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COMPLETIONS OF REGULAR RINGS. II

KENNETH R. GOODEARL

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This paper continues earlier investigations into the structure of completions of a (von Neumann) regular ring R with respect to pseudo-rank functions, and into the connections between the ring-theoretic structure of such completions and the geometric structure of the compact convex set $P(R)$ of all pseudo-rank functions on R . In particular, earlier results on the completion of R with respect to a single $N \in P(R)$ are extended to completions with respect to any nonempty subset $X \subseteq P(R)$. Completions in this generality are proved to be regular and self-injective by reducing to the case of a single pseudo-rank function, using a theorem that the lattice of σ -convex faces of $P(R)$ forms a complete Boolean algebra. Given a completion \bar{R} with respect to some $X \subseteq P(R)$, it is shown that the Boolean algebra of central idempotents of \bar{R} is naturally isomorphic to the lattice of those σ -convex faces of $P(R)$ which are contained in the σ -convex face generated by X . Consequently, conditions on X are obtained which tell when \bar{R} is a direct product of simple rings, and how many simple ring direct factors \bar{R} must have. Also, it is shown that the X -completion of R contains a natural copy of the completion with respect to any subset of X , so in particular the $P(R)$ -completion of R contains copies of all the X -completions of R . The final section investigates the question of when a regular self-injective ring is complete with respect to some family of pseudo-rank functions. It is proved that a regular, right and left self-injective ring R is complete with respect to a family $X \subseteq P(R)$ provided only that the Boolean algebra of central idempotents of R is complete with respect to X .

1. Completions. All rings in this paper are associative with unit, and ring maps are assumed to preserve the unit. This paper is a direct continuation of [7], and the reader should consult [7] for definitions which are not discussed here. A family of pseudo-rank functions on a regular ring R induces a uniform topology on R , and the purpose of this paper is to study the resulting completion of R . We begin by recalling the appropriate topological concepts.

Let S be a nonempty set, and let D be a nonempty family of pseudo-metrics on S . The (uniform) topology induced by D on S has as a subbasis the balls $\{x \in S \mid d(x, y) < \varepsilon\}$, for various $y \in S$, $d \in D$, $\varepsilon > 0$. Thus the basic open neighborhoods of a point $y \in S$ are the sets $\{x \in S \mid d_i(x, y) < \varepsilon \text{ for } i = 1, \dots, n\}$ for various $\varepsilon > 0$ and $d_1, \dots, d_n \in D$. A net in S is a Cauchy net (with respect to D)

provided it is Cauchy with respect to each $d \in D$. The space S is *complete* (with respect to D) if the topology on S is Hausdorff and every Cauchy net in S converges in S .

The *completion* of S (with respect to D) is constructed from the set of all Cauchy nets in S by factoring out an equivalence relation \sim , where $\{x_i\} \sim \{y_j\}$ if and only if $d(x_i, y_j) \rightarrow 0$ for all $d \in D$. Each $d \in D$ extends to a pseudo-metric \bar{d} on the completion \bar{S} , and the family $\{\bar{d} | d \in D\}$ induces a complete Hausdorff uniform topology on \bar{S} . There is a natural map $\phi: S \rightarrow \bar{S}$, where $\phi(x)$ is the equivalence class of the constant net $\{x, x, \dots\}$. This map ϕ is continuous, and $\phi(S)$ is dense in \bar{S} . For $x, y \in S$, $\phi(x) = \phi(y)$ if and only if $d(x, y) = 0$ for all $d \in D$.

Now consider another space T topologized by a family E of pseudo-metrics. A function $f: S \rightarrow T$ is *uniformly continuous* (with respect to D and E) provided that for any $\varepsilon > 0$ and any $e \in E$, there exist $\delta > 0$ and $d_1, \dots, d_n \in D$ such that for all $x, y \in S$, $\max \{d_i(x, y)\} < \delta$ implies $e(f(x), f(y)) < \varepsilon$. Any such f extends uniquely to a continuous map \bar{f} from the completion \bar{S} to the completion \bar{T} , and \bar{f} is uniformly continuous.

DEFINITION. Let R be a regular ring, and let X be a nonempty subset of $P(R)$. Each $N \in X$ induces a pseudo-metric δ_N on R , where $\delta_N(x, y) = N(x - y)$ [19, pp. 231, 232]. The family $\{\delta_N | N \in X\}$ then induces a uniform topology on R , which we call the *X-topology*.

In general, the *X-topology* has a basis of open sets of the form $\{x \in R | N_i(x - y) < \varepsilon \text{ for } i = 1, \dots, k\}$ for various $y \in R$, $\varepsilon > 0$, and $N_1, \dots, N_k \in X$. However, if X is convex, then the *X-topology* has a basis of open sets of the form $\{x \in R | N(x - y) < \varepsilon\}$. Namely, given an open set $U \subseteq R$ and an element $y \in U$, we first find $\varepsilon > 0$ and $N_1, \dots, N_k \in X$ such that $y \in V \subseteq U$, where $V = \{x \in R | N_i(x - y) < \varepsilon \text{ for } i = 1, \dots, k\}$. Setting $N = (N_1 + \dots + N_k)/k \in X$ and $W = \{x \in R | N(x - y) < \varepsilon/k\}$, we obtain $y \in W \subseteq V \subseteq U$.

DEFINITION. Let R be a regular ring, and let $X \subseteq P(R)$. The *kernel* of X , denoted $\ker(X)$, is the set $\{x \in R | P(x) = 0 \text{ for all } P \in X\}$. If X is empty, then $\ker(X) = R$, while if X is nonempty, then we see from [6, Lemma 5] that $\ker(X)$ is a proper two-sided ideal of R . For nonempty X , note that the *X-topology* on R is Hausdorff if and only if $\ker(X) = 0$.

LEMMA 1.1. *Let R be a regular ring, and let X, Y be nonempty subsets of $P(R)$. Then the following conditions are equivalent:*

(a) *The identity map $(R, Y\text{-topology}) \rightarrow (R, X\text{-topology})$ is (uniformly) continuous.*

(b) For each $P \in X$, the map $P: (R, Y\text{-topology}) \rightarrow [0, 1]$ is (uniformly) continuous.

(c) Given $\varepsilon > 0$ and $P \in X$, there exist $\delta > 0$ and $N_1, \dots, N_k \in Y$ such that for all $x \in R$, $\max \{N_i(x)\} < \delta$ implies $P(x) < \varepsilon$.

Proof. (a) \Leftrightarrow (c): It is clear from the definitions that if the identity map $(R, Y) \rightarrow (R, X)$ is continuous, then (c) holds; and if (c) holds, then the identity map $(R, Y) \rightarrow (R, X)$ is uniformly continuous.

(b) \Leftrightarrow (c): If $P: (R, Y) \rightarrow [0, 1]$ is continuous for all $P \in X$, then (c) clearly holds. Conversely, assume (c) and consider any $P \in X$. Given $\varepsilon > 0$, there exist $\delta > 0$ and $N_1, \dots, N_k \in Y$ as in (c). For any $x, y \in R$, we see that if $\max \{N_i(x - y)\} < \delta$, then $|P(x) - P(y)| \leq P(x - y) < \varepsilon$, using [19, Corollary, p. 231]. Thus $P: (R, Y) \rightarrow [0, 1]$ is uniformly continuous.

DEFINITION. Let R be a regular ring, and let $X, Y \subseteq P(R)$. We say that X is *continuous with respect to* Y , denoted $X \ll Y$, provided condition (c) of Lemma 1.1 is satisfied. In particular, $\emptyset \ll Y$ for any Y , whereas $X \ll \emptyset$ only for $X = \emptyset$. In case $X = \{P\}$, we write $P \ll Y$ in place of $\{P\} \ll Y$, and similarly when $Y = \{Q\}$. Note in general that $X \ll Y$ if and only if $P \ll Y$ for all $P \in X$. Note also that $X \ll Y$ implies $\ker(Y) \leq \ker(X)$.

THEOREM 1.2. Let R be a regular ring, and let $X, Y \subseteq P(R)$. Then the following conditions are equivalent:

- (a) $X \ll Y$.
- (b) X is contained in the σ -convex face generated by Y in $P(R)$.
- (c) X is contained in the σ -convex hull of the face generated by Y in $P(R)$.

Proof. (b) \Leftrightarrow (c) by [7, Theorem 3.9].

(b) \Rightarrow (a): Given $P \in X$, [7, Theorem 3.9] says that $P \ll Q$ for some Q in the σ -convex hull of Y . There is a σ -convex combination $Q = \sum \alpha_k Q_k$ with all $Q_k \in Y$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $Q(x) < \delta$ implies $P(x) < \varepsilon$. Choose a positive integer n such that $\sum_{k=n+1}^{\infty} \alpha_k < \delta/2$. Then whenever $x \in R$ and $\max \{Q_1(x), \dots, Q_n(x)\} < \delta/2$, we have

$$Q(x) = \sum_{k=1}^n \alpha_k Q_k(x) + \sum_{k=n+1}^{\infty} \alpha_k Q_k(x) \leq \sum_{k=1}^n \alpha_k (\delta/2) + \sum_{k=n+1}^{\infty} \alpha_k < \delta,$$

whence $P(x) < \varepsilon$. Thus $P \ll Y$. Since this holds for all $P \in X$, we obtain $X \ll Y$.

(a) \Rightarrow (b): Given $P \in X$, we have $P \ll Y$. Thus there exist real

numbers $\delta_1, \delta_2, \dots > 0$, positive integers $n(1) = 1 < n(2) < \dots$, and $Q_1, Q_2, \dots \in Y$ with the following property: whenever $x \in R$ and $Q_i(x) < \delta_k$ for $i = n(k), \dots, n(k+1) - 1$, then $P(x) < 1/k$. Now set $Q = \sum_{i=1}^{\infty} Q_i/2^i$, which lies in the σ -convex hull of Y . We claim that $P \ll Q$.

Given $\varepsilon > 0$, choose a positive integer $k > 1/\varepsilon$, and set $n = n(k+1) - 1$, $\delta = \delta_k/2^n$. Whenever $x \in R$ and $Q(x) < \delta$, we have

$$Q_i(x) \leq 2^i Q(x) \leq 2^n Q(x) < 2^n \delta = \delta_k$$

for $i = n(k), \dots, n$, whence $P(x) < 1/k < \varepsilon$. Thus $P \ll Q$, hence [7, Theorem 3.9] says that P lies in the σ -convex face generated by Q . Therefore P lies in the σ -convex face generated by Y .

COROLLARY 1.3. *Let R be a regular ring, let X and Y be non-empty subsets of $P(R)$, and assume that X and Y generate the same σ -convex face in $P(R)$. Then the X -topology and the Y -topology on R are identical. Moreover, Cauchy-ness and uniform continuity are the same whether considered relative to X or relative to Y .*

Proof. By Theorem 1.2, $X \ll Y$ and $Y \ll X$, whence Lemma 1.1 shows that the identity map $(R, X\text{-topology}) \rightarrow (R, Y\text{-topology})$ is a homeomorphism. Thus the topologies are identical. The equivalence of Cauchy-ness and uniform continuity relative to X and Y also follows from the relation $X \ll Y \ll X$.

DEFINITION. Let R be a regular ring, and let X be a nonempty subset of $P(R)$. The X -completion of R is the completion of R with respect to the uniform topology induced by X . By the standard properties of pseudo-rank functions [19, p. 232], the ring operations on R and the maps $N \in X$ are all uniformly continuous with respect to X . Thus the X -completion \bar{R} is a ring, the natural map $R \rightarrow \bar{R}$ is a ring map, and each $N \in X$ extends uniquely to a continuous map $\bar{N}: \bar{R} \rightarrow [0, 1]$. The pseudo-metrics $\bar{\delta}_N$ on \bar{R} which are part of the completion construction are of course induced by the \bar{N} , i.e., $\bar{\delta}_N(x, y) = \bar{N}(x - y)$ for all $N \in X$ and all $x, y \in \bar{R}$.

Because of the continuity of the ring operations, we obtain a slight simplification in the construction of \bar{R} . Namely, the set C of Cauchy nets in R forms a ring, the subset C_0 of null nets (i.e., nets which converge to zero) forms a two-sided ideal in C , and $\bar{R} = C/C_0$. The kernel of the natural map $\phi: R \rightarrow \bar{R}$ is thus the ideal $\{x \in R \mid N(x) = 0 \text{ for all } N \in X\}$, i.e., $\ker \phi = \ker(X)$.

These properties of \bar{R} are standard consequences of the general theory of completions of uniform spaces. By analogy with the case of a single pseudo-rank function—[11, Theorem 3.7] and [6, Corollary

15]—we should expect \bar{R} to be a regular self-injective ring, and the maps \bar{N} should be pseudo-rank functions on \bar{R} . While these properties do hold, the only one we are able to prove directly is that each \bar{N} is a pseudo-rank function on \bar{R} . It is possible to prove self-injectivity in a fairly straightforward manner once it is established that \bar{R} is regular, but regularity seems impossible to prove directly, mainly because the proofs in the case of a single pseudo-rank function depend so heavily on the use of sequences that they do not generalize to nets.

DEFINITION. Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let \bar{R} denote the X -completion of R . Each $N \in X$ extends uniquely to a continuous map $\bar{N}: \bar{R} \rightarrow [0, 1]$. For now, we refer to \bar{N} as the *natural extension of N to \bar{R}* . Once we have proved that \bar{R} is regular and that \bar{N} is a pseudo-rank function on \bar{R} , we shall refer to \bar{N} as the *natural extension of N to $P(\bar{R})$* . For all $x, y \in R$, we have $N(xy) \leq N(x), N(y)$ by definition and $N(x + y) \leq N(x) + N(y)$ by [19, Corollary, p. 231]. By continuity, we obtain $\bar{N}(xy) \leq \bar{N}(x), \bar{N}(y)$ and $\bar{N}(x + y) \leq \bar{N}(x) + \bar{N}(y)$ for all $x, y \in \bar{R}$.

LEMMA 1.4. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let \bar{R} denote the X -completion of R . Any idempotent $e \in \bar{R}$ can be obtained as the limit of a net of idempotents from R .*

Proof. Let $\phi: R \rightarrow \bar{R}$ be the natural map, and for each $N \in X$ let \bar{N} denote the natural extension of N to \bar{R} . Now e has basic open neighborhoods of the form $B = \{x \in \bar{R} \mid \bar{N}_i(x - e) < \varepsilon \text{ for } i = 1, \dots, k\}$, for suitable $\varepsilon > 0$ and $N_1, \dots, N_k \in X$. We must show that for any such B , there exists an idempotent $f \in R$ with $\phi f \in B$.

There exists a net $\{a_j\} \subseteq R$ such that $\phi a_j \rightarrow e$, and of course $\phi(a_j^2) \rightarrow e^2 = e$ as well. Thus there is some $a = a_j \in R$ such that $\bar{N}_i(\phi a - e) < \varepsilon/3$ and $\bar{N}_i(\phi(a^2) - e) < \varepsilon/3$ for all i . Note that

$$N_i(a^2 - a) = \bar{N}_i(\phi(a^2) - \phi a) \leq \bar{N}_i(\phi(a^2) - e) + \bar{N}_i(\phi a - e) < 2\varepsilon/3$$

for all i . According to [11, Lemma 2.3], there exists an idempotent $f \in R$ such that $f - a \in aR(a^2 - a)$. Thus $N_i(f - a) \leq N_i(a^2 - a) < 2\varepsilon/3$ for all i , and consequently

$$\bar{N}_i(\phi f - e) \leq \bar{N}_i(\phi f - \phi a) + \bar{N}_i(\phi a - e) = N_i(f - a) + \bar{N}_i(\phi a - e) < \varepsilon$$

for all i . Therefore $\phi f \in B$.

