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# ON THE INTEGRAL MEANS OF UNIVALENT, MEROMORPHIC FUNCTIONS

ALBERT EDWARD LIVINGSTON

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# ON THE INTEGRAL MEANS OF UNIVALENT, MEROMORPHIC FUNCTIONS

## Albert E. Livingston

We consider two classes of functions, univalent and meromorphic in the unit disk  $\Delta$ . The first class is normalized by requiring that the functions be nonzero in  $\Delta$  with f(0) = 1and a pole at a fixed point, p, 0 . In the second classthe functions are allowed to have a zero with fixed magnitude. Theorems concerning the integral means of functionsin both classes are proven and consequences of thesetheorems are considered.

1. Introduction. Let  $\Sigma(p)$ , 0 , be the class of functions <math>f(z), univalent and meromorphic in  $\Delta = \{z : |z| < 1\}$ , with a simple pole at z = p and such that  $f(z) \neq 0$  for z in  $\Delta$  and f(0) = 1. Also, if 0 and <math>0 < q < 1, we let  $\Sigma(p, q)$  be the class of functions f(z), univalent and meromorphic in  $\Delta$ , with a simple pole at z = p such that  $f(z_0) = 0$  for some  $z_0$  with  $|z_0| = q$  and f(0) = 1. Recently Libera and the author [4] and the author [5] have studied a subclass of  $\Sigma(p)$ , namely the class of weakly starlike meromorphic functions  $\Lambda^*(p)$  which have the representation

$$f(z)=rac{z}{\Big(1-rac{z}{p}\Big)(1-pz)}g(z)$$

where g(z) is in  $\Sigma^*$ , the class of normalized meromorphic starlike functions. In this paper we will extend many of the results obtained for  $\Lambda^*(p)$  to the class  $\Sigma(p)$ . In particular it was proven in [5] that if f is in  $\Lambda^*(p)$  and  $F(z) = (1 + z)^2/(1 - z/p)(1 - pz)$ , then

$$\int_{-\pi}^{\pi} |f(re^{i heta})|^{\lambda} d heta \leq \int_{-\pi}^{\pi} |F(re^{i heta})|^{\lambda} d heta$$

for 0 < r < 1 and  $\lambda > 0$ . Using a powerful method of Baernstein [1], we will extend and generalize this result to the class  $\Sigma(p)$ . Similar results are also obtained for the class  $\Sigma(p, q)$ .

2. The class  $\Sigma(p)$ . The proof of the theorem concerning the integral means of a function in  $\Sigma(p)$  follows the proof given by Kirwan and Schober [3] who consider the class S(p) of functions f(z), univalent and meromorphic in  $\Delta$ , with a simple pole at z = p and such that f(0) = 0 and f'(0) = 1. The proof relies on results of Baernstein [1] which we now state.

For this purpose we need to introduce some notation. If g is a measurable, extended real valued function on  $[-\pi, \pi]$ , then we define

$$g^*( heta) = \sup_{_E} \int_{_E} g( heta) d heta$$

where the supremum is taken over all Lebesque measurable sets  $E \subset [-\pi, \pi]$  with measure  $m(E) = 2\theta$ . In particular, if  $u(re^{i\theta})$  is defined in an annulus  $r_1 < |z| < r_2$  and the \* operation is performed in the  $\theta$  variable, then  $u^*(re^{i\theta})$  is defined in  $\{re^{i\theta}: r_1 < r < r_2, 0 \leq \theta \leq \pi\}$ . Baernstein [1] has proven the following.

PROPOSITION 1 ([1, Theorems A and A' and Proposition 5]).

(i) Let D be a domain containing  $r_0 > 0$  and having a classical Green's function. Let u be the Green's function of D with pole at  $r_0$ . (It is assumed here that u is defined on the extended plane by defining it to be zero on the complement of D.) Then

$$u^{*}(re^{i heta}) = u^{*}(re^{i heta}) + 2\pi\log^{+}rac{r}{r_{\scriptscriptstyle 0}}$$

is subharmonic in the upper half-plane.

(ii) Let D and u be as in (i) and suppose further that D is circularly symmetric. Let  $D^+ = D \cap \{z: \text{Im } z > 0\}$ . Then  $u^{\sharp}(re^{i\theta})$  is harmonic in  $D^+$ .

**PROPOSITION 2** ([1, Proposition 2]). For  $g \in L^1[-\pi, \pi]$ ,

$$g^*( heta) = \int_{- heta}^{ heta} G(x) dx$$
 ,  $0 \leq heta \leq \pi$  ,

where G(x) is the symmetric nonincreasing rearrangement of g. (For the definition of G(x) see [1] and [2].)

