.

Proof. Using a simple argment involving a partition of unity we see that it suffices to prove the proposition locally.

More explicitly it suffices to show that for all $\zeta_0 \in B$ there exists Ω some Nhd of ζ_0 in C^2 and a kernel $\Lambda(z, \zeta) = \Lambda_\Omega(z, \zeta)$ as in the proposition such that the conclusion of the proposition holds for that Λ_Ω and all μ as above that satisfy the additional hypothesis

$$\operatorname{supp} \mu \subset \Omega . \tag{3.4.1}$$

Let us fix therefore some $\zeta_0 \in B$ and let us choose $\omega \in C^\infty_{1,0}(\Omega)$ a normalised (i.e., of length 1) differential form in a small enough Nhd Ω of ζ_0 such that

$$\partial\rho,\ \bar{\partial}\rho,\ \omega,\ \bar{\omega}$$

form a basis of the complexified cotangent space at every point of Ω (here of course $\rho(z) = |z|^2 - 1$ but we prefer to keep the general notation).

Let also $\mu \in \sum_{0,1}$ be arbitrary but satisfying (3.4.1):

$$\mu = \mu_1 d\bar{z}_1 + \mu_2 d\bar{z}_2 = \tilde{\mu}_1 \bar{\partial}\rho + \tilde{\mu}_2 \bar{\omega} .$$

By our hypothesis we have then:

$$|\tilde{\mu}_1|,\ |\tilde{\mu}_2|,\ |\rho|^{-1/2}|\tilde{\mu}_2| \in M(\mathring{B}) \tag{3.4.2}$$

and if we suppose in addition that $\mu \in C_{0,1}$ (satisfying the Carleson condition) then the measures (3.4.2) are Carleson measures.

Let now f be as in part (iii) of our proposition. We have then:

$$\bar{\partial} Tf = X\bar{\partial}\rho + Y\bar{\omega} .$$

Using now (3.2.7), (3.2.8), and (3.2.9) we see that

$$X(z) = |\rho(z)|^{-1/2} \sum_{\alpha,\beta,S} \beta(z) \int_{\partial B} S(z, \zeta)\alpha(\zeta)f(\zeta) \tag{3.4.3}$$

where the summation extends over a finite number of kernels $S(z, \zeta)$ taken out of the 12 kernels (3.3.5), (3.3.6), (3.3.7), and also a finite number of $\alpha(\zeta)$ that are polynomials in ζ and $\bar{\zeta}$ and a finite number of $\beta(z) \in C^\infty(\mathring{B})$ that are bounded and continuous in B.

Using Lemma 3.2.1 we also see that

$$Y(z) = \sum_{\alpha^*,\beta^*,S^*} \beta^*(z) \int_{\partial B} S^*(z, \zeta)\alpha^*(\zeta)f(\zeta)$$

where the summation is as in (3.4.3).

We conclude therefore that for μ and f as above we have:

$$\mu \wedge \bar{\partial} Tf = (\tilde{\mu}_1 Y - \tilde{\mu}_2 X)\bar{\partial}\rho \wedge \bar{\omega} = \int_{\partial B} \Theta(z, \zeta)f(\zeta)$$

where

$$(3.4.4)\qquad \begin{aligned}\Theta(z,\zeta) = [\tilde\mu_1(z) \sum_{\alpha^*,\beta^*,S^*} \beta^*(z)S^*(z,\zeta)\alpha^*(\zeta) \\ - |\rho(z)|^{-1/2}\tilde\mu_2(z) \sum_{\alpha,\beta,S} \beta(z)S(z,\zeta)\alpha(\zeta)]\bar\partial\rho \wedge \bar\omega .\end{aligned}$$

If we express $\tilde\mu_1$, $\tilde\mu_2$, $\bar\partial\rho$, and $\bar\omega$ in terms of μ_1, μ_2, $d\bar z_1$ and $d\bar z_2$ in (3.4.4) we obtain

$$\Theta(z,\zeta) = (\mu_1 A_2 - \mu_2 A_1) d\bar z_1 \wedge d\bar z_2$$

where $A_1(z,\zeta)$ and $A_2(z,\zeta)$ $(z \in \mathring{B},\ \zeta \in \partial B)$ are two well determined kernels.