LEMMA 1.5. *Let R be a regular ring, let X be a nonempty*

subset of $P(R)$, let \bar{R} denote the X -completion of R , and let $\phi: R \rightarrow \bar{R}$ be the natural map. If e, f are orthogonal idempotents in \bar{R} , then there exists a net $\{(e_j, f_j)\} \subseteq R \times R$ such that $(\phi e_j, \phi f_j) \rightarrow (e, f)$ in $\bar{R} \times \bar{R}$, and for all j , e_j and f_j are orthogonal idempotents.

Proof. For each $N \in X$, let \bar{N} denote the natural extension of N to \bar{R} . In $\bar{R} \times \bar{R}$, (e, f) has basic open neighborhoods of the form

$$B = \{(x, y) \in \bar{R} \times \bar{R} \mid \bar{N}_i(x - e), \bar{N}_i(y - f) < \varepsilon \text{ for } i = 1, \dots, k\},$$

for suitable $\varepsilon > 0$ and $N_1, \dots, N_k \in X$. We must show that for any such B , there exist orthogonal idempotents $e', f' \in R$ such that $(\phi e', \phi f') \in B$.

According to Lemma 1.4, there exist nets $\{g_j\}, \{h_j\}$ (which we may arrange to be indexed by the same directed set) of idempotents in R such that $\phi g_j \rightarrow e$ and $\phi h_j \rightarrow f$. In addition, $\phi(g_j h_j) \rightarrow ef = 0$ and $\phi(h_j g_j) \rightarrow fe = 0$. Thus there exist idempotents $g = g_j$ and $h = h_j$ in R such that $\bar{N}_i(\phi g - e) < \varepsilon/2$, $\bar{N}_i(\phi h - f) < \varepsilon/2$, $\bar{N}_i(\phi(gh)) < \varepsilon/2$, and $\bar{N}_i(\phi(hg)) < \varepsilon/2$ for all i . Note that $N_i(gh), N_i(hg) < \varepsilon/2$ for all i . According to [11, Lemma 2.4], there exist orthogonal idempotents $e', f' \in R$ such that $e' - g \in ghR$ and $f' - h \in hgR$. Thus $N_i(e' - g) \leq N_i(gh) < \varepsilon/2$ and likewise $N_i(f' - h) < \varepsilon/2$ for all i . Consequently, $\bar{N}_i(\phi e' - e) \leq N_i(e' - g) + \bar{N}_i(\phi g - e) < \varepsilon$ and likewise $\bar{N}_i(\phi f' - f) < \varepsilon$ for all i . Therefore $(\phi e', \phi f') \in B$.

PROPOSITION 1.6. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let \bar{R} denote the X -completion of R . For each $N \in X$, let \bar{N} denote the natural extension of N to \bar{R} . Then \bar{N} is a pseudo-rank function on \bar{R} .*

Proof. Let $\phi: R \rightarrow \bar{R}$ denote the natural map, and note that $\bar{N}(1) = \bar{N}(\phi(1)) = N(1) = 1$. We have observed above that $\bar{N}(xy) \leq \bar{N}(x), \bar{N}(y)$ for all $x, y \in \bar{R}$. Now consider any orthogonal idempotents $e, f \in \bar{R}$. By Lemma 1.5, there exists a net $\{(e_j, f_j)\} \subseteq R \times R$ such that $\phi e_j \rightarrow e, \phi f_j \rightarrow f$, and e_j, f_j are orthogonal idempotents for each j . Observing that $\bar{N}\phi(e_j + f_j) = \bar{N}\phi(e_j) + \bar{N}\phi(f_j)$ for all j , we conclude that $\bar{N}(e + f) = \bar{N}(e) + \bar{N}(f)$. Thus \bar{N} is a pseudo-rank function on \bar{R} .

In order to prove that the X -completion \bar{R} of a regular ring R is a regular self-injective ring, we must use the following circuitous procedure. The first step, which we develop in the next section, is to prove that the lattice of σ -convex faces of $P(R)$ is a complete Boolean algebra. Using this, we reduce the problem to the case when the $N \in X$ are facially independent. In this case, we prove that \bar{R} is isomorphic to the direct product of the N -completions of

R , from which the required properties of \bar{R} are immediate.

PROPOSITION 1.7. *Let R be a regular ring, let X and Y be nonempty subsets of $P(R)$, and assume that X and Y generate the same σ -convex face in $P(R)$. Then $\ker(X) = \ker(Y)$ and the X -completion of R coincides with the Y -completion.*

Proof. By Theorem 1.2, $X \ll Y$ and $Y \ll X$, hence we see that $\ker(X) = \ker(Y)$. In addition, Corollary 1.3 shows that the X -topology and the Y -topology on R are the same, and that nets in R are Cauchy (null) with respect to X exactly when they are Cauchy (null) with respect to Y . Thus the two completions of R , constructed as the ring of Cauchy nets modulo the ideal of null nets, are identical.

THEOREM 1.8. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let \bar{R} denote the X -completion of R . For each $N \in X$, let \bar{R}_N denote the N -completion of R . If X is a facially independent subset of $P(R)$, then $\bar{R} \cong \prod_{N \in X} \bar{R}_N$.*

Proof. For each $N \in X$, let $\phi_N: R \rightarrow \bar{R}_N$ be the natural map, and let \bar{N} be the natural extension of N to $P(\bar{R}_N)$. Set $S = \prod_{N \in X} \bar{R}_N$, and for each $N \in X$ let p_N denote the projection $S \rightarrow \bar{R}_N$. The maps ϕ_N induce a map $\phi: R \rightarrow S$, and we note that $\ker \phi = \bigcap \ker \phi_N = \bigcap \ker(N) = \ker(X)$.

For each $N \in X$, we have a pseudo-rank function $N^* = \bar{N}p_N$ on S , and we note that $N^*\phi = \bar{N}p_N\phi = \bar{N}\phi_N = N$, i.e., N^* is an extension of N to $P(S)$. Setting $X^* = \{N^* | N \in X\}$, we see also that S is complete with respect to X^* . Thus to show that $S \cong \bar{R}$, it suffices to show that $\phi(R)$ is dense in S in the X^* -topology.

Now let $s \in S$, $\varepsilon > 0$, and $N_1, \dots, N_k \in X$. Set $N = (N_1 + \dots + N_k)/k$ in $P(R)$. Inasmuch as the N_i are facially independent, [7, Theorem 4.3] says that the natural map from the N -completion of R into $T = \bar{R}_{N_1} \times \dots \times \bar{R}_{N_k}$ is an isomorphism. We have a natural map $\psi: R \rightarrow T$ (induced by $\phi_{N_1}, \dots, \phi_{N_k}$), and we have a rank function N' on T defined by the rule $N'(x_1, \dots, x_k) = [\bar{N}_1(x_1) + \dots + \bar{N}_k(x_k)]/k$. By virtue of the isomorphism of the N -completion of R onto T , we see that $\psi(R)$ is dense in T in the N' -metric. Applying this information to the element $t = (p_{N_1}(s), \dots, p_{N_k}(s))$ in T , there must exist an element $r \in R$ such that $N'(\psi(r) - t) < \varepsilon/k$. Inasmuch as

$$\begin{aligned} N'(\psi(r) - t) &= N'(\phi_{N_1}(r) - p_{N_1}(s), \dots, \phi_{N_k}(r) - p_{N_k}(s)) \\ &= [\bar{N}_1(\phi_{N_1}(r) - p_{N_1}(s)) + \dots + \bar{N}_k(\phi_{N_k}(r) - p_{N_k}(s))]/k \\ &= [\bar{N}_1 p_{N_1}(\phi(r) - s) + \dots + \bar{N}_k p_{N_k}(\phi(r) - s)]/k \\ &= [N_1^*(\phi(r) - s) + \dots + N_k^*(\phi(r) - s)]/k, \end{aligned}$$

we conclude that $N_i^*(\phi(r) - s) < \varepsilon$ for all $i = 1, \dots, k$.

Therefore $\phi(R)$ is dense in S in the X^* -topology, as desired.

2. σ -Convex faces in $P(R)$. We show in this section that for any regular ring R , the lattice of σ -convex faces of $P(R)$ forms a complete Boolean algebra.

LEMMA 2.1. *Let R be a regular ring, and let $\{F_i\}$ be a collection of faces of $P(R)$.*

- (a) *The convex hull of $\bigcup F_i$ is a face of $P(R)$.*
- (b) *The σ -convex hull of $\bigcup F_i$ is a σ -convex face of $P(R)$.*
- (c) *If the F_i are all σ -convex and only finitely many of them are nonempty, then the convex hull of $\bigcup F_i$ is a σ -convex face of $P(R)$.*

Proof. (a) Since $P(R)$ is a Choquet simplex by [7, Corollary 3.6], this follows from [2, Proposition 3].

(b) In view of (a), the σ -convex hull of $\bigcup F_i$ is also the σ -convex hull of a face of $P(R)$. By Theorem 1.2, this is a σ -convex face of $P(R)$.

(c) We may assume that there are only finitely many F_i , say F_1, \dots, F_n . Let F denote the convex hull of $\bigcup F_i$, which is a face of $P(R)$ by (a).

Now consider any σ -convex combination $N = \sum \alpha_k P_k$ where all $P_k \in F$. For each k , there is a convex combination $P_k = \beta_{k1} P_{k1} + \dots + \beta_{kn} P_{kn}$ with each $P_{ki} \in F_i$. Set $\gamma_i = \sum_k \alpha_k \beta_{ki} \geq 0$ for each $i = 1, \dots, n$, and note that $\gamma_1 + \dots + \gamma_n = 1$. After renumbering, we may assume that $\gamma_1, \dots, \gamma_r > 0$ and $\gamma_{r+1}, \dots, \gamma_n = 0$, for some $1 < r \leq n$. For each $i = 1, \dots, r$, set $N_i = \sum_k (\alpha_k \beta_{ki} / \gamma_i) P_{ki}$, which lies in F_i because F_i is σ -convex. Then $N = \gamma_1 N_1 + \dots + \gamma_r N_r$ is a convex combination of the N_i , whence $N \in F$.

Therefore F is σ -convex.

DEFINITION. As in [7, 8], we use $B(R)$ to denote the Boolean algebra of central idempotents in a ring R . In case R is regular and right (or left) self-injective, $B(R)$ is complete [8, Proposition 4.1]: for $\{e_i\} \subseteq B(R)$, $\bigwedge e_i$ is the central idempotent which generates the ideal $\bigcap e_i R$.

LEMMA 2.2. *Let R be a regular ring, let $N \in P(R)$, and let $E \subseteq B(R)$. If $e_0 R \cap \ker(N) = 0$ for some $e_0 \in E$, then there exists a countable sequence $\{e_1, e_2, \dots\} \subseteq E$ such that $\bigcap_{e \in E} eR = \bigcap_{n=1}^{\infty} e_n R$.*

Proof. Replacing E by $\{e_0 e \mid e \in E\}$, we may assume that $eR \cap$

$\ker(N) = 0$ for all $e \in E$. Thus we may transfer the problem to $R/\ker(N)$, i.e., we may assume, without loss of generality, that $\ker(N) = 0$. Now N is a rank function on R , from which it follows that R does not contain any uncountable direct sums of nonzero right ideals.

Set $F = \{1 - e \mid e \in E\}$ and $X = \{xR \mid x \in \bigcup_{f \in F} fR\}$. Given any nonzero $y \in FR = \sum_{f \in F} fR$, we must have $yf \neq 0$ for some $f \in F$, whence yfR is a nonzero member of X which is contained in yR . Thus every nonzero submodule of $(FR)_R$ contains a nonzero member of X , hence $(FR)_R$ must have an essential submodule which is a direct sum of members of X . Inasmuch as R contains no uncountable direct sums of nonzero right ideals, this direct sum must be countable, hence we obtain an independent sequence $\{x_1R, x_2R, \dots\} \subseteq X$ such that $\bigoplus x_nR$ is an essential submodule of $(FR)_R$.

Since R is a right nonsingular ring, the left annihilator of $\bigoplus x_nR$ must coincide with the left annihilator of FR . For each n , $x_nR \leq (1 - e_n)R$ for some $e_n \in E$. Consequently, we see that $\bigcap_{n=1}^{\infty} e_nR$ is contained in the left annihilator of FR , i.e., $\bigcap_{n=1}^{\infty} e_nR \subseteq \bigcap_{e \in E} eR$. The opposite inclusion is automatic.

PROPOSITION 2.3. *Let R be a regular ring, let $N \in P(R)$, and let \bar{R} denote the N -completion of R . Let $X, Y \subseteq P(R)$ such that $X, Y \ll N$, and for each $P \in X \cup Y$ let \bar{P} be the continuous extension of P to $P(\bar{R})$. Set $\bar{X} = \{\bar{P} \mid P \in X\}$ and $\bar{Y} = \{\bar{P} \mid P \in Y\}$. Then the following conditions are equivalent:*

- (a) $X \ll Y$.
- (b) $\bar{X} \ll \bar{Y}$.
- (c) $\ker(\bar{Y}) \leq \ker(\bar{X})$.

Proof. Let ϕ denote the natural map $R \rightarrow \bar{R}$, and let \bar{N} denote the natural extension of N to $P(\bar{R})$.

(a) \Rightarrow (b): Given $P \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $Q_1, \dots, Q_k \in Y$ such that for all $y \in R$, $\max\{Q_i(y)\} < \delta$ implies $P(y) < \varepsilon/2$. Since $\bar{P}, \bar{Q}_1, \dots, \bar{Q}_k \ll \bar{N}$, there also exists $\delta' > 0$ such that for all $z \in \bar{R}$, $\bar{N}(z) < \delta'$ implies both $\bar{P}(z) < \varepsilon/2$ and $\max\{\bar{Q}_i(z)\} < \delta/2$.

Now consider any $x \in \bar{R}$ for which $\max\{\bar{Q}_i(x)\} < \delta/2$. There is some $y \in R$ for which $\bar{N}(\phi y - x) < \delta'$, whence $\bar{P}(\phi y - x) < \varepsilon/2$ and $\max\{\bar{Q}_i(\phi y - x)\} < \delta/2$. Then $Q_i(y) = \bar{Q}_i(\phi y) \leq \bar{Q}_i(\phi y - x) + \bar{Q}_i(x) < \delta$ for all $i = 1, \dots, k$, whence $P(y) < \varepsilon/2$ and so $\bar{P}(x) \leq \bar{P}(x - \phi y) + \bar{P}(\phi y) = \bar{P}(\phi y - x) + P(y) < \varepsilon$. Thus for all $x \in \bar{R}$, $\max\{\bar{Q}_i(x)\} < \delta/2$ implies $\bar{P}(x) < \varepsilon$.

(b) \Rightarrow (a) and (b) \Rightarrow (c) are clear.