**PROPOSITION 3** ([1, Proposition 3]). For  $g, h \in L^1[-\pi, \pi]$  the following are equivalent.

(a) For every convex nondecreasing function  $\Phi$  on  $(-\infty, \infty)$ ,

$$\int_{-\pi}^{\pi} \varPhi(g( heta)) d heta \leqq \int_{-\pi}^{\pi} \varPhi(h( heta)) d heta$$
 .

(b) For every  $t \in (-\infty, \infty)$ ,

$$\int_{-\pi}^{\pi} [g( heta)-t]^+ d heta \leqq \int_{-\pi}^{\pi} [h( heta)-t]^+ d heta \; .$$

 $(\mathbf{c}) \quad g^*(\theta) \leq h^*(\theta), \ \mathbf{0} \leq \theta \leq \pi.$ 

We can now state and prove the following theorem.

THEOREM 1. Let  $\Phi$  be a convex nondecreasing function on  $(-\infty, \infty)$ . Then for all  $f \in \Sigma(p)$  and 0 < r < 1,

$$\int_{-\pi}^{\pi} arPsi(\pm \log |f(re^{i heta})|) d heta \leq \int_{-\pi}^{\pi} arPsi(\pm \log |F_p(re^{i heta})|) d heta$$

where

$${F}_{p}(z) = rac{(1+z)^2}{\Big(1-rac{z}{p}\Big)\!(1-pz)} \; .$$

*Proof.* We first consider the inequality.

(2.1) 
$$\int_{-\pi}^{\pi} \Phi(\log |f(re^{i\theta})|) d\theta \leq \int_{-\pi}^{\pi} \Phi(\log |F_p(re^{i\theta})|) d\theta .$$

With  $f^{-1}$  denoting the inverse function of f we define

(2.2) 
$$u(w) = \begin{bmatrix} -\log |f^{-1}(w)|, & w \in f(\varDelta) \\ 0, & \text{otherwise} \end{bmatrix}$$

and

$$v(w) = egin{bmatrix} -\log |F_p^{-1}(w)|\,, & w \in F_p(arDelta) \ 0 &, & ext{otherwise} \;. \end{cases}$$

According to Proposition 1(i) the function  $u^*(re^{i\theta}) = u^*(re^{i\theta}) + 2\pi \log^+ r$ is subharmonic in the upper half-plane and by ([1, Theorem A']) is continuous on the real line with 0 deleted. The function  $F_p$  maps  $\varDelta$  onto the extended plane slit along the interval  $[-4p/(1-p)^2, 0]$ . Thus  $F_p(\varDelta)$  is circularly symmetric and according to Proposition 1(ii) the function  $v^*(re^{i\theta}) = v^*(re^{i\theta}) + 2\pi \log^+ r$  is harmonic in the upper half-plane and by [1, Theorem A'] is continuous on the real line with 0 deleted. It follows then that  $u^* - v^* = u^* - v^*$  is subharmonic in the upper half-plane and continuous on the real line with 0 deleted.

The inequality (2.1) will follow from Proposition  $3(b \Rightarrow a)$  if it can be proven that for  $f \in \Sigma(p)$ , 0 < r < 1 and  $0 < \rho < \infty$ ,

(2.4) 
$$\int_{-\pi}^{\pi} \log^{+} \frac{|f(re^{i\theta})|}{\rho} d\theta \leq \int_{-\pi}^{\pi} \log^{+} \frac{|F_{p}(re^{i\theta})|}{\rho} d\theta .$$

At this point we have need of a lemma analogous to Proposition 4 in [1] and one which appears in [3].

LEMMA. Let 
$$f \in \Sigma(p)$$
,  $0 < r < 1$  and  $0 < \rho < \infty$ . Then,

$$\int_{-\pi}^{\pi} \log^+ rac{|f(re^{i\phi})|}{
ho} d\phi + 2\pi \log^+ rac{r}{p} = \int_{-\pi}^{\pi} [u(
ho e^{i\phi}) + \log r]^+ d\phi + 2\pi \log^+ rac{1}{
ho} \, .$$

Because of this Lemma, we see that (2.4) is equivalent to the inequality

(2.5) 
$$\int_{-\pi}^{\pi} [u(\rho e^{i\phi}) + \log r]^{+} d\phi \leq \int_{-\pi}^{\pi} [v(\rho e^{i\phi}) + \log r]^{+} d\phi.$$