Let us then set:

$$A_\Omega(z,\zeta) = (A_1(z,\zeta)d\bar z_1 + A_2(z,\zeta)d\bar z_2) \wedge dz_1 \wedge dz_2 .$$

We have then

$$(3.4.5)\qquad \Theta(z,\zeta) \wedge dz_1 \wedge dz_2 = \mu(z) \wedge A_\Omega(z,\zeta)$$

and A_Ω satisfies the conditions of (i) and (ii) in our proposition. To see that, we observe that all the kernels in the summations of (3.4.4) satisfy the conditions of Lemma 3.3.3. It suffices then to use (3.4.2) to obtain our result.

(3.4.5) on the other hand implies that

$$\mu \wedge Qf(z) = \int_{\zeta\in\partial B} \mu(z) \wedge A_\Omega(z,\zeta) \wedge f(\zeta) .$$

It therefore follows that:

$$\begin{aligned}\int_{\mathring{B}} \mu \wedge Qf &= \int_{z\in\mathring{B}} \int_{\zeta\in\partial B} \mu(z) \wedge A_\Omega(z,\zeta) \wedge f(\zeta) \\ &= \int_{\zeta\in\partial B} \left(\int_{z\in B} \mu(z) \wedge A_\Omega(z,\zeta)\right) \wedge f(\zeta) = \int_{\partial B} \check\mu_\Omega(\zeta) f(\zeta)\end{aligned}$$

and this proves part (iii) of the proposition.

Before we give the proof of Theorem 3.1.1 (i) we shall need two lemmas.

LEMMA 3.4.1. *For every $f \in C^\infty_{2,1}(\partial B)$ that satisfies $H_b(f) = 0$ we have:*

$$\int_{\partial B} f \wedge \varphi = \lim_{r\to 1} \int_{\partial B_r} Qf \wedge \varphi ; \qquad \forall \varphi \in C^\infty(B)$$

where $B_r = \{z \in \boldsymbol{C}^2;\ |z| \leq r\}$.

Proof. We have by Stoker's theorem

$$\int_{\partial B_r} Qf \wedge \varphi = -\int_{\partial B_r} Tf \wedge \bar\partial\varphi \wedge dz_1 \wedge dz_2 .$$

But by Lemma 3.3.2 we also have

$$\lim_{r\to 1}\int_{\partial B_r} Tf\wedge\bar\partial\varphi\wedge dz_1\wedge dz_2=\int_{\partial B} Tf\wedge\bar\partial\varphi\wedge dz_1\wedge dz_2\,.$$

We deduce therefore that:

$$\lim_{r\to 1}\int_{\partial B_r} Qf\wedge\varphi=-\int_{\partial B} Tf\wedge\bar\partial\varphi\wedge dz_1\wedge dz_2\,.$$

But by Henkin's theorem in § 3.2 we also have:

$$\int_{\partial B} Tf\wedge\bar\partial\varphi\wedge dz_1\wedge dz_2=-\int_{\partial B} f\wedge\varphi\,.$$

From this our lemma follows.

LEMMA 3.4.2. *Let f be as in Lemma 3.4.1. Then for all $u\in C^\infty(B)$ the integral $\int_B \bar\partial u\wedge Qf$ is absolutely convergent and it satisfies*

$$\int_{\partial B} u\wedge f=\int_B \bar\partial u\wedge Qf\,. \tag{3.4.6}$$

Proof. The first part of the lemma follows from (3.2.7), (3.2.8), and (3.2.9) which show that the behavior of Qf near the boundary is controlled by $|\rho|^{-1/2}$. To obtain (3.4.6) we use our previous lemma and Stokes's theorem in B_r and then let $r\to 1$; we have

$$\int_{\partial B} u\wedge f=\lim_{r\to 1}\int_{\partial B_r} u\wedge Qf=\lim_{r\to 1}\int_{B_r}\bar\partial u\wedge Qf=\int_{\mathring{B}}\bar\partial u\wedge Qf\,.$$

The following theorem is essentially due to Henkin [14].

THEOREM (Henkin). *For every $\mu\in\sum_{0,1}$ such that $\bar\partial\mu=0$. We have:*

$$\int_{\mathring{B}}\mu\wedge\psi=\int_{\partial B}\check\mu\psi$$

for all $\psi\in C^\infty_{2,1}(B)$ that satisfies $\bar\partial\psi=0$ in some Nhd *of B* ($\check\mu$ is the function defined in Proposition 3.4.1).

We shall give a quick proof of the above result making the additional hypothesis that $\mu\in C^\infty_{0,1}(B)$. Let $\mu\in C^\infty_{0,1}(B)$ and let us suppose that $\bar\partial\mu=0$ in some Nhd of B. Using then standard methods we can find some $u\in C^\infty(B)$ such that

$$\bar\partial u=\mu$$

in some Nhd of B.