(c) \Rightarrow (b): According to [7, Lemma 3.7], each $\ker(\bar{Q})$ (for $Q \in Y$) is generated by a central idempotent. Since we have a rank func-

tion \bar{N} on \bar{R} , we see from Lemma 2.2 that there exists a countable sequence $\{Q_1, Q_2, \dots\} \subseteq Y$ such that $\ker(\bar{Y}) = \bigcap_{n=1}^{\infty} \ker(\bar{Q}_n)$. Set $\bar{Q} = \sum_{n=1}^{\infty} \bar{Q}_n/2^n$, which lies in the σ -convex hull of \bar{Y} , and note from Theorem 1.2 that $\bar{Q} \ll \bar{Y}$. Inasmuch as each $\bar{Q}_n \ll \bar{N}$, we also see from Theorem 1.2 that $\bar{Q} \ll \bar{N}$.

Now $\ker(\bar{Q}) = \bigcap_{n=1}^{\infty} \ker(\bar{Q}_n) = \ker(\bar{Y}) \leq \ker(\bar{X})$, hence $\ker(\bar{Q}) \leq \ker(\bar{P})$ for all $P \in X$. According to [7, Proposition 3.8], \bar{X} is contained in the σ -convex hull of the face generated by \bar{Q} . Therefore $\bar{X} \ll \bar{Q}$ by Theorem 1.2, whence $\bar{X} \ll \bar{Y}$.

LEMMA 2.4. *Let R be a regular ring, and let $F \subseteq G$ be σ -convex faces of $P(R)$. If $F \neq G$, then there exists $Q \in G$ such that the σ -convex face generated by Q is disjoint from F .*

Proof. Choose some $N \in G - F$, and let H be the intersection of F with the σ -convex face generated by N . We are done if H is empty, hence we may assume that H is nonempty. Let \bar{R} denote the N -completion of R , and let \bar{N} denote the natural extension of N to $P(\bar{R})$. By Theorem 1.2, $H \ll N$, hence each $P \in H$ extends continuously to some $\bar{P} \in P(\bar{R})$. Set $\bar{H} = \{\bar{P} | P \in H\}$.

Inasmuch as N does not lie in the σ -convex face H , we see from Theorem 1.2 that N is not continuous with respect to H . According to Proposition 2.3, it follows that $\ker(\bar{H}) \not\leq \ker(\bar{N})$, whence $\ker(\bar{H}) \neq 0$. Using [7, Lemma 3.7], we thus obtain a nonzero central idempotent $e \in B(\bar{R})$ such that $e\bar{R} = \ker(\bar{H})$.

Since $e \neq 0$, $\bar{N}(e) \neq 0$, hence we can define a pseudo-rank function $\bar{Q} \in P(\bar{R})$ by the rule $\bar{Q}(x) = \bar{N}(ex)/\bar{N}(e)$. Pulling \bar{Q} back to $Q \in P(R)$, we have $Q \leq [1/\bar{N}(e)]N$, whence Q lies in the face generated by N [7, Corollary 3.3]. Thus $Q \in G$.

Now consider any P in the σ -convex face generated by Q , and note that P also lies in the σ -convex face generated by N . By Theorem 1.2, $P \ll Q, N$, hence P extends continuously to some $\bar{P} \in P(\bar{R})$. According to Proposition 2.3, $(1 - e)\bar{R} = \ker(\bar{Q}) \leq \ker(\bar{P})$, hence $\bar{P}(e) = 1$. Since $e\bar{R} = \ker(\bar{H})$, we conclude that $P \notin H$ and so $P \notin F$.

Therefore the σ -convex face generated by Q is disjoint from F .

LEMMA 2.5. *Let R be a regular ring, let F, G be faces in $P(R)$, and let F_1, G_1 be the σ -convex hulls of F, G . If F and G are disjoint, then F_1 and G_1 are disjoint.*

Proof. Suppose there exists $N \in F_1 \cap G_1$. Then there is a σ -convex combination $N = \sum \alpha_k P_k$ with all $P_k \in F$. By renumbering, we may assume that $\alpha_1 > 0$. If $\alpha_1 = 1$, then $P_1 = N \in G_1$. If $\alpha_1 < 1$,

then

$$\alpha_1 P_1 + (1 - \alpha_1) \sum_{k=2}^{\infty} \alpha_k P_k / (1 - \alpha_1) = \sum_{k=1}^{\infty} \alpha_k P_k = N \in G_1.$$

Since G_1 is a face of $P(R)$ by Theorem 1.2, $P_1 \in G_1$ in this case also, hence we obtain a σ -convex combination $P_1 = \sum \beta_k Q_k$ with all $Q_k \in G$. Again, we may assume that $\beta_1 > 0$. Since $\sum \beta_k Q_k = P_1$ lies in the face F , we conclude as above that $Q_1 \in F$. But then $Q_1 \in F \cap G$, which is impossible.

THEOREM 2.6. *Let R be a regular ring, and let \mathcal{F} denote the lattice of σ -convex faces of $P(R)$. Then \mathcal{F} is a complete Boolean algebra. For $\{F_i\} \subseteq \mathcal{F}$, $\bigwedge F_i = \bigcap F_i$ and $\bigvee F_i$ is the σ -convex hull of $\bigcup F_i$. For $F, G \in \mathcal{F}$, $F \vee G$ is the convex hull of $F \cup G$.*

Proof. It is clear that \mathcal{F} is a complete lattice in which arbitrary infima are given by intersections. For $\{F_i\} \subseteq \mathcal{F}$, Lemma 2.1 shows that the σ -convex hull of $\bigcup F_i$ belongs to \mathcal{F} , whence the σ -convex hull of $\bigcup F_i$ equals $\bigvee F_i$. For $F, G \in \mathcal{F}$, Lemma 2.1 shows that the convex hull of $F \cup G$ belongs to \mathcal{F} , whence the convex hull of $F \cup G$ equals $F \vee G$.

Given $F, G, H \in \mathcal{F}$, we automatically have $(F \wedge G) \vee (F \wedge H) \subseteq F \wedge (G \vee H)$. Now consider any $N \in F \wedge (G \vee H)$. Since $N \in G \vee H$, we obtain a convex combination $N = \alpha P + (1 - \alpha)Q$ with $P \in G$, $Q \in H$. If $\alpha = 0$, then $N = Q \in F \wedge H$, while if $\alpha = 1$, then $N = P \in F \wedge G$. If $0 < \alpha < 1$, then since N lies in the face F , we obtain $P, Q \in F$, and consequently $P \in F \wedge G$, $Q \in F \wedge H$. Thus $N \in (F \wedge G) \vee (F \wedge H)$ in any case, whence $F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H)$. Therefore \mathcal{F} is a distributive lattice.

Now let $F \in \mathcal{F}$, and let X denote the set of those $G \in \mathcal{F}$ which are disjoint from F . Given any nonempty chain $\{G_i\} \subseteq X$, we see that $\bigcup G_i$ is a face of $P(R)$ which is disjoint from F . According to Lemma 2.5, the σ -convex hull of $\bigcup G_i$ is also disjoint from F . Thus $\bigvee G_i \in X$, which provides the chain $\{G_i\}$ with an upper bound in X . Now Zorn's Lemma gives us a maximal element $G \in X$.

If $F \vee G \neq P(R)$, then by Lemma 2.4 there is a nonempty $H \in \mathcal{F}$ which is disjoint from $F \vee G$. In particular, H is disjoint from both F and G . Inasmuch as \mathcal{F} is distributive, we obtain $F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H) = \emptyset$ and so $G \vee H \in X$, which contradicts the maximality of G . Thus $F \vee G = P(R)$, whence G is a complement for F in \mathcal{F} .

Therefore \mathcal{F} is a complete, complemented, distributive lattice, i.e., a complete Boolean algebra.

COROLLARY 2.7. *Let R be a regular ring, and let $X \subseteq P(R)$.*

Then there exists a facially independent set $Y \subseteq P(R)$ such that Y and X generate the same σ -convex face in $P(R)$. In particular, any σ -convex face in $P(R)$ can be generated by a facially independent subset of $P(R)$.

Proof. If X is empty, then X itself is facially independent, hence we may assume that X is nonempty. Let \mathcal{F} denote the lattice of σ -convex faces of $P(R)$, which is a complete Boolean algebra by Theorem 2.6. For each $P \in P(R)$, let $F(P)$ denote the σ -convex face generated by P in $P(R)$, and set $\mathcal{F}_0 = \{F(P) \mid P \in P(R)\}$.

Note that every nonempty face in \mathcal{F} contains a (nonempty) face from \mathcal{F}_0 . Since \mathcal{F} is a complete Boolean algebra, we may thus express F as the supremum of some family $\{F(P_i)\}$ of pairwise disjoint members of \mathcal{F}_0 . Since the $F(P_i)$ are pairwise disjoint, the set $Y = \{P_i\}$ is facially independent. Since each P_i generates $F(P_i)$, we see that Y generates $\bigvee F(P_i) = F$. Thus Y and X generate the same σ -convex face in $P(R)$.

The results of Theorem 2.6 and Corollary 2.7 do not hold in general for non- σ -convex faces. That is, the lattice of faces of $P(R)$ may not be a complete Boolean algebra (although it must be a complete distributive lattice), and there may exist faces in $P(R)$ which cannot be generated by facially independent sets. In fact, in the following example we construct a regular ring R such that $P(R)$ cannot be generated (as a face) by facially independent pseudo-rank functions.

Let K be a field, let K_1, K_2, \dots be copies of K , and let R be the K -subalgebra of $\prod K_n$ generated by 1 and $J = \bigoplus K_n$. Note that R is regular and that $R/J \cong K$.

Since $R/J \cong K$, there exists a unique $P_0 \in P(R)$ such that $\ker(P_0) = J$. For $n = 1, 2, \dots$, let e_n denote the identity of K_n . Since $R/(1 - e_n)R \cong K$, there exists a unique $P_n \in P(R)$ such that $\ker(P_n) = (1 - e_n)R$. Given any $P \in P(R)$ and $n = 1, 2, \dots$, we claim that $P(e_n x) = P(e_n)P_n(x)$ for all $x \in R$, which is clear if $P(e_n) = 0$. If $P(e_n) \neq 0$, then the rule $Q(x) = P(e_n x)/P(e_n)$ defines $Q \in P(R)$ such that $\ker(Q) = (1 - e_n)R$. In this case, we obtain $Q = P_n$ by uniqueness of P_n , from which the claim follows.

We now claim that every $P \in P(R)$ is a σ -convex combination of P_0, P_1, P_2, \dots . More specifically, we claim that

$$P = [1 - \sum_{n=1}^{\infty} P(e_n)]P_0 + \sum_{n=1}^{\infty} P(e_n)P_n.$$

If $\sum P(e_n) = 0$, then $P(J) = 0$, whence $P = P_0$ by uniqueness and the claim holds. Now assume that $\sum P(e_n) = \gamma > 0$, and set $Q =$

$\sum_{n=1}^{\infty} [P(e_n)/\gamma]P_n$ in $P(R)$. Given $x \in R$ and $n = 1, 2, \dots$, we have $(e_1x + \dots + e_nx)R \leq xR$ and so $P(e_1x) + \dots + P(e_nx) \leq P(x)$, whence

$$\gamma Q(x) = \sum_{n=1}^{\infty} P(e_n)P_n(x) = \sum_{n=1}^{\infty} P(e_nx) \leq P(x),$$

using the claim above. As a result, $\gamma Q \leq P$, hence [7, Proposition 3.2] says that $P - \gamma Q = \beta Q'$ for some $\beta \geq 0$ and some $Q' \in P(R)$. Note that $\beta = 1 - \gamma = 1 - \sum_{n=1}^{\infty} P(e_n)$. If $\beta = 0$, then $\gamma = 1$ and $P = Q = \sum_{n=1}^{\infty} P(e_n)P_n$. If $\beta > 0$, then $Q'(J) = \beta^{-1}(P - \gamma Q)(J) = 0$, whence $Q' = P_0$ (by uniqueness) and so

$$P = \gamma Q + \beta P_0 = \beta P_0 + \sum_{n=1}^{\infty} P(e_n)P_n,$$

as required.

Since every $P \in P(R)$ is a σ -convex combination of the P_n , every nonempty face of $P(R)$ must contain at least one P_n . As a result, any collection of nonempty pairwise disjoint faces of $P(R)$ must be countable, whence every facially independent subset of $P(R)$ must be countable.

Now consider any facially independent subset $X \subseteq P(R)$. We claim that the face F generated by X is not equal to $P(R)$. Write $X = \{Q_1, Q_2, \dots\}$, repeating some Q_i if necessary in order to get an infinite sequence. For each $i = 1, 2, \dots$, there is a σ -convex combination $Q_i = \sum_{n=0}^{\infty} \alpha_{i,n}P_n$. Inasmuch as $\lim_{n \rightarrow \infty} \alpha_{i,n} = 0$ for each i , we can find positive integers $n(1) < n(2) < \dots$ such that for all $k = 1, 2, \dots$, $\alpha_{1,n(k)}, \alpha_{2,n(k)}, \dots, \alpha_{k,n(k)} < 1/2^{2k}$. Define β_1, β_2, \dots by setting $\beta_{n(k)} = 1/2^k$ for all k and all other $\beta_n = 0$, and set $Q = \sum_{n=1}^{\infty} \beta_n P_n$ in $P(R)$. We shall prove that $Q \notin F$.

If $Q \in F$, then by [1, (1.9)] there are convex combinations $\alpha Q + (1 - \alpha)Q' = \alpha_1 Q_1 + \dots + \alpha_t Q_t$ for some $0 < \alpha < 1$, some $Q' \in P(R)$, and some t . Now choose a positive integer $k \geq t$ such that $2^k > \alpha^{-1}$. Then

$$\beta_{n(k)} = 1/2^k = 2^k/2^{2k} > \alpha^{-1}\alpha_{i,n(k)}$$

for $i = 1, \dots, k$, whence

$$\begin{aligned} Q(e_{n(k)}) &= \beta_{n(k)} = (\alpha_1 + \dots + \alpha_t)\beta_{n(k)} \\ &> \alpha^{-1}(\alpha_1\alpha_{1,n(k)} + \dots + \alpha_t\alpha_{t,n(k)}) \\ &= \alpha^{-1}[\alpha_1 Q_1(e_{n(k)}) + \dots + \alpha_t Q_t(e_{n(k)})] \\ &= \alpha^{-1}[\alpha Q(e_{n(k)}) + (1 - \alpha)Q'(e_{n(k)})] \geq Q(e_{n(k)}), \end{aligned}$$

which is impossible. Thus $Q \notin F$, hence $F \neq P(R)$.

Thus the faces generated by facially independent subsets of $P(R)$ are all proper, so that $P(R)$ cannot be generated (as a face) by facially independent pseudo-rank functions.

Also, if X is a maximal facially independent subset of $P(R)$, then X generates a face F which is proper, yet F intersects every nonempty face of $P(R)$. Thus F has no complement in the lattice of faces of $P(R)$, hence the lattice of faces of $P(R)$ is not a Boolean algebra.

3. Structure of completions.

THEOREM 3.1. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let \bar{R} denote the X -completion of R . For each $N \in X$, let \bar{N} denote the natural extension of N to \bar{R} .*

- (a) \bar{R} is a regular, right and left self-injective ring.
- (b) For each $N \in X$, \bar{N} is a pseudo-rank function on \bar{R} .
- (c) If $\bar{X} = \{\bar{N} | N \in X\}$, then $\ker(\bar{X}) = 0$ and \bar{R} is complete with respect to \bar{X} .