However, applying Proposition  $3(c \rightarrow b)$  we see that (2.5) will hold provided

$$(2.6) \qquad (u^*-v^*)(\rho e^{i\phi}) \leq 0 \ , \qquad 0 < \rho < \infty \ , \qquad 0 \leq \theta \leq \pi \ .$$

As we have already noted  $u^* - v^*$  is subharmonic in the upper half-plane and continuous on the real line with 0 deleted. In a neighborhood of w = 0 both u(w) and v(w) are continuous with u(0) = v(0) = 0. Thus given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|u(w)| < \varepsilon/2\pi$  if  $|w| < \delta$ . Thus if  $|w| < \delta$ ,  $w = \rho e^{i\phi} (0 \le \phi \le \pi)$  and  $m(E) = 2\phi$  we have

$$\int_{\scriptscriptstyle E} u(
ho e^{i heta}) d heta < rac{arepsilon}{2\pi} m(E) \leqq arepsilon$$
 .

Therefore

$$u^*(
ho e^{i\phi}) = \sup_{_E} \int_{_E} u(
ho e^{i heta}) d heta \leqq arepsilon$$
 .

It follows then that  $u^*(w)$  approaches 0 as w approaches 0. A similar statement holds for  $v^*(w)$ . Thus

(2.7) 
$$\lim_{w\to 0} (u^* - v^*)(w) = 0.$$

We also have

$$\lim_{w\to\infty} u(w) = \lim_{w\to\infty} v(w) = -\log p .$$

Thus given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|u(w) + \log p| < \varepsilon/2\pi$ and  $|v(w) + \log p| < \varepsilon/2\pi$  if  $|w| > \delta$ . Thus if  $|w| > \delta$ ,  $w = \rho e^{i\phi} (0 \le \phi \le \pi)$  and  $m(E) = 2\phi$ ,

$$\left|\int_{E} (u(
ho e^{i heta}) + \log p)d heta
ight| < rac{arepsilon}{2\pi}m(E) \leq arepsilon$$
 .

It follows that

 $-arepsilon \leq u^*(
ho e^{i\phi}) + 2\phi \log p \leq arepsilon$  .

Similarly, we have

 $-arepsilon \leq v^*(
ho e^{i\phi}) + 2\phi \log p \leq arepsilon$  .

Thus

$$-2arepsilon \leq (u^*-v^*)(
ho e^{i\phi}) \leq 2arepsilon$$
 .

It follows then that

(2.8) 
$$\lim_{w\to\infty} (u^* - v^*)(w) = 0$$

From (2.8) and previous remarks it follows that the subharmonic function  $u^* - v^*$  is bounded in the upper half-plane. Thus, by the maximum principle, it is enough to prove that  $(u^* - v^*)(s) \leq 0$  for s on the real axis R.

For this purpose we let

$$D_f = \sup_{w \notin f(J)} |w|$$

and divide the real line into 3 intervals,

$$R=(-\infty,\ -D_f)\cup [-D_f,\ 0)\cup [0,\ +\infty)$$
 .

Case (i).  $s \in [0, +\infty)$ . Because of (2.7) we need only consider  $s \in (0 + \infty)$ . But then  $u^*(s) = v^*(s) = 0$  by definition, if s > 0.

Case (ii).  $s \in (-\infty, -D_f)$ . We first note that u(w) is harmonic for max  $\{1, D_f\} < |w| \leq \infty$  and v(w) is subharmonic in the same region. Thus (u - v)(w) is superharmonic in max  $\{1, D_f\} < |w| \leq \infty$ . In general,  $u(w) + \log |w - 1|$  is harmonic in  $|w| > D_f$  and  $v(w) + \log |w - 1|$  is subharmonic in  $|w| > D_f$ . It follows that (u - v)(w)is superharmonic for  $D_f < |w| \leq \infty$ . Thus we have

$$(u^* - v^*)(s) = \int_{-\pi}^{\pi} (u - v)(|s|e^{i\theta})d\theta \leq 2\pi(u - v)(\infty) = 0$$
.

Case (iii).  $s \in [-D_f, 0)$ . Following Kirwan and Schober [3], for a given  $\varepsilon > 0$  we introduce the subharmonic function

 $Q(\rho e^{i\phi}) = (u^* - v^*)(\rho e^{i\phi}) - \varepsilon \phi$   $(0 \leq \rho < \infty, 0 \leq \phi \leq \pi)$ .