It follows therefore from Stokes's theorem that

$$\int_{\partial B} u \wedge \psi = \int_{\mathring{B}} \mu \wedge \psi \tag{3.4.7}$$

for all ψ as in the statement of the theorem.

But by Lemma 3.4.2 and Proposition 3.4.1 we also have:

$$\int_{\partial B} u \wedge \psi = \int_{\mathring{B}} \mu \wedge Q\psi = \int_{\partial B} \check{\mu}\psi \tag{3.4.8}$$

for the same class of ψ's. From (3.4.7) and (3.4.8) therefore our theorem follows. (Observe that the above ψ's satisfy $H_b(\psi) = 0$.)

The difficulty in obtaing the general case from the above lies in the fact that the condition that determines the space $\sum_{0,1}$ has a singularity at the boundary. It is therefore not trivial to regularise in that space (say by convolution) and to approximate a general element of $\sum_{0,1}$ by one that satisfies our special conditions. It can be done however, the interested reader should look, for example, in Skoda [16]. (In [16] Skoda has obtained formulas that are equivalent to Henkin's and for the same purpose. Although I have not gone through the details I am convinced that one could obtain the BMO estimates from Skoda's formulas as well.)

It should be observed, however, that the above special case which gives the solution with a priori estimates is sufficient for most practical purposes.

Proof of Theorem 3.1.1 (i). It now suffices to combine the above theorem (of Henkin) with Proposition (3.4.1) (ii) to obtain our theorem.

3.5. Proof of the second part of Theorem 3.1.1 and the Corona problem.

Proof of Theorem 3.1.1 (ii). Let $g \in \mathrm{BMO}(\partial B)$. Then by Theorem 1.1.1 there exists some $F \in C^\infty(\mathring{B})$ and $f \in L^\infty(\partial B)$ such that

$$\int_{\partial B} (g - f) \wedge \varphi = \lim_{r \to 1} \int_{\partial B_r} F \wedge \varphi \qquad \forall \varphi \in C^\infty_{(2,1)}(B) \tag{3.5.1}$$

where $B_r = \{z \in \boldsymbol{C}^2; |z| \leqq r\}$, and such that

$$|DF|\, d(\mathrm{Vol})$$

is a Carleson measure in $\mathring{B}$.

It follows therefore by Stokes's theorem and a simple passage to the limit as $r \to 1$ that

$$\int_{\partial B}(g-f)\wedge\varphi=\int_{\mathring{B}}\bar\partial F\wedge\varphi \qquad \forall\varphi\in C^\infty(B)$$

provided that $\bar\partial\varphi=0$ in some Nhd of B.

But by Lemma 2.5.1 and our hypothesis on F it follows that the form $\mu=\bar\partial F$ satisfies the Carleson condition, and being trivially $\bar\partial$-closed we see that μ satisfies all the conditions of Theorem 3.1.1 (ii). This completes the proof.

Let us now consider the Hopf mapping

$$\pi\colon (z_1, z_2)\longrightarrow z_1/z_2$$

which is defined for all $(z_1, z_2)\in\partial B$, $z_2\neq 0$, and takes its values in the complex plane $\boldsymbol{C}=\boldsymbol{R}^2$. (We can in fact define π from the whole of ∂B into the compactified complex plane $\boldsymbol{C}\cup\infty$, i.e., the Riemann sphere, but we shall not need to do that here.)

Using that mapping we can give the following coordinates on $\{\partial B; z_2\neq 0\}$.

$$z_1=\frac{e^{i\theta}}{\sqrt{1+|u|^2}}$$

$$z_2=\frac{ue^{i\theta}}{\sqrt{1+|u|^2}}$$

$\theta\in[0, 2\pi)$, $u\in\boldsymbol{C}$.

We have then:

LEMMA 3.5.1. *Let $f(u)\in\mathrm{BMO}(\boldsymbol{R}^2)$, $f\notin L^\infty(\boldsymbol{R}^2)$ and of compact support, and let us define*:

$$\tilde f(z_1, z_2)=f(z_1/z_2) \qquad \forall(z_1, z_2)\in\partial B \qquad z_2\neq 0$$

$$\tilde f(z_1, 0)=0 \qquad \forall(z_1, 0)\in\partial B\,.$$

Then $\tilde f\in\mathrm{BMO}(\partial B)$ and it cannot be decomposed in the form:

$$\tilde f=\varphi+\psi\,;\quad \varphi\in L^\infty(\partial B)\,,\quad \psi\in H^2(\partial B)\,. \tag{3.5.2}$$

Proof. To test the BMO condition on $\tilde f$ observe that the vector field μ_0 runs along the fibers of the Hopf mapping, and that therefore $B_t(\zeta_0)$, a ball in ∂B centered at $\zeta_0\in\partial B$, is essentially the cartesian product of a ball $B_{\sqrt t}(H(\zeta_0))$ in $\boldsymbol{C}=\boldsymbol{R}^2$ centered at $H(\zeta_0)$ and of radius $\sqrt t$ and of a segment of length t centered at ζ_0 along the fiber. Using the above the verification of the BMO condition is immediate.