Proof. (a) According to Corollary 2.7, there exists a facially independent set $Y \subseteq P(R)$ such that Y and X generate the same σ -convex face in $P(R)$. Then Proposition 1.7 shows that \bar{R} coincides with the Y -completion of R . For each $N \in Y$, let \bar{R}_N denote the N -completion of R , which by [11, Theorem 3.7] and [6, Corollary 15] is a regular, right and left self-injective ring. According to Theorem 1.8, $\bar{R} \cong \prod_{N \in Y} \bar{R}_N$, whence \bar{R} is regular and right and left self-injective.

(b) Proposition 1.6.

(c) is clear from the completion process.

Our major tool for investigating the structure of an X -completion \bar{R} is Theorem 3.7, which provides a complete description of the Boolean algebra $B(\bar{R})$ of central idempotents of \bar{R} . In order to prove this theorem, we first require generalizations of several of the results of [7].

DEFINITION. Let $\{e_i | i \in I\}$ be a nonempty family of pairwise orthogonal idempotents in a ring R . There is a standard net of idempotents in R formed from $\{e_i\}$ as follows. For index set, we take the family \mathcal{F} of all nonempty finite subsets of I , ordered by inclusion. For each $F \in \mathcal{F}$, we write $e_F = \sum_{i \in F} e_i$, thus obtaining a net $\{e_F\}$ of idempotents indexed by the directed set \mathcal{F} . We abbreviate this net as $\sum e_i$, and refer to it as the *net of partial sums of the e_i* .

LEMMA 3.2. *Let R be a regular ring, let X be a nonempty subset of $P(R)$ such that $\ker(X) = 0$, and assume that R is complete*

with respect to X . Let J be a right ideal of R which is closed in the X -topology, and let $\{e_i\}$ be a nonempty family of orthogonal idempotents in J .

(a) $\sum e_i$ converges to an idempotent $e \in J$.

(b) If $\bigoplus e_i R$ is essential in J , then $eR = J$. If, in addition, J is a two-sided ideal, then e is central in R .

(c) J is generated by an idempotent. If J is a two-sided ideal, then J is generated by a central idempotent.

Proof. (a) Let I be the index set for the e_i , let \mathcal{F} be the family of all nonempty finite subsets of I , and set $e_F = \sum_{i \in F} e_i$ for all $F \in \mathcal{F}$. We claim that the net $\sum e_i = \{e_F\}$ is Cauchy with respect to any $N \in X$.

Whenever $F \subseteq G$ in \mathcal{F} , we have $e_F = e_F e_G$ and so $N(e_F) \leq N(e_G) \leq 1$. Thus the net $\{N(e_F)\}$ of real numbers is increasing and bounded above, hence it must converge. As a result, given any $\varepsilon > 0$ there must exist $F \in \mathcal{F}$ such that $|N(e_G) - N(e_H)| < \varepsilon/2$ whenever $G, H \supseteq F$ in \mathcal{F} . In particular, when $G \supseteq F$ we see that e_F and $e_G - e_F$ are orthogonal idempotents, whence $N(e_G - e_F) = N(e_G) - N(e_F) < \varepsilon/2$. Consequently, $N(e_G - e_H) < \varepsilon$ whenever $G, H \supseteq F$ in \mathcal{F} . Thus the net $\sum e_i$ is indeed Cauchy with respect to N .

By completeness, $\sum e_i$ converges to some $e \in R$, and of course e is an idempotent. Since each e_F lies in the closed set J , we also have $e \in J$.

(b) Given any $i \in I$, we have $e_F e_i = e_i$ for all $F \supseteq \{i\}$ in \mathcal{F} , whence $ee_i = e_i$. Thus $\bigoplus e_i R \subseteq eR \subseteq J$. Since $\bigoplus e_i R$ is essential in J , it follows that eR is essential in J , from which we infer that $eR = J$.

If J is two-sided, then eR is a two-sided ideal in a semiprime ring, whence e must be central.

(c) Choose a maximal independent family $\{x_j R\}$ of principal right ideals contained in J , so that $\bigoplus x_j R$ is essential in J . Also, choose a right ideal K such that $J \oplus K$ is essential in R_R , whence $(\bigoplus x_j R) \oplus K$ is essential as well. Inasmuch as R is regular and right self-injective by Theorem 3.1, we see that for each k ,

$$R_R = E((\bigoplus x_j R) \oplus K) = x_k R \oplus E((\bigoplus_{j \neq k} x_j R) \oplus K).$$

As a result, there exists an idempotent $f_k \in R$ such that $f_k R = x_k R$ and $f_j x_j = 0$ for all $j \neq k$. Thus we obtain orthogonal idempotents f_j such that $\bigoplus f_j R = \bigoplus x_j R$ is essential in J .

According to (a) and (b), $\sum f_j$ converges to an idempotent f such that $fR = J$, and if J is two-sided, then f is central.

LEMMA 3.3. *Let R be a regular ring, let X be a nonempty*

subset of $P(R)$, and let \bar{R} denote the X -completion of R . If $P \in P(R)$ and $P \ll X$, then P extends (uniquely) to a continuous $\bar{P} \in P(\bar{R})$. In addition, $\ker(\bar{P})$ is generated by a central idempotent in \bar{R} .

Proof. By continuity, P extends uniquely to a continuous map $\bar{P}: \bar{R} \rightarrow [0, 1]$. Exactly as in Proposition 1.6, we infer that $\bar{P} \in P(\bar{R})$. Now $\ker(\bar{P})$ is a two-sided ideal of \bar{R} which is topologically closed, hence Lemma 3.2 says that $\ker(\bar{P})$ is generated by a central idempotent.

LEMMA 3.4. *Let R be a regular ring, let X be a nonempty subset of $P(R)$ such that $\ker(X) = 0$, and assume that R is complete with respect to X . Let $P \in P(R)$ such that $P \ll X$.*

(a) *If $x, x_1, x_2, \dots \in R$ such that $x_1R \leq x_2R \leq \dots$ and $\cup x_nR$ is essential in xR , then $P(x) = \sup P(x_n)$.*

(b) *If $y, y_1, y_2, \dots \in R$ such that $y_1R \geq y_2R \geq \dots$ and $\cap y_nR = yR$, then $P(y) = \inf P(y_n)$.*

Proof. (a) Proceeding as in [6, Lemma 12], we construct orthogonal idempotents $e_1, e_2, \dots \in R$ such that $e_1R \oplus \dots \oplus e_nR = x_nR$ for all n . Each $e_n \in xR$, and xR is closed in the X -topology (because it is an annihilator). Thus by Lemma 3.2, $\sum e_n$ converges to an idempotent $e \in R$ such that $eR = xR$. Since P is continuous, we thus obtain

$$P(x) = P(e) = \sum P(e_n) = \sup \{P(e_1) + \dots + P(e_n)\} = \sup P(x_n).$$

(b) Choose idempotents $e_1, e_2, \dots \in R$ such that $(1 - e_n)R = y_nR$ for all n , and note that $Re_1 \leq Re_2 \leq \dots$. Since R is left self-injective by Theorem 3.1, some left ideal of R is an injective hull for $\cup Re_n$. Thus there is an idempotent $e \in R$ such that $\cup Re_n$ is essential in Re . Observing that Re and $\cup Re_n$ have the same right annihilator, we see that $(1 - e)R = yR$. According to (a), $1 - P(y) = P(e) = \sup P(e_n) = \sup \{1 - P(y_n)\}$, whence $P(y) = \inf P(y_n)$.

PROPOSITION 3.5. *Let R be a regular ring, let X be a nonempty subset of $P(R)$ such that $\ker(X) = 0$, and assume that R is complete with respect to X . Let $P, Q \in P(R)$ such that $P, Q \ll X$. If $\ker(Q) \leq \ker(P)$, then $P \ll Q$.*

Proof. If not, then there exist $\varepsilon > 0$ and $x_1, x_2, \dots \in R$ such that for all n , $Q(x_n) < 1/2^n$ but $P(x_n) \geq \varepsilon$. Set $y_{kn}R = x_kR + \dots + x_nR$ for all $n \geq k$. Since R is right self-injective by Theorem 3.1, there exist elements $z_1, z_2, \dots \in R$ such that $\bigcup_{k=1}^{\infty} y_{kn}R$ is essential in z_kR for all k , and there exists $z \in R$ such that $\bigcap_{k=1}^{\infty} z_kR = zR$. Using

Lemma 3.4, we obtain

$$\begin{aligned} Q(z) &\leq Q(z_k) = \sup \{Q(y_{kk}), Q(y_{k,k+1}), \dots\} \\ &\leq \sup \{Q(x_k) + \dots + Q(x_n)\} = \sum_{n=k}^{\infty} Q(x_n) < \sum_{n=k}^{\infty} 1/2^n = 1/2^{k-1} \end{aligned}$$

for all $k = 1, 2, \dots$. Thus $Q(z) = 0$, whence $P(z) = 0$. However, $z_k R \geq y_{kk} R = x_k R$ for all k and so $P(z_k) \geq P(x_k) \geq \varepsilon$ for all k , hence Lemma 3.4 says that $P(z) = \inf P(z_k) \geq \varepsilon$, which is a contradiction. Therefore $P \ll Q$.

COROLLARY 3.6. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let \bar{R} denote the X -completion of R . Let $Y, W \subseteq P(R)$ such that $Y, W \ll X$, and for each $P \in Y \cup W$ let \bar{P} be the continuous extension of P to $P(\bar{R})$. Set $\bar{Y} = \{\bar{P} | P \in Y\}$ and $\bar{W} = \{\bar{P} | P \in W\}$. Then the following conditions are equivalent:*

- (a) $Y \ll W$.
- (b) $\bar{Y} \ll \bar{W}$.
- (c) $\ker(\bar{W}) \leq \ker(\bar{Y})$.

Proof. Let $\phi: R \rightarrow \bar{R}$ be the natural map. For each $N \in X$, let \bar{N} denote the natural extension of N to $P(\bar{R})$, and set $\bar{X} = \{\bar{N} | N \in X\}$

(a) \Rightarrow (b): Given $P \in Y$ and $\varepsilon > 0$, there exist $\delta > 0$ and $Q_1, \dots, Q_k \in W$ such that for all $y \in R$, $\max \{Q_i(y)\} < \delta$ implies $P(y) < \varepsilon/2$. Since $\bar{P}, \bar{Q}_1, \dots, \bar{Q}_k \ll \bar{X}$, there also exist $\delta' > 0$ and $N_1, \dots, N_s \in X$ such that for all $z \in \bar{R}$, $\max \{\bar{N}_j(z)\} < \delta'$ implies both $\bar{P}(z) < \varepsilon/2$ and $\max \{\bar{Q}_i(z)\} < \delta/2$.

Now consider any $x \in \bar{R}$ for which $\max \{\bar{Q}_i(x)\} < \delta/2$. There is some $y \in R$ for which $\max \{\bar{N}_j(\phi y - x)\} < \delta'$, whence $\bar{P}(\phi y - x) < \varepsilon/2$ and $\max \{\bar{Q}_i(\phi y - x)\} < \delta/2$. Then $\bar{Q}_i(y) = \bar{Q}_i(\phi y) \leq \bar{Q}_i(\phi y - x) + \bar{Q}_i(x) < \delta$ for all $i = 1, \dots, k$, whence $P(y) < \varepsilon/2$ and so $\bar{P}(x) \leq \bar{P}(x - \phi y) + \bar{P}(\phi y) = \bar{P}(\phi y - x) + P(y) < \varepsilon$. Thus for all $x \in \bar{R}$, $\max \{\bar{Q}_i(x)\} < \delta/2$ implies $\bar{P}(x) < \varepsilon$.

(b) \Rightarrow (a) and (b) \Rightarrow (c) are clear.

(c) \Rightarrow (b): Given any $P \in Y$, Lemma 3.3 gives us a central idempotent $e \in \bar{R}$ such that $(1 - e)\bar{R} = \ker(\bar{P})$. Since $\ker(\bar{W}) \leq \ker(\bar{P})$, we thus obtain $\bigcap_{Q \in W} e[\ker(\bar{Q})] = 0$. Lemma 3.3 also shows that each of the ideals $e[\ker(\bar{Q})]$ is generated by a central idempotent, hence Lemma 2.2 says that there exists a countable sequence $\{Q_1, Q_2, \dots\} \subseteq W$ such that $\bigcap_{n=1}^{\infty} e[\ker(\bar{Q}_n)] = 0$, i.e., $\bigcap_{n=1}^{\infty} \ker(\bar{Q}_n) \leq \ker(\bar{P})$. Set $\bar{Q} = \sum_{n=1}^{\infty} \bar{Q}_n/2^n$, which lies in the σ -convex hull of \bar{W} , and note from Theorem 1.2 that $\bar{Q} \ll \bar{W}$. Inasmuch as each $\bar{Q}_n \ll \bar{X}$ we also see from Theorem 1.2 that $\bar{Q} \ll \bar{X}$. Observing that $\ker(\bar{Q}) \leq \ker(\bar{P})$, we see from Proposition 3.5 that $\bar{P} \ll \bar{Q}$, whence $\bar{P} \ll \bar{W}$. Therefore $\bar{Y} \ll \bar{W}$.

Let K be a convex subset of a real vector space, and let F be a face of K . It is clear from the definitions that a subset of F is a face of F if and only if it is a face of K . Thus the lattice of faces of F is just the lattice of those faces of K which are contained in F .

THEOREM 3.7. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let \bar{R} denote the X -completion of R . Let F be the σ -convex face generated by X in $P(R)$, and let \mathcal{F} be the lattice of σ -convex faces of F . Then $B(\bar{R}) \cong \mathcal{F}$.*

Proof. For each $N \in F$, we have $N \ll X$ by Theorem 1.2, and we let \bar{N} denote the continuous extension of N to $P(\bar{R})$.

Given $e \in B(\bar{R})$, set $\theta(e) = \{N \in F \mid \bar{N}(e) = 1\}$, and note that $\theta(e)$ is a σ -convex subset of F . Suppose that we have $0 < \alpha < 1$ and $N_1, N_2 \in F$ with $\alpha N_1 + (1 - \alpha)N_2 \in \theta(e)$. Then $\alpha \bar{N}_1(e) + (1 - \alpha)\bar{N}_2(e) = 1$, whence $\bar{N}_1(e) = \bar{N}_2(e) = 1$ and so $N_1, N_2 \in \theta(e)$. Thus $\theta(e)$ is a face of F , i.e., $\theta(e) \in \mathcal{F}$. Now suppose that $e \leq f$ in $B(\bar{R})$, i.e., $e = ef$. For any $N \in \theta(e)$, we have $1 = \bar{N}(e) \leq \bar{N}(f)$ and so $\bar{N}(f) = 1$, whence $N \in \theta(f)$. Thus $\theta(e) \subseteq \theta(f)$. Therefore we have a monotone map $\theta: B(\bar{R}) \rightarrow \mathcal{F}$.

Given any $G \in \mathcal{F}$, set $\bar{G} = \{\bar{N} \mid N \in G\}$. According to Lemma 3.2, there is some $\mu(G) \in B(\bar{R})$ such that $\ker(\bar{G}) = (1 - \mu(G))\bar{R}$. If $G \subseteq H$ in \mathcal{F} , then $(1 - \mu(H))\bar{R} = \ker(\bar{H}) \leq \ker(\bar{G}) = (1 - \mu(G))\bar{R}$ and so $1 - \mu(H) \leq 1 - \mu(G)$, whence $\mu(G) \leq \mu(H)$. Therefore we have a monotone map $\mu: \mathcal{F} \rightarrow B(\bar{R})$.