From previous cases we have,

(2.9) 
$$\lim_{w \to s} \sup Q(w) \leq 0$$

for all  $s \in \{R - [-D_f, 0)\} \cup \{\infty\}$ . Suppose  $\sup_{\operatorname{Im} w>0} Q(w) = M > 0$ . Then as in [3] we have by the maximum principle and (2.9) the existence of some  $\hat{s} \in [-D_f, 0)$  such that

$$(2.10) Q(\hat{s}) \ge Q(|\hat{s}|e^{i\phi}), 0 \le \phi \le \pi.$$

Thus,

(2.11)  
$$0 \leq \lim_{\phi \to \pi} \frac{Q(|\hat{s}|e^{i\phi}) - Q(\hat{s})}{\phi - \pi} \\ = \lim_{\phi \to \pi} \frac{u^*(|\hat{s}|e^{i\phi}) - u^*(\hat{s})}{\phi - \pi} - \lim_{\phi \to \pi} \frac{v^*(|\hat{s}|e^{i\phi}) - v^*(\hat{s})}{\phi - \pi} - \varepsilon.$$

From Proposition 2 and the definition of G(x) [1] it follows that

(2.12) 
$$\lim_{\phi \to \pi} \frac{u^*(|\hat{s}|e^{i\phi}) - u^*(\hat{s})}{\phi - \pi} = 2 \min_{0 \le \phi \le \pi} u(|\hat{s}|e^{i\phi}).$$

A similar equality holds for  $v^*$ . Combining (2.11) and (2.12) we obtain

$$(2.13) 0 \leq 2 \min_{0 \leq \varphi \leq \pi} u(|\hat{s}|e^{i\phi}) - 2 \min_{0 \leq \phi \leq \pi} v(|\hat{s}|e^{i\phi}) - \varepsilon \leq -\varepsilon.$$

Inequality (2.13) follows since the circle  $|w| = |\hat{s}|$  intersects the complement of  $f(\varDelta)$  and thus  $u(|\hat{s}|e^{i\phi}) = 0$  for some  $\phi$  and since  $v(|\hat{s}|e^{i\phi}) \ge 0$  for all  $\phi$ .

However (2.13) is obviously contradictory and thus we must have  $\sup_{\operatorname{Im} w>0} Q(w) \leq 0$ . Letting  $\varepsilon \to 0$  we obtain  $(u^* - v^*)(s) \leq 0$  for all  $s \in [-D_f, 0)$ . This then completes the proof of (2.6) and hence (2.1). The proof that

$$\int_{-\pi}^{\pi} \varPhi(-\log |f(re^{i\theta})|) d\theta \leq \int_{-\pi}^{\pi} \varPhi(-\log |F_p(re^{i\theta})|) d\theta$$

follows the proofs given in [1] and [3]. The only difference is that (52) of [1] is replaced by

$$\int_{-\pi}^{\pi} \log^+{(
ho \,|\, f(re^{i heta})\,|)} d heta\,=\,2\pi \Bigl(\log \,
ho\,-\,\log^+rac{r}{p}\Bigr) +\,\int_{-\pi}^{\pi}\log^+rac{1}{
ho \,|\, f(re^{i heta})\,|} d heta\,\,.$$

This then completes the proof of Theorem 1.

We have the following theorem as an immediate consequence of Theorem 1.

THEOREM 2. Let  $f \in \Sigma(p)$ , then for all  $\lambda$ ,  $-\infty < \lambda < \infty$ , and 0 < r < 1,

(2.14) 
$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta \leq \int_{-\pi}^{\pi} |F_{p}(re^{i\theta})|^{\lambda} d\theta .$$

## 3. Applications of Theorem 2.

THEOREM 3. Let 
$$f \in \Sigma(p)$$
 and  $0 < r < 1$ , then for  $|z| = r$ .  
(3.1)  $F_p(-r) \leq |f(z)| \leq |F_p(r)|$ .

REMARK. Inequality (3.1) was obtained earlier by Libera and the author [4] for the class  $\Lambda^*(p) \subset \Sigma(p)$ .

*Proof.* The right side of (3.1) follows upon taking the  $\lambda$ th root of both sides of (2.14) and letting  $\lambda \rightarrow +\infty$ . To obtain the left side of (3.1) we note that 2.1 gives for  $\lambda > 0$ 

$$\int_{-\pi}^{\pi} \left|rac{1}{f(re^{i heta})}
ight|^{\lambda} d heta \leq \int_{-\pi}^{\pi} \left|rac{1}{F_p(re^{i heta})}
ight|^{\lambda} d heta \; .$$

Taking the  $\lambda$ th root in the last inequality and letting  $\lambda \rightarrow +\infty$  we obtain

$$rac{1}{|f(z)|} \leq \max_{|z|=r} rac{1}{|f(z)|} \leq \max_{|z|=r} rac{1}{|F_p(z)|} = rac{ig(1+rac{r}{p}ig)(1+pr)}{(1-r)^2} = rac{1}{F_p(-r)}\,.$$

The last inequality is equivalent to the left side of (3.1).