To see that the decomposition 3.5.2 is impossible, let us suppose by contradiction that we could write

$$\tilde f(u)=\varphi(u,\theta)+\psi(u,\theta)\,;\quad \varphi\in L^\infty(\partial B)\,,\quad \psi\in H^2(\partial B)$$

where we use the coordinates of the Hopf mapping introduced above.

It would then follow that:

$$2\pi\tilde{f}(u) = \int_0^{2\pi} \varphi(u, \theta)d\theta + \int_0^{2\pi} \psi(u, \theta)d\theta \tag{3.5.3}$$

but ψ being analytic it follows that $\int_0^{2\pi} \psi(u, \theta)d\theta$ is at consant (independent of u). The expression (3.5.3) implies therefore that $\tilde{f} \in L^\infty(\partial B)$ which is in contradiction with the hypothesis. We can now give the:

Proof of Theorem 3.5.2. Let us suppose, by contradiction, that Theorem 3.5.2 fails.

Let $v \in \mathrm{BMO}\,(\partial B)$ be an arbitrary BMO function. Then we know by Theorem 3.1.1 (ii) that there exists some $\mu \in C_{0,1}$ and some $w \in L^\infty(\partial \mathscr{D})$ that satisfy

$$\bar{\partial}\mu = 0\,; \quad \int_{\partial B} v \wedge \varphi = \int_{\mathring{B}} \mu \wedge \varphi + \int_{\partial B} w \wedge \varphi$$

for all $\varphi \in C^\infty_{2,1}(B)$ that is $\bar{\partial}$-closed in some Nhd of B. By our contradictory hypothesis it follows that there also exists some $u \in L^\infty(\partial B)$ such that:

$$\int_{\partial B} u \wedge \varphi = \int_{\mathring{B}} \mu \wedge \varphi$$

for the same class of φ's as above. We conclude therefore that

$$\int_{\partial B} (v - u - w) \wedge \varphi = 0$$

for the same φ's as above.

But from Proposition 2.5.1 it follows then that $w + u - v$ is of analytic type and that therefore we can write

$$v = u_1 + \alpha\,; \quad u_1 \in L^\infty(\partial B)\,, \quad \alpha \in H^2(\partial B)$$

and v being arbitrary, this contradicts Lemma 3.5.1 and proves our theorem.

REMARK. If we choose v as in Lemma 3.5.1 we see that the measure μ can be constructed without the use of Theorem 2.1.1.

Indeed to construct the function $F \in C^\infty(\mathring{B})$ whose boundary values are v and for which $|DF|\,d(\mathrm{Vol})$ is a Carleson measure, it suffices to construct the corresponding F in the interior of the Riemann sphere (or in $\mathring{\boldsymbol{R}}^3_+$) and lift it up by the Hopf mapping (which readily extends to the interior of the ball). All one needs then to do the above construction is the much easier Theorem 1.1.1 (i).

The above considerations are not conclusive as far as the Corona problem is concerned, they do show however that the classical approach breaks down at a very essential point!

In the positive direction we can use Theorem 3.1.1 (i) to prove the following.

THEOREM. *Let $f_1, f_2 \in H^\infty(\mathring{B})$ be two bounded analytic functions in $\mathring{B}$ such that*

$$|f_1(z)| + |f_2(z)| \geqq \delta > 0 \qquad \forall z \in \mathring{B}\,.$$

Then there exist two holomorphic functions φ_1, φ_2 in $\mathring{B}$ that satisfy:

$$f_1\varphi_1 + f_2\varphi_2 \equiv 1$$

$$\sup_r \{\|\varphi_1(r\zeta)\|_{\mathrm{BMO}(\partial B)}, \|\varphi_2(r\zeta))_{\mathrm{BMO}(\partial B)}\} < +\infty\,.$$

The same holds for general strictly pseudoconvex domains.

The proof which is a straightforward but lengthy adaptation of L. Carleson's one-dimensional proof (with the modifications of L. Hörmander) will be omitted.

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