Consider any $e \in B(\bar{R})$. Since $\bar{N}(e) = 1$ for all $N \in \theta(e)$, we obtain $\bar{N}(1 - e) = 0$ for all $N \in \theta(e)$, whence $1 - e \in \ker(\bar{\theta(e)}) = (1 - \mu\theta(e))\bar{R}$. Thus $1 - e \leq 1 - \mu\theta(e)$, hence $\mu\theta(e) \leq e$. Set $f = e - \mu\theta(e)$, which is a central idempotent in \bar{R} , and assume that $f \neq 0$. Then $\bar{Q}(f) > 0$ for some $Q \in X$, and we may define $P^* \in P(\bar{R})$ by the rule $P^*(x) = \bar{Q}(fx)/\bar{Q}(f)$. Pulling P^* back to $P \in P(R)$, we see that $P \leq [1/\bar{Q}(f)]Q$, whence [7, Corollary 3.3] shows that $P \in F$. Clearly $P^* \ll \bar{Q}$ and so $P^* \ll \{\bar{N} \mid N \in X\}$, hence $P^* = \bar{P}$. Thus $\bar{P}(x) = \bar{Q}(fx)/\bar{Q}(f)$ for all $x \in \bar{R}$. Since $ef = f$, we obtain $\bar{P}(e) = 1$, whence $P \in \theta(e)$ and so $1 - \mu\theta(e) \in \ker(\bar{P})$. Now $f = f(1 - \mu\theta(e))$ belongs to $\ker(\bar{P})$, which is impossible, because $\bar{P}(f) = 1$. Therefore $f = 0$, i.e., $\mu\theta(e) = e$.

Finally, consider any $G \in \mathcal{F}$. Since $1 - \mu(G) \in \ker(\bar{N})$ for all $N \in G$, we have $\bar{N}(\mu(G)) = 1$ for all $N \in G$, whence $G \subseteq \theta\mu(G)$. Given any $P \in \theta\mu(G)$, we have $\bar{P}(\mu(G)) = 1$, hence $\ker(\bar{G}) = (1 - \mu(\bar{G}))\bar{R} \leq \ker(\bar{P})$. According to Corollary 3.6, $P \ll G$, and consequently $P \in G$, by Theorem 1.2. Therefore $\theta\mu(G) = G$.

Therefore θ and μ are inverse order isomorphisms, hence lattice isomorphisms.

LEMMA 3.8. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, let \bar{R} denote the X -completion of R , and let $e \in B(\bar{R})$. Then $e\bar{R}$ is a simple ring if and only if e is an atom of $B(\bar{R})$.*

Proof. Obviously simplicity of $e\bar{R}$ implies atomicity of e . Conversely, assume that e is an atom, so that $e\bar{R}$ is indecomposable as a ring. Since \bar{R} is a regular, right and left self-injective ring by Theorem 3.1, [18, Theorems 4.7, 5.1] show that $e\bar{R}$ is directly finite, whence [16, Proposition 2.7] shows that $e\bar{R}$ is simple.

The following corollaries of Theorem 3.7 extend [6, Theorems 19, 22, 23] to the case of X -completions.

COROLLARY 3.9. *Let R be a regular ring, and let X be a nonempty subset of $P(R)$. Then the following conditions are equivalent:*

- (a) *The X -completion of R is a simple ring.*
- (b) *X consists of a single extreme point of $P(R)$.*
- (c) *The σ -convex face generated by X is minimal among the nonempty σ -convex faces of $P(R)$.*

Proof. Let \bar{R} denote the X -completion of R , let F denote the σ -convex face generated by X in $P(R)$, and let \mathcal{F} denote the lattice of σ -convex faces of F .

(b) \Rightarrow (c): We have $X = \{N\}$ for some extreme point $N \in P(R)$, hence $F = \{N\}$ as well, from which minimality is clear.

(c) \Rightarrow (a): According to (c), $\mathcal{F} = \{\emptyset, F\}$, hence Theorem 3.7 shows that $B(\bar{R}) = \{0, 1\}$. By Lemma 3.8, \bar{R} is simple.

(a) \Rightarrow (b): Obviously $B(\bar{R}) = \{0, 1\}$, hence $\mathcal{F} = \{\emptyset, F\}$, by Theorem 3.7. Choosing $N \in F$, we see that F is the σ -convex face generated by N . According to Proposition 1.7, \bar{R} equals the N -completion of R , whence [6, Corollary 20] shows that N is an extreme point of $P(R)$. Then $\{N\} \in \mathcal{F}$, whence $F = \{N\}$, and consequently $X = \{N\}$.

COROLLARY 3.10. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let F be the σ -convex face generated by X in $P(R)$. Then the set of simple ring direct factors of the X -completion of R has the same cardinality as the set of extreme points of F . This is also the same cardinality as that of the set of extreme points of the face generated by X .*

Proof. Let \bar{R} denote the X -completion of R . According to Lemma 3.8, the set of simple ring direct factors of \bar{R} has the same cardinality as the set of atoms of $B(\bar{R})$. Using Theorem 3.7, we

can put the atoms of $B(\bar{R})$ in one-to-one correspondence with the minimal (nonempty) σ -convex faces of F . Finally, we see from Corollary 3.9 that the set of minimal σ -convex faces of F has the same cardinality as the set of extreme points of F .

If G is the face generated by X , then clearly any extreme point of G is also an extreme point of F . Inasmuch as F is the σ -convex hull of G (by Theorem 1.2), we conclude that any extreme point of F must lie in G . Therefore F and G have the same extreme points.

COROLLARY 3.11. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let k be a positive integer. Then the following conditions are equivalent:*

- (a) *The X -completion of R is a direct product of k simple rings.*
- (b) *The σ -convex face generated by X can be generated by k distinct extreme points of $P(R)$.*
- (c) *The face generated by X is the convex hull of k distinct extreme points of $P(R)$.*
- (d) *The face generated by X has dimension $k - 1$.*

Proof. Let \bar{R} denote the X -completion of R , let F denote the σ -convex face generated by X in $P(R)$, and let G denote the face generated by X in $P(R)$.

(a) \Rightarrow (b): Clearly $B(\bar{R})$ is an atomic Boolean algebra with k atoms, hence by Theorem 3.7 the same is true of the lattice of σ -convex faces of F . Thus F contains k distinct minimal (nonempty) σ -convex faces F_1, \dots, F_k , and F is generated by $F_1 \cup \dots \cup F_k$. According to Corollary 3.9, each $F_i = \{N_i\}$ for some extreme point $N_i \in P(R)$. Then N_1, \dots, N_k are distinct extreme points of $P(R)$, and F is the σ -convex face generated by $\{N_1, \dots, N_k\}$.

(b) \Rightarrow (a): There exist distinct extreme points $N_1, \dots, N_k \in P(R)$ such that F is the σ -convex face generated by $\{N_1, \dots, N_k\}$. Then the lattice of σ -convex faces of F is atomic with k atoms (namely $\{N_1\}, \dots, \{N_k\}$), hence by Theorem 3.7 the same is true of $B(\bar{R})$. Thus \bar{R} is a direct product of k nonzero indecomposable rings, and by Lemma 3.8 each of these indecomposable rings is simple.

(b) \Rightarrow (c): There exist distinct extreme points $N_1, \dots, N_k \in P(R)$ such that F is the σ -convex face generated by $\{N_1, \dots, N_k\}$. Since each $\{N_i\}$ is a σ -convex face of $P(R)$, we see from Lemma 2.1 that F equals the convex hull of $\{N_1, \dots, N_k\}$. Thus $F = G$, so that G is the convex hull of $\{N_1, \dots, N_k\}$.

(c) \Rightarrow (b): There exist distinct extreme points $N_1, \dots, N_k \in P(R)$ such that G is the convex hull of $\{N_1, \dots, N_k\}$. Since each $\{N_i\}$ is a σ -convex face of $P(R)$, we see from Lemma 2.1 that G is σ -

convex. Thus $F = G$, and F is the σ -convex face generated by $\{N_1, \dots, N_k\}$.

(c) \Rightarrow (d): There exist distinct extreme points $N_1, \dots, N_k \in P(R)$ such that G is the convex hull of $\{N_1, \dots, N_k\}$. Thus the affine span of G equals the affine span of $\{N_1, \dots, N_k\}$, whence $\dim(G) \leq k - 1$. If $\dim(G) < k - 1$, then the N_i must be affinely dependent. After renumbering, we obtain $N_1 = \alpha_2 N_2 + \dots + \alpha_k N_k$ for some real numbers $\alpha_2, \dots, \alpha_k$ whose sum is 1. Renumbering once again, we obtain an index t with $2 \leq t < k$ such that $\alpha_2, \dots, \alpha_t \leq 0$ and $\alpha_{t+1}, \dots, \alpha_k > 0$. Now $N_1 - \alpha_2 N_2 - \dots - \alpha_t N_t = \alpha_{t+1} N_{t+1} + \dots + \alpha_k N_k$, and we note that $1 - \alpha_2 - \dots - \alpha_t = \alpha_{t+1} + \dots + \alpha_k = \beta > 0$. Thus

$$(\alpha_{t+1}/\beta)N_{t+1} + \dots + (\alpha_k/\beta)N_k = \beta^{-1}N_1 - (\alpha_2/\beta)N_2 - \dots - (\alpha_t/\beta)N_t,$$

so that some positive convex combination of N_{t+1}, \dots, N_k equals a convex combination of N_1, \dots, N_t .

Let H be the convex hull of $\{N_1, \dots, N_t\}$, which is a face of $P(R)$ by Lemma 2.1. Since a positive convex combination of N_{t+1}, \dots, N_k lies in this face, we obtain $N_{t+1}, \dots, N_k \in H$, whence $G = H$. Using the implication (c) \Rightarrow (a), we find that \bar{R} is a direct product of t simple rings as well as a direct product of k simple rings. Since $t < k$, this is impossible. Therefore $\dim(G) = k - 1$.

(d) \Rightarrow (c): Let A denote the affine span of G in R^R . Since $\dim(A) = k - 1 < \infty$, A is closed in R^R , hence $A \cap P(R)$ is closed in $P(R)$. Given any $P \in A \cap P(R)$, we have $P = \alpha_1 N_1 + \dots + \alpha_s N_s$ for some $N_1, \dots, N_s \in G$ and some real numbers $\alpha_1, \dots, \alpha_s$ whose sum is 1. After renumbering, we obtain an index $t < s$ such that $\alpha_1, \dots, \alpha_t \leq 0$ and $\alpha_{t+1}, \dots, \alpha_s > 0$. Proceeding as above, we obtain a convex combination $\beta_0 P + \beta_1 N_1 + \dots + \beta_t N_t$ with $\beta_0 > 0$ which equals a convex combination of N_{t+1}, \dots, N_s . Thus $\beta_0 P + \beta_1 N_1 + \dots + \beta_t N_t$ lies in the face G , whence $P \in G$. Therefore $A \cap P(R) = G$, so that G is closed in $P(R)$.

Now G is a compact convex subset of R^R , hence the Krein-Milman Theorem [14, p. 131] says that G is the closure of the convex hull of its extreme points. Suppose G contains $k + 1$ distinct extreme points P_1, \dots, P_{k+1} . If H is the convex hull of these extreme points, then H is a face of $P(R)$ by Lemma 2.1, and the implication (c) \Rightarrow (d) says that $\dim(H) = k$. Since $H \subseteq G$, this is impossible. Thus G must have only $h \leq k$ distinct extreme points P_1, \dots, P_h . Since the convex hull of the finite set $\{P_1, \dots, P_h\}$ is closed, G must be the convex hull of $\{P_1, \dots, P_h\}$. Using the implication (c) \Rightarrow (d) again, we find that $\dim(G) = h - 1$, whence $h = k$.

COROLLARY 3.12. *Let R be a regular ring, let X be a nonempty*

subset of $P(R)$, and let F be the σ -convex face generated by X in $P(R)$. Then the X -completion of R is a direct product of simple rings if and only if F can be generated by some collection of extreme points of $P(R)$.

Proof. Let \bar{R} denote the X -completion of R , and let \mathcal{F} denote the lattice of σ -convex faces of F .

If \bar{R} is a direct product of simple rings, then $B(\bar{R})$ must be atomic, whence Theorem 3.7 shows that \mathcal{F} is atomic. Thus there exist minimal (nonempty) σ -convex faces $F_i \subseteq F$ such that $F = \vee F_i$ in \mathcal{F} . According to Corollary 3.9, each F_i consists of a single extreme point N_i , hence F is the σ -convex face generated by the collection $\{N_i\}$ of extreme points.

Conversely, assume that F is generated by a collection of extreme points of $P(R)$. Then F is the supremum of a collection of atoms in \mathcal{F} , whence \mathcal{F} is atomic. By Theorem 3.7, $B(\bar{R})$ is atomic, hence there exist orthogonal atoms $e_j \in B(\bar{R})$ such that $\vee e_j = 1$. Each $e_j \bar{R}$ is a simple ring by Lemma 3.8. Since $\wedge (1 - e_j) = 0$ generates the ideal $\cap (1 - e_j) \bar{R}$, we see that the ideal $\oplus e_j \bar{R}$ has zero annihilator in \bar{R} . Consequently, we obtain an injective ring map $\phi: \bar{R} \rightarrow \coprod e_j \bar{R}$. As in [5, Theorem 18], we conclude that ϕ is an isomorphism, whence \bar{R} is a direct product of simple rings.

Let R be the simple regular ring of [6, Example C]. According to [6, Lemma 31], $P(R)$ has uncountably many distinct extreme points. If F is the σ -convex face generated by the extreme points of $P(R)$, then Corollaries 3.12 and 3.10 show that the F -completion of R is a direct product of uncountably many simple rings.

4. Decomposition of completions.

PROPOSITION 4.1. *Let R be a regular ring, let X_1, X_2 be non-empty subsets of $P(R)$ such that $X_1 \ll X_2$, and let \bar{R}_i denote the X_i -completion of R .*

(a) *The natural map $R/\ker(X_2) \rightarrow R/\ker(X_1)$ extends uniquely to a continuous map $\phi: \bar{R}_2 \rightarrow \bar{R}_1$. Moreover, ϕ is a ring map.*

(b) *For each $N \in X_1$, let \bar{N} denote the natural extension of N to $P(\bar{R}_1)$ and let N^* denote the continuous extension of N to $P(\bar{R}_2)$. Then $N^* = \bar{N}\phi$.*

(c) *If $X_1^* = \{N^* \mid N \in X_1\}$, then $\ker \phi = \ker(X_1^*)$.*

Proof. (a) The existence and uniqueness of ϕ are standard properties of completions. Since the ring operations in each \bar{R}_i are continuous, ϕ is a ring map.

(b) is exactly analogous to [7, Lemma 2.4].

(c) If $\bar{X}_1 = \{\bar{N} \mid N \in X_1\}$, then $\ker(\bar{X}_1) = 0$ because \bar{R}_1 is the X -completion of R . Thus it follows from (b) that $\ker \phi = \bigcap_{N \in X_1} \ker(\bar{N}\phi) = \ker(X_1^*)$.