Let  $f \in \Sigma(p)$  and  $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$  for |z| < p. It has been proven [4] that if  $f \in \Lambda^*(p) \subset \Sigma(p)$ , then

$$rac{(1-p)^{ ext{\tiny 2}}}{p} \leq |\, a_{ ext{\tiny 1}}\,| \leq rac{(1+p)^{ ext{\tiny 2}}}{p}\,.$$

The inequality  $|a_1| \leq (1+p)^2/p$  can be obtained for the class  $\Sigma(p)$  by considering the case  $\lambda = 2$  in Theorem 2 and letting  $r \to 0$ . However, making use of some results of Kirwan and Schober [3] we can obtain both the upper and lower bounds on  $|a_1|$ .

THEOREM 4. Let  $f \in \Sigma(p)$  and  $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$  for |z| < p, then

(3.2) 
$$\frac{(1-p)^2}{p} \leq |a_1| \leq \frac{(1+p)^2}{p}$$
.

The inequalities are sharp.

*Proof.* It is easily seen that if  $f \in \Sigma(p)$  with  $f'(0) = a_1$ , then we can write  $f(z) = a_1g(z) + 1$  where  $g \in S(p)$ . According to Kirwan and Schober [3],  $g(\Delta)$  contains  $\{w: |w| < p/(1+p)^2\}$  and  $\{w: |w| > p/(1-p)^2\}$ . It follows that  $f(\Delta)$  contains

$$\{w \colon | \, w - 1 \, |$$

and

$$\{w \colon |\, w-1\,| > p\,|\, a_{_1}|/(1-p)^2\}$$
 .

Since  $0 \notin f(\Delta)$  we must have  $1 \ge p |a_1|/(1+p)^2$  and  $1 \le p |a_1|/(1-p)^2$ , which gives (3.2).

The function  $F_p(z) = (1 + z)^2/(1 - z/p)(1 - pz)$  gives equality on the right side of (3.2) and  $f(z) = (1 - z)^2/(1 - z/p)(1 - pz)$  gives equality on the left side of (3.2).

REMARK. Using Theorem 4 and the representation  $f(z) = a_1g(z) + 1$  where  $g \in S(p)$ , estimates  $|a_n|$  similar to those given in [3] may be obtained. Estimates may also be obtained by using Theorem 2 directly.

In [4] sharp estimates on the quantity |f'(z)/f(z)| were obtained for  $f \in \Lambda^*(p)$ . Making use of Theorem 4, we can now extend the results to the class  $\Sigma(p)$ .

THEOREM 5. Let  $f \in \Sigma(p)$  and  $w \in A$ ,  $w \neq p$ , then

$$(3.3) \qquad \frac{1}{(1-|w|^2)} \frac{(1-|a|)^2}{|a|} \le \left| \frac{f'(w)}{f(w)} \right| \le \frac{1}{(1-|w|^2)} \frac{(1+|a|)^2}{|a|}$$

where  $a = (p - w)/(1 - p\bar{w})$ .

Moreover, given  $w \in \Delta$ ,  $w \neq p$ , there exists a function  $f \in \Sigma(p)$ for which equality is obtained on the right side of (3.3) and similarly for the left side of (3.3).

*Proof.* Let  $f \in \Sigma(p)$  and  $w \in A$ ,  $w \neq p$ , and let  $1 = e^{i\theta} z + w$ 

$$g(z)=rac{1}{f(w)}f\Bigl(rac{e^{iz}+w}{1+ar w e^{i heta}z}\Bigr)$$

where  $\theta = \arg (p - w)/(1 - p\overline{w})$ . Obviously g is univalent in  $\Delta$  with g(0) = 1 and letting  $a = (p - w)/(1 - p\overline{w})$  we see that g has a simple pole at z = |a|. Thus  $g \in \Sigma(|a|)$ . Therefore by Theorem 4 we have

$$\frac{(1-|a|)^2}{|a|} \le |g'(0)| \le \frac{(1+|a|)^2}{|a|} \,.$$

A straightforward computation now gives (3.3).

Suppose we are given  $w \in A$ ,  $w \neq p$ . Let  $a = (p - w)/(1 - p\overline{w})$ and  $\theta = \arg a$ . For  $z \in A$ , let

$$f(z) = rac{ig(1+rac{we^{-i heta}}{|a|}ig)(1+|a|we^{-i heta})(1+A(z))^2}{(1-we^{-i heta})^2ig(1-rac{A(z)}{|a|}ig)(1-|a|A(z))}$$

where

$$A(z)=rac{z-w}{e^{i heta}(1-ar w z)}$$
 .

The function f(z) is univalent in  $\Delta$ , different from 0 and f(0) = 1. Moreover, f has a pole at that value of z for which A(z) = |a|. That is, when z = p. Thus  $f \in \Sigma(p)$  and a straightforward computation gives equality on the right side of (3.3).

To obtain sharpness on the left side of (3.3) for a given  $w \neq p$ , we set

$$f(z) = rac{ig(1+rac{we^{-i heta}}{|a|}ig)(1+|a|we^{-i heta})(1-A(z))^2}{(1+we^{-i heta})^2ig(1-rac{A(z)}{|a|}ig)(1-|a|A(z))}$$

where a,  $\theta$ , and A(z) have the same meaning as before. Again it is easily seen that  $f \in \Sigma(p)$  and that equality is obtained on the left side of (3.3).

4. The class  $\Sigma(p, q)$ . In this section we extend the previous results to the class  $\Sigma(p, q)$  where the functions now take on the value 0. Here the function playing the role of  $F_p(z)$  is the function

$$G_{(p,q)} = rac{ig(1+rac{z}{q}ig)(1+qz)}{ig(1-rac{z}{p}ig)(1-pz)} \; .$$

It is easily seen that  $G_{(p,q)} \in \Sigma(p,q)$  and maps  $\varDelta$  onto the extended plane slit along the interval

$$[-p(1+q)^2/q(1-p)^2, -p(1-q)^2/q(1+p)^2]$$
 .

THEOREM 6. Let  $\Phi$  be a convex nondecreasing function on  $(-\infty, \infty)$ . Then for all  $f \in \Sigma(p, q)$  and 0 < r < 1,

$$\int_{-\pi}^{\pi} arPsi(\pm \log |f(re^{i heta})|) d heta \leqq \int_{-\pi}^{\pi} arPsi(\pm \log |G_{(p,q)}(re^{i heta})|) d heta$$
 .

*Proof.* We first consider the inequality

(4.1) 
$$\int_{-\pi}^{\pi} \Phi(\log |f(re^{i\theta})|) d\theta \leq \int_{-\pi}^{\pi} \Phi(\log |G_{(p,q)}(re^{i\theta})|) d\theta$$

Let

$$u(w) = egin{bmatrix} -\log \mid f^{-1}(w) \mid \,, \quad w \in f(arDelta) \ 0 \,\,, \qquad ext{otherwise} \end{cases}$$

and

$$v(w) = egin{bmatrix} -\log |G_{(p,q)}^{-1}(w)| \ , & w \in G_{(p,q)}(arDelta) \ 0 \ , & ext{otherwise} \ . \end{cases}$$

Arguing as in Theorem 1, inequality (4.1) will be proven if we can prove that

$$(4.2) (u^* - v^*)(s) \leq 0 , s \in R .$$

For this purpose we let

$$d_f = \inf_{w \, \epsilon \, f(\mathcal{J})} |w| \qquad ext{and} \qquad D_f = \sup_{w \, \epsilon \, f(\mathcal{J})} |w|$$

and

$$R=(-\infty,\,-D_f)\cup \llbracket -D_f,\,-d_f
brace\cup (-d_f,\,0)\cup \llbracket 0,\,+\infty)$$
 .

Case (i).  $s \in [0, +\infty)$ . This case is exactly as in Theorem 1.

Case (ii).  $s \in (-\infty, -D_f)$ . The argument is the same as the corresponding case in Theorem 1.

Case (iii).  $s \in (-d_f, 0)$ . Since  $\{w: |w| < d_f\} \subset f(\Delta)$ , we have that  $u(w) + \log |w - 1|$  is harmonic in  $|w| < d_f$  and  $v(w) + \log |w - 1|$  is subharmonic in  $|w| < d_f$ . (The term  $\log |w - 1|$  is only necessary when 1 < d.) It follows that (u - v) is superharmonic for  $|w| < d_f$  and therefore

$$(u^* - v^*)(s) = \int_{-\pi}^{\pi} (u - v)(|s|e^{i heta})d heta \leq 2\pi(u - v)(0) = 0 \; .$$

Case (iv).  $s[-D_f - d_f]$ . The argument in this case is the same as the argument given in case (iii) of the proof of Theorem 1.