DEFINITION. In the situation of Proposition 4.1, we refer to ϕ as the *natural map* from \bar{R}_2 to \bar{R}_1 .

THEOREM 4.2. *Let R be a regular ring, let X_1, X_2 be nonempty subsets of $P(R)$, and let \bar{R}_i denote the X_i -completion of R . If $X_1 \ll X_2$, then the natural map $\phi: \bar{R}_2 \rightarrow \bar{R}_1$ is surjective.*

Proof. For all $N \in X_i$, let \bar{N} denote the natural extension of N to $P(\bar{R}_i)$. For all $N \in X_1$, let N^* denote the continuous extension of N to $P(\bar{R}_2)$, and note from Proposition 4.1 that $N^* = \bar{N}\phi$. Set $\bar{X}_i = \{\bar{N} \mid N \in X_i\}$ and $X_1^* = \{N^* \mid N \in X_1\}$, and note from Proposition 4.1 that $\ker \phi = \ker(X_1^*)$.

According to Lemma 3.2, there is a central idempotent $e \in \bar{R}_2$ such that $(1 - e)\bar{R}_2 = \ker(X_1^*)$, and we note that $e \neq 0$. Set $X'_2 = \{N \in X_2 \mid \bar{N}(e) \neq 0\}$, which is nonempty because $\ker(\bar{X}_2) = 0$. For each $N \in X'_2$, we may define $\bar{N}' \in P(\bar{R}_2)$ by the rule $\bar{N}'(x) = \bar{N}(ex)/\bar{N}(e)$. Since $\bar{N}' \leq [1/\bar{N}(e)]\bar{N}$, we have $\bar{N}' \ll \bar{N}$, hence $\bar{N}' \ll \bar{X}_2$. Setting $\bar{X}'_2 = \{\bar{N}' \mid N \in X'_2\}$, we thus have $\bar{X}'_2 \ll \bar{X}_2$.

Obviously $1 - e \in \ker(\bar{X}'_2)$. Given any $x \in \bar{R}_2$ for which $ex \neq 0$, we have $\bar{N}(ex) \neq 0$ for some $N \in X_2$. For this N , $\bar{N}(e) \neq 0$ as well, whence $N \in X'_2$ and $\bar{N}'(x) \neq 0$. Thus $\ker(\bar{X}'_2) = (1 - e)\bar{R}_2 = \ker(X_1^*)$, hence Corollary 3.6 shows that $\bar{X}'_2 \ll X_1^*$.

Now let ψ_i denote the natural map $R \rightarrow \bar{R}_i$, and note that $\phi\psi_2 = \psi_1$. Given any $x \in \bar{R}_1$, there exists a net $\{x_j\} \subseteq R$ such that $\phi\psi_2(x_j) = \psi_1(x_j) \rightarrow x$ in the \bar{X}_1 -topology. Since $(1 - e)\bar{R}_2 = \ker(X_1^*) = \ker \phi$, we see that $\phi(e\psi_2(x_j)) \rightarrow x$ as well. Now

$$N^*(e\psi_2(x_j) - e\psi_2(x_k)) = \bar{N}(\phi(e\psi_2(x_j)) - \phi(e\psi_2(x_k)))$$

for all j, k and all $N \in X_1$, hence the net $\{e\psi_2(x_j)\} \subseteq \bar{R}_2$ must be Cauchy with respect to X_1^* . Inasmuch as $\bar{X}'_2 \ll X_1^*$, it follows that $\{e\psi_2(x_j)\}$ is also Cauchy with respect to \bar{X}'_2 . Since

$$\bar{N}(e\psi_2(x_j) - e\psi_2(x_k)) = \bar{N}(e)\bar{N}'(e\psi_2(x_j) - e\psi_2(x_k))$$

for all j, k and all $N \in X'_2$, $\{e\psi_2(x_j)\}$ is Cauchy with respect to \bar{N} for all $N \in X'_2$. In addition, we have $\bar{N}(e\psi_2(x_j) - e\psi_2(x_k)) \leq \bar{N}(e) = 0$ for all j, k and all $N \in X_2 - X'_2$, hence $\{e\psi_2(x_j)\}$ is Cauchy with respect to \bar{N} in this case as well. Therefore the net $\{e\psi_2(x_j)\} \subseteq \bar{R}_2$ is Cauchy with respect to \bar{X}_2 .

By completeness, there exists $y \in \bar{R}_2$ such that $e\psi_2(x_j) \rightarrow y$ in the \bar{X}_2 -topology. Since ϕ is continuous, $\phi(e\psi_2(x_j)) \rightarrow \phi(y)$ in the \bar{X}_1 -topology, and consequently $\phi(y) = x$.

Therefore ϕ is surjective.

DEFINITION. Let R be a regular ring, let $\{X_i\}$ be a nonempty family of nonempty subsets of $P(R)$, and for each i let \bar{R}_i denote the X_i -completion of R . If \bar{R} denotes the $(\cup X_i)$ -completion of R , then we have natural maps $\phi_i: \bar{R} \rightarrow \bar{R}_i$ for each i . Together, these maps induce a map $\phi: \bar{R} \rightarrow \prod \bar{R}_i$, which we of course call the *natural map*.

COROLLARY 4.3. *Let R be a regular ring, let $\{X_i\}$ be a nonempty family of nonempty subsets of $P(R)$, and for each i let \bar{R}_i denote the X_i -completion of R . If \bar{R} denotes the $(\cup X_i)$ -completion of R , then the natural map $\phi: \bar{R} \rightarrow \prod \bar{R}_i$ yields an isomorphism of \bar{R} onto a subdirect product of the \bar{R}_i .*

Proof. For each $N \in \cup X_i$, let \bar{N} denote the natural extension of N to $P(\bar{R})$. Set $\bar{X}_i = \{\bar{N} | N \in X_i\}$ for each i , and note from Proposition 4.1 that $\ker(\bar{X}_i)$ equals the kernel of the natural map $\phi_i: \bar{R} \rightarrow \bar{R}_i$. As a result, $\ker \phi = \cap \ker \phi_i = \cap \ker(\bar{X}_i) = \ker(\cup \bar{X}_i) = 0$, hence ϕ is injective. Inasmuch as each ϕ_i is surjective by Theorem 4.2, $\phi(\bar{R})$ is a subdirect product of the \bar{R}_i .

THEOREM 4.4. *Let R be a regular ring, let F be a nonempty σ -convex face of $P(R)$, and let \bar{R} denote the F -completion of R . Let \mathcal{F} denote the lattice of σ -convex faces of F , and for each nonempty $G \in \mathcal{F}$ let \bar{R}_G denote the G -completion of R . Then there is a lattice isomorphism $\mu: \mathcal{F} \rightarrow B(\bar{R})$ such that $\mu(G)\bar{R} \cong \bar{R}_G$ for all nonempty $G \in \mathcal{F}$.*

Proof. Set $\bar{G} = \{\bar{N} | N \in G\}$ for all $G \in \mathcal{F}$. Using Theorem 3.7, we obtain a lattice isomorphism $\mu: \mathcal{F} \rightarrow B(\bar{R})$ such that $(1 - \mu(G))\bar{R} = \ker(\bar{G})$ for all $G \in \mathcal{F}$. Given a nonempty $G \in \mathcal{F}$, the natural map $\phi_G: \bar{R} \rightarrow \bar{R}_G$ is surjective by Theorem 4.2. Since $\ker(\phi_G) = \ker(\bar{G}) = (1 - \mu(G))\bar{R}$ by Proposition 4.1, we conclude that ϕ_G restricts to an isomorphism of $\mu(G)\bar{R}$ onto \bar{R}_G .

Taking account of Proposition 1.7, Theorem 4.4 shows that whenever $X \subseteq Y$ are nonempty subsets of $P(R)$, then the Y -completion of R contains a copy of the X -completion of R . In particular, the $P(R)$ -completion of R is the "largest" completion, since it contains copies of all the X -completions of R .

PROPOSITION 4.5. *Let R be a regular ring, let $\{X_k\}$ be a non-empty family of nonempty subsets of $P(R)$, and for each k let \bar{R}_k denote the X_k -completion of R . Let \bar{R} denote the $(\cup X_k)$ -completion of R , for each $N \in \cup X_k$ let \bar{N} denote the natural extension of N to $P(\bar{R})$, and for each k set $\bar{X}_k = \{\bar{N} | N \in X_k\}$. Then the natural map $\phi: \bar{R} \rightarrow \prod \bar{R}_k$ is an isomorphism if and only if $\ker(\bar{X}_i) + \ker(\bar{X}_j) = \bar{R}$ for all $i \neq j$.*

Proof. Note that the natural map $\phi_i: \bar{R} \rightarrow \bar{R}_i$ is the composition of ϕ with the projection $\prod \bar{R}_k \rightarrow \bar{R}_i$. If ϕ is an isomorphism, then clearly $\ker(\phi_i) + \ker(\phi_j) = \bar{R}$ for all $i \neq j$, whence Proposition 4.1 shows that $\ker(\bar{X}_i) + \ker(\bar{X}_j) = \bar{R}$ for all $i \neq j$.

Conversely, assume that $\ker(\bar{X}_i) + \ker(\bar{X}_j) = \bar{R}$ for all $i \neq j$. Using Lemma 3.2, we obtain central idempotents $e_k \in \bar{R}$ such that $(1 - e_k)\bar{R} = \ker(\bar{X}_k)$. Inasmuch as $(1 - e_i)\bar{R} + (1 - e_j)\bar{R} = \bar{R}$ for all $i \neq j$, we see that the e_k are pairwise orthogonal. Since \bar{R} is the $(\cup X_k)$ -completion of R , we have $\cap \ker(\bar{X}_k) = 0$, so that $\cap (1 - e_k)\bar{R} = 0$. Thus the annihilator of the ideal $\bigoplus e_k \bar{R}$ is zero. Proceeding as in [5, Theorem 18], we see that the natural map $\psi: \bar{R} \rightarrow \prod e_k \bar{R}$ is an isomorphism.

For each k , $\ker(\phi_k) = \ker(\bar{X}_k) = (1 - e_k)\bar{R}$ by Proposition 4.1, hence ϕ_k induces a monomorphism $\theta_k: e_k \bar{R} \rightarrow \bar{R}/\ker(\bar{X}_k) \rightarrow \bar{R}_k$. According to Theorem 4.2, ϕ_k is surjective, whence θ_k is an isomorphism. As a result, these θ_k induce an isomorphism $\theta: \prod e_k \bar{R} \rightarrow \prod \bar{R}_k$. Observing that $\phi = \theta\psi$, we conclude that ϕ is an isomorphism.

THEOREM 4.6. *Let R be a regular ring, let $\{X_k\}$ be a nonempty family of nonempty subsets of $P(R)$, and let \bar{R} denote the $(\cup X_k)$ -completion of R . For each k , let \bar{R}_k denote the X_k -completion of R , and let F_k be the face generated by X_k in $P(R)$. Then the natural map $\phi: \bar{R} \rightarrow \prod \bar{R}_k$ is an isomorphism if and only if the faces F_k are pairwise disjoint.*

Proof. For each $N \in \cup X_k$, let \bar{N} denote the natural extension of N to $P(\bar{R})$. For each k , set $\bar{X}_k = \{\bar{N} | N \in X_k\}$.

First assume that there exists $P \in F_i \cap F_j$ for some $i \neq j$. By [7, Corollary 3.3], there exist Q_i in the convex hull of X_i and Q_j in the convex hull of F_j such that $P \leq \alpha Q_i, \alpha Q_j$ for some $\alpha > 0$. Now $P \ll Q_i \ll X_i \ll \cup X_k$, hence P has a continuous extension $\bar{P} \in P(\bar{R})$. By continuity, $\bar{P} \leq \alpha \bar{Q}_i, \alpha \bar{Q}_j$, whence

$$\ker(\bar{X}_i) + \ker(\bar{X}_j) \leq \ker(\bar{Q}_i) + \ker(\bar{Q}_j) \leq \ker(\bar{P}) < \bar{R}.$$

Then Proposition 4.5 says that ϕ is not an isomorphism.

Conversely, if ϕ is not an isomorphism, then by Proposition 4.5

we must have $\ker(\bar{X}_i) + \ker(\bar{X}_j) \neq \bar{R}$ for some $i \neq j$. By Lemma 3.2, $\ker(\bar{X}_i)$ and $\ker(\bar{X}_j)$ are each generated by a central idempotent, hence there is a central idempotent $e \neq 0$ in \bar{R} such that $(1 - e)\bar{R} = \ker(\bar{X}_i) + \ker(\bar{X}_j)$. Then $\bar{N}(e) \neq 0$ for some $N \in \cup X_k$, hence we may define $\bar{Q} \in P(\bar{R})$ by the rule $\bar{Q}(x) = \bar{N}(ex)/\bar{N}(e)$. Pulling \bar{Q} back to $Q \in P(R)$, we see that $Q \leq [1/\bar{N}(e)]N$, whence $Q \ll N \ll \cup X_k$. Inasmuch as $\ker(\bar{X}_i) + \ker(\bar{X}_j) = (1 - e)\bar{R} \leq \ker(\bar{Q})$, Corollary 3.6 says that $Q \ll X_i, X_j$. According to Theorem 1.2, Q lies in the σ -convex hulls of F_i and F_j . Therefore F_i and F_j are not disjoint, by Lemma 2.5.

COROLLARY 4.7. *Let R be a regular ring, and let $\{F_k\}$ be a nonempty family of nonempty faces of $P(R)$. Let \bar{R} denote the $(\cup F_k)$ -completion of R , and for each k let \bar{R}_k denote the F_k -completion of R . If the F_k are pairwise disjoint, then $\bar{R} \cong \prod \bar{R}_k$.*

Theorem 4.6 and Corollary 4.7 are generalizations of [7, Theorem 4.3 and Corollary 4.4], for if $N \in P(R)$ is a positive σ -convex combination of some $P_k \in P(R)$, then the σ -convex face generated by N coincides with the σ -convex face generated by the P_k .

5. Extending pseudo-rank functions to completions. [7, Theorem 7.4] gives a description of the closure of the face generated by a subset $X \subseteq P(R)$. This theorem is a bit awkward, because it is not constructed in terms of the X -completion of R . A more natural description of closures of faces is given by the following theorem.

THEOREM 5.1. *Let R be a regular ring, let X be a nonempty subset of $P(R)$, and let \bar{R} denote the X -completion of R . Let $\phi: R \rightarrow \bar{R}$ be the natural map, and let $P \in P(R)$. Then P lies in the closure of the face generated by X in $P(R)$ if and only if $P = P'\phi$ for some $P' \in P(\bar{R})$.*

Proof. Since \bar{R} is a regular, right and left self-injective ring by Theorem 3.1, [17, Theorems 4.7, 5.1] show that \bar{R} is directly finite.

Assume first that $P = P'\phi$ for some $P' \in P(\bar{R})$. If $\bar{X} = \{\bar{N} | N \in X\}$ (where \bar{N} denotes the natural extension of N to $P(\bar{R})$), then $\ker(\bar{X}) = 0 \leq \ker(P')$, hence [7, Theorem 7.1] says that P' lies in the closure of the face generated by \bar{X} in $P(\bar{R})$. As a result, we infer that $P = P'\phi$ lies in the closure of the face generated by $\bar{X}\phi = X$.