This then proves (4.2) and hence (4.1). The inequality

$$\int_{-\pi}^{\pi} \varPhi(-\log |f(re^{i\theta})|) d\theta \leq \int_{-\pi}^{\pi} \varPhi(-\log |G_{\scriptscriptstyle (p,q)}(re^{i\theta})|) d\theta$$

is obtained as in Theorem 1 except that (52) of [1] is now replaced by

$$egin{aligned} &\int_{-\pi}^{\pi}\log^+{(
ho\left|f(re^{i heta})
ight|)}d heta&=2\piigg[\log
ho+\log^+rac{r}{q}-\log^+rac{r}{p}igg]\ &+\int_{-\pi}^{\pi}\log^+rac{1}{
ho\left|f(re^{i heta})
ight|}d heta$$
 .

We have the following as an immediate consequence of Theorem 6.

THEOREM 7. Let  $f \in \Sigma(p, q)$ , 0 < r < 1,  $-\infty < \lambda < \infty$ , then $\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta \leq \int_{-\pi}^{\pi} |G_{(p,q)}(re^{i\theta})|^{\lambda} d\theta$ .

5. Applications of Theorem 7. Arguing as in Theorem 3 we obtain the following.

THEOREM 8. Let 
$$f \in \Sigma(p, q)$$
, then for  $|z| = r$ .  
 $|G_{(p,q)}(-r)| \leq |f(z)| \leq |G_{(p,q)}(r)|$ .

THEOREM 9. Let  $f \in \Sigma(p, q)$  and  $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ , |z| < p, then

(5.1) 
$$\frac{|p-q|(1-pq)}{pq} \le |a_1| \le \frac{(p+q)(1+pq)}{pq}.$$

Both inequalities are sharp.

*Proof.* Let  $f \in \Sigma(p, q)$  with  $f(z_0) = 0$  where  $|z_0| = q$ . Let  $g(z) = (f(z) - 1)/a_1$ , then  $g \in S(p)$ . We therefore have [3]

$$rac{|z_0|}{\Big(1+rac{|z_0|}{p}ig)(1+p\,|\,z_0|)} \leq |\,g(z_0)\,| \leq rac{|z_0|}{\Big|1-rac{|z_0|}{p}\Big|(1-p\,|\,z_0|)} \,.$$

Since  $g(z_0) = -1/a_1$  and  $|z_0| = q$ , we immediately obtain (5.1).

Equality on the right side of (5.1) is attained by the function  $G_{(p,q)}(z)$  and on the left side by the function

$$f(z) = (1 - z/q)(1 - qz)/(1 - z/p)(1 - pz)$$
.

REMARK. The right side of (5.1) could also be obtained by considering the case  $\lambda = 2$  of Theorem 7 and letting r approach 0.

REMARK. We may obtain estimates on  $|a_n|$ ,  $n \ge 2$ , by either using the case  $\lambda = 1$  of Theorem 7 or by using Theorem 9 and the fact that  $f(z) = a_1g(z) + 1$  where  $g \in S(p)$  and then using the estimate on the coefficients of a function in S(p) [3].

As an application of Theorem 9 we obtain the following analogue of Theorem 5.

THEOREM 10. Let  $f \in \Sigma(p, q)$  with  $f(z_0) = 0$ ,  $|z_0| = q$ , then for

 $w \in \Delta$ ,  $w \neq z_0$ ,  $w \neq p$ ,

(5.2) 
$$\frac{1}{1-|w|^2} \left[ \frac{||a|-|b||(1-|a||b|)}{|a||b|} \right] \leq \left| \frac{f'(w)}{f(w)} \right| \\ \leq \frac{1}{1-|w|^2} \left[ \frac{(|a|+|b|)(1+|a||b|)}{|a||b|} \right]$$

where

$$|a| = \left| rac{p-w}{1-p ar w} 
ight| \quad and \quad |b| = \left| rac{z_{\scriptscriptstyle 0}-w}{1-ar w z_{\scriptscriptstyle 0}} 
ight| \, .$$

The left hand side of (5.2) is sharp and the right side is sharp at least for |w| < q.