Conversely, let F denote the face generated by X in $P(R)$, and assume that P lies in the closure of F . By Theorem 1.2, $N \ll X$ for each $N \in F$, hence each such N has a continuous extension $\bar{N} \in P(\bar{R})$ such that $\bar{N}\phi = N$. If $\phi^*: P(\bar{R}) \rightarrow P(R)$ is the map induced by

ϕ , we thus have $F \subseteq \phi^*(P(\bar{R}))$. Now $\phi^*(P(\bar{R}))$ is a continuous image of a compact space and so is compact, hence closed in $P(R)$. Therefore $\phi^*(P(\bar{R}))$ contains the closure of F , whence $P \in \phi^*(P(\bar{R}))$, i.e., $P = P'\phi$ for some $P' \in P(\bar{R})$.

6. Completeness versus self-injectivity. Theorem 3.1 shows that any regular ring R which is complete with respect to a non-empty set X of pseudo-rank functions is right and left self-injective. Since self-injectivity may be viewed as an algebraic completeness property, it is natural to ask about the converse implication: If R is a regular, right and left self-injective ring, must R be complete with respect to some family of pseudo-rank functions? For indecomposable rings, the next theorem shows that the answer is yes. In general, we show that the answer depends on whether or not $B(R)$ is complete, and can be negative.

THEOREM 6.1. *Let R be a regular, right and left self-injective ring which is indecomposable (as a ring). Then there exists a unique rank function N on R , and R is complete in the N -metric.*

Proof. By [18, Theorems 4.7, 5.1], R is directly finite, whence [16, Proposition 2.7] shows that R is a simple ring. In addition, [5, Lemma 5', p. 832] shows that for any $x, y \in R$, either $xR \lesssim yR$ or $yR \lesssim xR$, i.e., R satisfies the "comparability axiom" of [9, p. 812]. As a result, [9, Corollary 3.15] shows that there exists a unique rank function N on R .

According to [17, Corollary to Theorem 1], the lattice $L(R)$ of principal right ideals of R is continuous, i.e., $L(R)$ is a continuous geometry. Since R is indecomposable, $L(R)$ is irreducible [19, Theorem 2.9, p. 76]. As a result, [19, Theorem 17.4, p. 230] says that R is complete in the N -metric.

In general, a regular ring may be complete with respect to some families of pseudo-rank functions but not others. As the following example shows, there exists a regular, right and left self-injective ring R with rank functions N, N' such that R is complete in the N -metric but not in the N' -metric.

Choose fields F_1, F_2, \dots and set $R = \coprod F_n$, which is a regular self-injective ring. If e_n denotes the unit of F_n , then $R/(1 - e_n)R \cong F_n$, hence there exists a unique pseudo-rank function $P_n \in P(R)$ with $\ker(P_n) = (1 - e_n)R$. Setting $N = \sum_{n=1}^{\infty} P_n/2^n$, we obtain a rank function N on R , and it is clear that R is complete in the N -metric. Now choose a maximal ideal M of R which contains $\bigoplus F_n$. There is a unique pseudo-rank function $P \in P(R)$ with $\ker(P) = M$, and

we set $N' = (N + P)/2$ which is a rank function on R . If R is complete in the N' -metric, then we see from Lemma 3.2 then $\sum e_n \rightarrow 1$ in the N' -metric. However, $\sum_{n=1}^{\infty} N'(e_n) = 1/2$, hence this is impossible. Therefore R is not complete in the N' -metric.

We now proceed to show that a regular ring R is complete with respect to a family X of pseudo-rank functions provided only that $B(R)$ is complete with respect to X . As with Theorem 3.1, we must first prove the case of a single pseudo-rank function. In this case, the proof of [19, Theorem 17.4, p. 230] may be applied, once we have shown that the pseudo-rank function involved satisfies a certain countable additivity property, as follows.

DEFINITION. Let R be a regular ring, let $N \in P(R)$, and let J be a right ideal of R . We shall say that N is *countably additive on J* provided that whenever x_1R, x_2R, \dots is a countable sequence of independent principal right ideals contained in J and $\bigoplus x_nR$ is essential in xR for some $x \in J$, then $N(x) = \sum N(x_n)$. If this holds for $J = R$, then we simply say that N is *countably additive*.

LEMMA 6.2. *Let R be a regular ring, let $N \in P(R)$, let J be a right ideal of R , and assume that N is countably additive on J . If $x, x_1, x_2, \dots \in J$ and $\sum x_nR$ is essential in xR , then $N(x) \leq \sum N(x_n)$.*

Proof. We may choose independent principal right ideals $y_1R, y_2R, \dots \leq J$ such that $y_1R \oplus \dots \oplus y_kR = x_1R + \dots + x_kR$ for all k . Since N is countably additive on J , we obtain $N(x) = \sum N(y_n)$. In addition, we have $y_1R \oplus \dots \oplus y_kR \leq x_1R \oplus \dots \oplus x_kR$ for each k and so $N(y_1) + \dots + N(y_k) \leq N(x_1) + \dots + N(x_k)$, by [7, Lemma 6.6]. Thus $N(x) = \sum N(y_n) \leq \sum N(x_n)$.

LEMMA 6.3. *Let R be a regular, right and left self-injective ring with a rank function N . Let e be an idempotent in R such that N is countably additive on $(1 - e)R$. Then $(1 - e)Re$ is complete in the N -metric.*

Proof. Let $L(R)$ denote the lattice of principal right ideals of R , which is continuous by [17, Corollary to Theorem 1].

Let $\{x_n\}$ be a Cauchy sequence in $(1 - e)Re$. By passing to a subsequence, we may assume that $N(x_i - x_j) < 1/2^{k+1}$ whenever $i, j \geq k$. Now define $a_nR, b_kR, cR \in L(R)$ as follows: $a_nR = (e + x_n)R$, $b_kR = E(\sum_{n=k}^{\infty} a_nR)$, $cR = \bigcap_{k=1}^{\infty} b_kR$. Note that $a_kR \leq b_kR$ for all k and that $b_1R \geq b_2R \geq \dots$. Since $x_k \in (1 - e)R$ and $e + x_k \in a_kR$, we see that $a_kR + (1 - e)R = R$, whence $b_kR + (1 - e)R = R$ for all k . Inasmuch as $L(R)$ is lower continuous, we thus obtain

$$cR + (1 - e)R = (\bigcap_{k=1}^{\infty} b_k R) + (1 - e)R = \bigcap_{k=1}^{\infty} [b_k R + (1 - e)R] = R.$$

As a result, there exists an idempotent $f \in R$ such that $fR \leq cR$ and $(1 - f)R = (1 - e)R$.

Since $(1 - f)R = (1 - e)R$, we have $Rf = Re$, hence $f = fe$ and $e = ef$. As a result, we see that the element $x = f - e$ lies in $(1 - e)Re$. Note also that $e + x = f \in cR$. We shall show that $x_n \rightarrow x$.

Whenever $n \geq k$,

$$\begin{aligned} a_n R &= (e + x_n)R = [e + x_k + \sum_{j=k+1}^n (x_j - x_{j-1})]R \\ &\leq (e + x_k)R + \sum_{j=k+1}^{\infty} (x_j - x_{j-1})R \leq a_k R + \sum_{j=k+1}^{\infty} (x_j - x_{j-1})R. \end{aligned}$$

Defining $d_k R = E(\sum_{j=k+1}^{\infty} (x_j - x_{j-1})R) \leq (1 - e)R$, we thus have $a_n R \leq a_k R + d_k R$ for all $n \geq k$. As a result, $\sum_{n=k}^{\infty} a_n R \leq a_k R + d_k R$, whence $b_k R \leq a_k R + d_k R$. We also have $b_k R = a_k R \oplus u_k R$ for some u_k , whence $a_k R \oplus u_k R \leq a_k R + d_k R \leq a_k R \oplus d_k R$. According to [18, Theorems 4.7, 5.1], R is directly finite, hence [8, Corollary 3.9] implies that $u_k R \leq d_k R$. Since $d_k \in (1 - e)R$ and all $x_j - x_{j-1} \in (1 - e)R$, we may use Lemma 6.2 to obtain

$$N(u_k) \leq N(d_k) \leq \sum_{j=k+1}^{\infty} N(x_j - x_{j-1}) < \sum_{j=k+1}^{\infty} 1/2^j = 1/2^k$$

for all k .

Now $f \in cR \leq b_k R = a_k R + u_k R = (e + x_k)R + u_k R$, hence $f = (e + x_k)r + u_k s$ for some $r, s \in R$. Since $x_k \in (1 - e)Re$, $e + x_k$ is idempotent, so that $(e + x_k)f = (e + x_k)r + (e + x_k)u_k s$. We also have $e + x_k \in Re = Rf$, hence $e + x_k = (e + x_k)f = (e + x_k)r + (e + x_k)u_k s = f - u_k s + (e + x_k)u_k s = f + (e + x_k - 1)u_k s$. Consequently,

$$x_k - x = (e + x_k) - (e + x) = e + x_k - f = (e + x_k - 1)u_k s,$$

and so $N(x_k - x) \leq N(u_k) < 1/2^k$.

Therefore $x_n \rightarrow x$.

THEOREM 6.4. *Let R be a regular, right and left self-injective ring with a rank function N . Then N is countably additive if and only if R is complete in the N -metric.*

Proof. First assume that R is complete, and let $x_1 R, x_2 R, \dots$ be independent principal right ideals such that $\bigoplus x_n R$ is essential in some principal right ideal xR . For each k , choose $y_k \in R$ such that $y_k R = x_1 R \oplus \dots \oplus x_k R$. Then $y_1 R \leq y_2 R \leq \dots$ and $\bigcup y_k R$ is essential in xR , whence Lemma 3.4 says that $N(x) = \sup N(y_k)$. Since $N(y_k) = N(x_1) + \dots + N(x_k)$ for all k , we obtain $N(x) = \sum N(x_n)$. Thus N

is countably additive.

Conversely, assume that N is countably additive, and let T denote the ring of all 2×2 matrices over R . By [12, Theorem 1], N induces a rank function P on T such that $P\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = N(x)$ for all $x \in R$. Given $x \in R$, $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}T$ and $\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}T$ are isomorphic principal right ideals of T such that $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}T \oplus \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}T = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}T$, from which we see that $P\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = N(x)/2$. Also, $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}T \cong \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}T$, hence $P\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = N(x)/2$ as well.

The rule $xR \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}T$ defines an isomorphism from the lattice of principal right ideals of R onto the lattice of those principal right ideals of T which are contained in $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}T$. Inasmuch as $P\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = N(x)/2$ for all $x \in R$, we infer from the countable additivity of N that P must be countably additive on $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}T$. As a result, Lemma 6.3 shows that $\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$ is complete in the P -metric, from which we conclude that R is complete in the N -metric.

The result of Theorem 6.4 is used in the proof of [10, Corollaire 2.8], although the reference quoted there only covers the case in which the ring is indecomposable.

DEFINITION. Let R be a regular ring, and let X be a nonempty subset of $P(R)$ such that $\ker(X) = 0$. We shall say that $B(R)$ is *orthogonally complete with respect to X* provided that for any orthogonal family $\{e_i\} \subseteq B(R)$, $\sum e_i$ converges to some $e \in B(R)$. Note that when $\sum e_i \rightarrow e$, we have $e = \vee e_i$. Thus if $B(R)$ is orthogonally complete with respect to X , then $B(R)$ is also complete as a lattice.

For the case of a rank function N , we proceed to show that if R is self-injective and $B(R)$ is orthogonally complete with respect to N , then R is complete in the N -metric. In order to accomplish this, we must consider the Type I and Type II cases separately. (See [8, 15] for the definitions.)

PROPOSITION 6.5. *Let R be a regular, right and left self-injective ring of Type I with a rank function N . If $B(R)$ is orthogonally complete with respect to N , then R is complete in the N -metric.*

Proof. **Case I.** R is abelian.

Let x_1R, x_2R, \dots be an independent family of principal right ideals of R , and let $\bigoplus x_nR$ be essential in some principal right ideal xR . Choose idempotents $e, e_1, e_2, \dots \in R$ such that $eR = xR$ and $e_nR = x_nR$ for all n . Since R is abelian, we have $e, e_1, e_2, \dots \in B(R)$,

the e_n are pairwise orthogonal, and $e = \vee e_n$. Inasmuch as $B(R)$ is orthogonally complete with respect to N , $\sum e_n \rightarrow \vee e_n = e$ in the N -metric, whence $\sum N(x_n) = \sum N(e_n) = N(e) = N(x)$. Therefore N is countably additive, hence Theorem 6.4 says that R is complete in the N -metric.

Case II. R is Type I_n for some n .

There exist $n \times n$ matrix units $e_{ij} \in R$ such that the ring $T = e_{11}Re_{11}$ is abelian. We may define a rank function P on T by the rule $P(x) = N(x)/N(e_{11})$. Inasmuch as the rule $e \mapsto e_{11}e$ defines an isomorphism of $B(R)$ onto $B(T)$, we infer that $B(T)$ must be orthogonally complete with respect to P . As a result, Case I shows that T is complete in the P -metric, hence also in the N -metric. For any i, j , there is an additive isomorphism of T onto $e_{ii}Re_{jj}$ given by the rule $x \mapsto e_{ii}xe_{jj}$, and we observe that $N(x) = N(e_{ii}xe_{jj})$ for all $x \in T$. Thus each $e_{ii}Re_{jj}$ must be complete in the N -metric, whence R is complete in the N -metric.

Case III. General case.

According to [17, Theorems 4.7, 5.1], R is directly finite, hence Type I_f . Consequently, R is isomorphic to a direct product of rings of Type I_n [8, Corollary 6.5], [16, Corollaire 3.5]. Thus there exist orthogonal central idempotents $e_1, e_2, \dots \in B(R)$ such that $\vee e_n = 1$, each $e_n R$ is Type I_n , and $R = \prod e_n R$.

Whenever $e_n \neq 0$, we may define a rank function P_n on $e_n R$ by the rule $P_n(x) = N(x)/N(e_n)$. Since $B(e_n R) = B(R) \cap e_n R$, $B(e_n R)$ is orthogonally complete with respect to P_n , whence Case II shows that $e_n R$ is complete in the P_n -metric and thus in the N -metric.

Given any Cauchy sequence $\{x_n\} \subseteq R$, it follows that for each n , the sequence $\{e_n x_1, e_n x_2, \dots\}$ converges to some $y_n \in e_n R$. Inasmuch as $R = \prod e_n R$, we thus have $y \in R$ such that $e_n y = y_n$ for all n , i.e., $e_n x_k \rightarrow e_n y$ for each n . Also, because $B(R)$ is orthogonally complete, we have $\sum e_n \rightarrow \vee e_n = 1$, whence $\sum_n e_n x_k \rightarrow x_k$ for all k and $\sum_n e_n y \rightarrow y$. Thus $x_k \rightarrow y$.

LEMMA 6.6. *Let R be a regular, right self-injective ring, and let X be a nonempty subset of $P(R)$ such that $\ker(X) = 0$. Let $x, y \in R$ and $g \in B(R)$.*

(a) *If $N(ex) \leq N(ey)$ for all $e \leq g$ in $B(R)$ and all $N \in X$, then $gxR \leq gyR$.*

(b) *If $N(ex) = N(ey)$ for all $e \leq g$ in $B(R)$ and all $N \in X$, then $gxR \cong gyR$.*

Proof. (a) By [16, Théorème 1.1] or [8, Theorem 3.3], there exists $e \in B(R)$ such that $egyR \leq egxR$ and $(1 - e)gxR \leq (1 - e)gyR$. Then $egxR = aR \oplus bR$ with $aR \cong egyR$, and $N(b) = N(egx) - N(egy) \leq 0$

for all $N \in X$. Since $\ker(X) = 0$, we obtain $b = 0$, hence $egxR \cong egyR$. Thus $gxR \lesssim gyR$.