*Proof.* Let  $f \in \Sigma(p, q)$  with  $f(z_0) = 0$ ,  $|z_0| = q$ . For  $w \in A$ ,  $w \neq p$ ,  $w \neq z_0$ , let  $a = (p - w)/(1 - p\overline{w})$  and  $\theta = \arg a$ . Let

$$h(oldsymbol{z}) = rac{1}{f(w)} figg[rac{e^{i heta}oldsymbol{z}+w}{1+ar{w}e^{i heta}oldsymbol{z}}igg]$$

The function h is univalent and meromorphic in  $\Delta$  with h(0) = 1. Moreover h has a pole at z = |a| and h(z) = 0 when

$$z = (z_{\scriptscriptstyle 0} - w)/e^{i heta}(1 - ar w z_{\scriptscriptstyle 0}) = b$$
 .

Thus  $h \in \Sigma(|a|, |b|)$ . By Theorem 9 we then have

$$\frac{||a| - |b||(1 - |a||b|)}{|a||b|} \le |h'(0)| \le \frac{(|a| + |b|)(1 + |a||b|)}{|a||b|}$$

which gives (5.2).

With p and q fixed let  $w \neq p$  be such that |w| < q. Let  $a = (p - w)/(1 - p\overline{w})$  and  $\theta = \arg a$ . Choose  $z_0$  with  $|z_0| = q$  such that  $(z_0 - w)/e^{i\theta}(1 - \overline{w}z_0) < 0$ . Such a choice is possible since |w| < q. With this choice of  $z_0$  let  $b = (z_0 - w)/e^{i\theta}(1 - \overline{w}z_0)$  and define

$$f(z) = rac{ig(1+rac{we^{-i heta}}{|a|}ig)(1+|a|w\,e^{-i heta}ig)ig(1+rac{A(z)}{|b|}ig)(1+|b|\,A(z))}{ig(1-rac{we^{-i heta}}{|b|}ig)(1-|b|\,we^{-i heta}ig)ig(1-rac{A(z)}{|a|}ig)(1-|a|\,A(z))}$$

where

$$A(z) = rac{z-w}{e^{i heta}(1-ar w z)} \; .$$

The function f is univalent and meromorphic in  $\Delta$  with f(0) = 1.

Moreover, f has a pole when A(z) = |a|, that is when z = p. f has a zero when A(z) = -|b|. By the choice of  $z_0$ ,

$$A(z_{\scriptscriptstyle 0}) = (z_{\scriptscriptstyle 0} - w_{\scriptscriptstyle 0})/e^{i heta}(1 - ar w z_{\scriptscriptstyle 0}) = - |b|$$
 .

Thus  $f(z_0) = 0$  and  $f \in \Sigma(p, q)$ . A straightforward calculation gives equality on the right side of (5.2).

For equality on the left side of (5.2), let |w| < q,  $w \neq p$  and a and  $\theta$  be as before. Choose  $z_c$  so that  $(z_0 - w)/e^{i\theta}(1 - \bar{w}z_0) > 0$  and set  $b = (z_0 - w)/e^{i\theta}(1 - \bar{w}z_0)$ . With this choice of  $z_0$ , let

$$f(z) = rac{ig(1 + rac{w e^{-i heta}}{|a|}ig)(1 + |a| \, w e^{-i heta}ig)ig(1 - rac{A(z)}{|b|}ig)(1 - |b| \, A(z))}{ig(1 + rac{w e^{-i heta}}{|b|}ig)(1 + |b| \, w e^{-i heta}ig)ig(1 - rac{A(z)}{|a|}ig)(1 - |a| \, A(z))}\,.$$

It is easily seen that  $f \in \Sigma(p, q)$  and that we get equality on the left side of (5.2).

Suppose q < r < p. Let a = (p - r)/(1 - pr) and b = (q + r)/(1 + qr) and let

$$f(z) = rac{ig(1+rac{r}{a}ig)(1+ar)ig(1+rac{A(z)}{b}ig)(1+bA(z))}{ig(1-rac{r}{b}ig)(1-br)}$$

where

$$A(z)=\frac{z-r}{1-rz}.$$

The function f has a pole at z = p and a zero at z = -q. Thus  $f \in \Sigma(p, q)$  and a straightforward computation gives equality on the right side of (5.2) when w = r.

Let p and q be fixed and r > 0. Let a = (p + r)/(1 + pr) and b = (q + r)/(1 + qr) and

$$f(z) = rac{ig(1-rac{r}{a}ig)(1-ar)ig(1-rac{A(z)}{b}ig)(1-bA(z))}{ig(1-rac{r}{b}ig)(1-br)ig(1-rac{A(z)}{a}ig)(1-aA(z))}$$

where

$$A(z)=\frac{z+r}{1+rz}.$$

The function f has a pole at z = p and a zero at z = q. Thus  $f \in \Sigma(p, q)$  and we get equality on the left side of (5.2) when w = -r.

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