(b) is proved in the same manner.

LEMMA 6.7. *Let R be a regular ring, let X be a nonempty subset of $P(R)$ such that $\ker(X) = 0$, and assume that $B(R)$ is orthogonally complete with respect to X . Let $\phi: B(R) \rightarrow R$ be a continuous map such that $\phi(e + f) = \phi(e) + \phi(f)$ for all orthogonal $e, f \in B(R)$. Then there exists $g \in B(R)$ such that $\phi(e) \geq 0$ for all $e \leq 1 - g$ in $B(R)$ and $\phi(e) < 0$ for all nonzero $e \leq g$ in $B(R)$.*

Proof. Set $A = \{f \in B(R) \mid \phi(e) \geq 0 \text{ for all } e \leq f \text{ in } B(R)\}$, and choose a maximal orthogonal family $\{h_i\} \subseteq A$. By orthogonal completeness, $\sum h_i$ converges to some $h \in B(R)$. Given $e \leq h$ in $B(R)$, we note that $\{eh_i\}$ is an orthogonal family in $B(R)$ such that $\sum eh_i \rightarrow e$. For any finite set F of indices, we have $\phi(\sum_{i \in F} eh_i) = \sum_{i \in F} \phi(eh_i) \geq 0$ since each $h_i \in A$. Thus $\phi(e) \geq 0$, by continuity.

Setting $g = 1 - h \in B(R)$, we now have $\phi(e) \geq 0$ for all $e \leq 1 - g$ in $B(R)$.

Now consider any nonzero $e \leq g$ in $B(R)$. Since e is orthogonal to each h_i , it follows from the maximality of the family $\{h_i\}$ that e does not lie above any nonzero member of A . As a result, each nonzero $f \leq e$ in $B(R)$ must lie above some member of the set $B = \{f \in B(R) \mid \phi(f) < 0\}$. Consequently, there exists an orthogonal family $\{f_j\} \subseteq B$ such that $\vee f_j = e$, and by orthogonal completeness we obtain $\sum f_j \rightarrow e$. Choose a particular index k . Given any finite set F of indices such that $k \in F$, we have $\phi(\sum_{j \in F} f_j) = \sum_{j \in F} \phi(f_j) \leq \phi(f_k)$ since each $f_j \in B$. By continuity, $\phi(e) \leq \phi(f_k) < 0$.

PROPOSITION 6.8. *Let R be a regular, right self-injective ring of Type II with a rank function N . If $B(R)$ is orthogonally complete with respect to N , then N is countably additive.*

Proof. Let x_1R, x_2R, \dots be independent principal right ideals of R , and let $\bigoplus x_nR$ be essential in some principal right ideal xR . For $k = 1, 2, \dots$, we have $x_1R \oplus \dots \oplus x_kR \leq xR$, whence $N(x_1) + \dots + N(x_k) \leq N(x)$. Thus $\sum N(x_n) \leq N(x)$. Suppose that $\sum N(x_n) < N(x)$, and choose a positive integer t such that $\sum N(x_n) < N(x) - (1/t)$.

The rule $\phi(e) = \sum N(ex_n) - N(ex) + N(e)/t$ defines a continuous map $\phi: B(R) \rightarrow R$ such that $\phi(e + f) = \phi(e) + \phi(f)$ for all orthogonal idempotents $e, f \in B(R)$. Applying Lemma 6.7, we obtain $g \in B(R)$ such that $\sum N(ex_n) \geq N(ex) - N(e)/t$ for all $e \leq 1 - g$ in $B(R)$ and $\sum N(ex_n) < N(ex) - N(e)/t$ for all nonzero $e \leq g$ in $B(R)$. Inasmuch as $\sum N(x_n) < N(x) - (1/t)$, we see that $g \neq 0$.

Since R is Type II, it contains no nonzero abelian idempotents, hence [8, Proposition 5.8] says that there is some $y \in R$ for which $t(yR) \cong R$. Note that $t(gyR) \cong gR \neq 0$, whence $gy \neq 0$. Note also that $N(ey) = N(e)/t$ for all $e \in B(R)$. For all nonzero $e \leq g$ in $B(R)$,

$$N(ey) = N(e)/t \leq N(e)/t + \sum N(ex_n) < N(ex),$$

hence $N(ey) \leq N(ex)$ for all $e \leq g$ in $B(R)$. According to Lemma 6.6, $gyR \leq gxR$, hence $gyR \cong zR$ for some nonzero $z \in gxR$. Write $gxR = zR \oplus wR$ for some w , and note that

$$\sum N(ex_n) < N(ex) - N(e)/t = N(ex) - N(ez) = N(ew)$$

for all nonzero $e \leq g$ in $B(R)$.

In particular, $N(ex_1) \leq N(ew)$ for all $e \leq g$ in $B(R)$, hence Lemma 6.6 shows that $gx_1R \cong w_1R$ for some $w_1 \in wR$. Next, $wR = w_1R \oplus u_1R$ for some u_1 , and

$$N(ex_2) \leq \sum N(ex_n) - N(ex_1) \leq N(ew) - N(ew_1) = N(eu_1)$$

for all $e \leq g$ in $B(R)$, hence Lemma 6.6 shows that $gx_2R \cong w_2R$ for some $w_2 \in u_1R$. Continuing in this manner, we obtain an independent sequence $w_1R, w_2R, \dots \leq wR$ such that $gx_nR \cong w_nR$ for all n . Thus $\bigoplus gx_nR \leq wR$. Inasmuch as $\bigoplus gx_nR$ is essential in gxR , it follows that $gxR \leq wR$. But then $N(z) + N(w) = N(gx) \leq N(w)$ and so $N(z) = 0$, which contradicts the fact that $z \neq 0$.

Therefore $\sum N(x_n) = N(x)$, so that N is countably additive.

THEOREM 6.9. *Let R be a regular, right and left self-injective ring with a rank function N . Then R is complete in the N -metric if and only if $B(R)$ is orthogonally complete with respect to N .*

Proof. Obviously completeness of R implies orthogonal completeness of $B(R)$. Conversely, assume that $B(R)$ is orthogonally complete.

According to [18, Theorems 4.7, 5.1], R is directly finite, hence [8, Corollary 7.6] shows that there is some $g \in B(R)$ such that gR is Type I_f and $(1 - g)R$ is Type II_f . If $g \neq 0$, then we may define a rank function P on gR by the rule $P(x) = N(x)/N(g)$. Observing that $B(gR)$ is orthogonally complete with respect to P , we see from Proposition 6.5 that gR is complete in the P -metric, hence also in the N -metric. If $1 - g \neq 0$, then we may define a rank function Q on $(1 - g)R$ by the rule $Q(x) = N(x)/N(1 - g)$. According to Proposition 6.8, Q is countably additive, whence Theorem 6.4 shows that $(1 - g)R$ is complete in the Q -metric, and thus also in the N -metric.

Therefore gR and $(1 - g)R$ are both complete in the N -metric,

whence R is complete in the N -metric.

THEOREM 6.10. *Let R be a regular, right and left self-injective ring, and let X be a nonempty subset of $P(R)$ such that $\ker(X) = 0$. Then the following conditions are equivalent:*

- (a) R is complete with respect to X .
- (b) $B(R)$ is orthogonally complete with respect to X .
- (c) Every ideal of $B(R)$ which is closed in the X -topology is principal.

Proof. (a) \Rightarrow (c): If I is an ideal of $B(R)$ which is closed in the X -topology, then we check that IR is a two-sided ideal of R which is closed in the X -topology. According to Lemma 3.2, $IR = eR$ for some $e \in B(R)$, whence $I = eB(R)$.

(c) \Rightarrow (b): Let $\{e_i | i \in I\}$ be a family of pairwise orthogonal idempotents in $B(R)$. Let \mathcal{F} be the family of nonempty finite subsets of I , and set $e_F = \sum_{i \in F} e_i$ for all $F \in \mathcal{F}$. Set $J = \{e \in B(R) | e \leq e_F \text{ for some } F \in \mathcal{F}\}$, and note that J is an ideal of $B(R)$. If K is the X -closure of J , then K is an ideal of $B(R)$, and (c) says that K is generated by some $f \in B(R)$. In particular, note that $e_f \leq f$ for all $F \in \mathcal{F}$.

Given $N \in X$ and $\varepsilon > 0$, there is some $e \in J$ such that $N(e - f) < \varepsilon$, and $e \leq e_F$ for some $F \in \mathcal{F}$. Whenever $G \supseteq F$ in \mathcal{F} , we have $e \leq e_F \leq e_G \leq f$, hence $f - e_G = (f - e_G)(f - e)$ and so $N(f - e_G) \leq N(f - e) < \varepsilon$. Thus $\sum e_i \rightarrow f$, so that $B(R)$ is orthogonally complete.

(b) \Rightarrow (a): According to Corollary 2.7, there exists a facially independent set $Y = \{N_k\} \subseteq P(R)$ such that Y and X generate the same σ -convex face in $P(R)$. In view of Corollary 1.3 and Proposition 1.7, we see that $B(R)$ is orthogonally complete with respect to Y , and that it suffices to prove that R is complete with respect to Y . Therefore we may assume, without loss of generality, that $X = Y$. For each k , let F_k be the face generated by N_k in $P(R)$.

For each k , $\ker(N_k)$ is a two-sided ideal of R which is closed in the X -topology. Using (b), we see (as in Lemma 3.2) that $\ker(N_k) = (1 - e_k)R$ for some $e_k \in B(R)$. Now N_k restricts to a rank function on $e_k R$, and since $B(R)$ is orthogonally complete with respect to X we see that $B(e_k R)$ is orthogonally complete with respect to N_k . As a result, Theorem 6.9 shows that $e_k R$ is complete in the N_k -metric. If ϕ_k denotes the natural map from R into its N_k -completion \bar{R}_k , we thus have shown that ϕ_k is surjective. Recall that $\ker(\phi_k) = \ker(N_k) = (1 - e_k)R$.

Suppose that $e_j e_k \neq 0$ for some $j \neq k$. Then we may define pseudo-rank functions $N'_j, N'_k \in P(R)$ by the rules $N'_j(x) = N_j(e_j e_k x) / N_j(e_j e_k)$ and $N'_k(x) = N_k(e_j e_k x) / N_k(e_j e_k)$. By [7, Corollary 3.3], $N'_j \in F_j$ and

$N'_k \in F_k$. Set $N = (N'_j + N'_k)/2$, and note that N, N'_j , and N'_k all restrict to rank functions on $e_j e_k R$. Given orthogonal idempotents $\{f_n\} \subseteq B(e_j e_k R)$, (b) says that $\sum f_n$ must converge (in the X -topology) to some $f \in B(R)$, and we note that $f \in B(e_j e_k R)$. In particular, $\sum f_n \rightarrow f$ in the N_j -metric and the N_k -metric, from which we infer the $\sum f_n \rightarrow f$ in the N -metric. Therefore $B(e_j e_k R)$ is orthogonally complete with respect to N , hence Theorem 6.9 says that $e_j e_k R$ is complete in the N -metric. Note that $N'_j, N'_k \ll N$. Inasmuch as N'_j and N'_k both restrict to rank functions on $e_j e_k R$, it now follows from [7, Lemma 4.1] that these restrictions are facially dependent in $P(e_j e_k R)$. Consequently, there exist $P \in P(e_j e_k R)$ and $\alpha > 0$ such that $P \leq \alpha N'_j, \alpha N'_k$ on $e_j e_k R$. Defining P to be zero on $(1 - e_j e_k)R$, we obtain $P \in P(R)$ such that $P \leq \alpha N'_j, \alpha N'_k$. Using [7, Corollary 3.3] again, we find that $P \in F_j \cap F_k$, which is impossible.

Therefore $e_j e_k = 0$ for all $j \neq k$. We thus have pairwise orthogonal central idempotents e_k such that the annihilator of the ideal $\bigoplus e_k R$ is $\bigcap (1 - e_k)R = \bigcap \ker(N_k) = \ker(X) = 0$. As in [5, Theorem 18], it follows that the natural map $R \rightarrow \prod e_k R$ is an isomorphism. Inasmuch as each $\phi_k: R \rightarrow \bar{R}_k$ is surjective with kernel $(1 - e_k)R$, we now see that the map $\phi: R \rightarrow \prod \bar{R}_k$ induced by the ϕ_k must be an isomorphism.

Finally, let \bar{R} denote the X -completion of R , let $\psi: R \rightarrow \bar{R}$ and $\theta: \bar{R} \rightarrow \prod \bar{R}_k$ be the natural maps, and note that $\theta\psi = \phi$. Since the faces F_k are pairwise disjoint, we conclude from Theorem 4.6 that θ is an isomorphism. Therefore the inclusion map $\psi = \theta^{-1}\phi: R \rightarrow \bar{R}$ is an isomorphism, whence R is complete with respect to X .

Returning to our original question, we now see that in order for a regular self-injective ring R to be complete with respect to some nonempty $X \subseteq P(R)$, we need only find such an X such that $B(R)$ is orthogonally complete with respect to X . However, this is not always possible, as the following example shows.

By [4, Theorem 2.2], there exists a nonzero Boolean algebra B with the countable chain condition such that no direct summand of B has a strictly positive finitely additive measure. Considering B as a (commutative) regular ring in the usual way, this says that B contains no uncountable direct sums of nonzero ideals, and that there does not exist a rank function on any direct summand of B .

Now let R be the maximal quotient ring of B , which is a regular self-injective ring. In fact, R is the Boolean completion of B [3, Theorem 5], so that $B(R) = R$. Since B_B is essential in R_B , we see that R does not contain any uncountable direct sums of nonzero ideals (i.e., as a Boolean algebra, R satisfies the countable chain condition). Suppose there is an idempotent $e \in R$ such that there is a rank function N on eR . Then $e \neq 0$, hence there exists a nonzero

idempotent $f \in eR \cap B$. But then N induces a rank function on fB , which cannot happen. Thus there does not exist a rank function on any direct summand of R .

If R is complete with respect to some family of pseudo-rank functions, then using Theorem 4.6 we see that R must be isomorphic to a direct product $\prod R_k$, where each R_k is complete with respect to a rank function N_k . But then there exist rank functions on some direct summands of R , which is false. Therefore R is not complete with respect to any family of pseudo-rank functions.

Returning to the general case, we are left with the following problem: Given a regular, right and left self-injective ring R , when is $B(R)$ orthogonally complete with respect to some family of pseudo-rank functions? Since all pseudo-rank functions on $B(R)$ extend to pseudo-rank functions on R by [7, Corollary 6.10], we need only look for a suitable family of pseudo-rank functions on $B(R)$. This reduces the problem to Boolean algebras. For the case of a single pseudo-rank function, we thus have the following problem: Given a Boolean algebra B , when does there exist a rank function N on B such that B is complete in the N -metric? Obviously B must be complete and satisfy the countable chain condition, but the example above shows that these conditions are not sufficient. Rather complicated necessary and sufficient conditions on B may be found in [13, Theorems 4, 9] and [15, Theorem 4].

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