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## **A NOTE ON EBERLEIN COMPACTS**

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**A new proof is given of the theorem that every continuous image of an Eberlein compact is an Eberlein compact, and a new characterization of Eberlein compact sets is obtained.**

1. Introduction. According to D. Amir and J. Lindenstrauss [1], an *Eberlein compact*, or *EC*, is a space<sup>1</sup> which is homeomorphic to a weakly compact (i.e., compact in the weak topology) subset of a Banach space. The following theorem, which had been conjectured by Lindenstrauss in [6, Problem 5], was proved in [2, Theorem 2.1] by Y. Benyamini, M. E. Rudin and M. Wage.

**THEOREM 1.1.** [2]. *Every continuous image of an EC is again an EC.*

The proof of Theorem 1.1 in [2] uses a Banach space characterization of *EC*'s due to Amir and Lindenstrauss [1, Theorem 1] (see Theorem 1.4 (b) below), as well as a purely topological characterization—based on the former—which was obtained by H. P. Rosenthal in [9, Theorem 3.1] (see Theorem 1.4 (d)). The main purpose of this note is to give a somewhat more direct proof of Theorem 1.1 which depends only on Rosenthal's characterization, and which yields a new topological characterization of *EC*'s—formally weaker than Rosenthal's—which is given in Theorem 1.4 (e).

Before stating Theorem 1.4, we need two definitions.

**DEFINITION 1.2.** A collection  $\mathcal{U}$  of subsets of  $X$  is *point-finite* if each  $x \in X$  is in only finitely many  $U \in \mathcal{U}$ ; it is  *$\sigma$ -point-finite* if  $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$  with each  $\mathcal{U}_n$  point-finite.

**DEFINITION 1.3.** A collection  $\mathcal{U}$  of subsets of  $X$  *separates* (resp.  *$F$ -separates*)  $X$  if, whenever  $x \neq x'$  are in  $X$ , then there is a  $U \in \mathcal{U}$  such that  $x \in U$  and  $x' \notin U$  (resp.  $x \in U$  and  $x' \notin \bar{U}$ ), or vice versa.<sup>2</sup>

In the following theorem,  $c_0(\Gamma)$  denotes the Banach space of real-valued functions vanishing at infinity on the set  $\Gamma$ , and  $\mathcal{T}_w$

<sup>1</sup> All spaces in this paper are Hausdorff.

<sup>2</sup> Many authors use the term " $\mathcal{U}$  separates  $X$ " to denote the preceding concept without the phrase "or vice versa." In this paper, a collection  $\mathcal{U}$  with this stronger property is called *strongly separating* (see (1.6)). It should be remarked that, unlike strongly separating collections, separating collections need not be coverings.

(resp.  $\mathcal{T}_p$ ) the weak topology (resp. topology of pointwise convergence) on  $c_0(\Gamma)$ .

**THEOREM 1.4.** *The following properties of a compact space  $X$  are equivalent.*

- (a)  $X$  is an *EC*.
- (b)  $X$  is homeomorphic to a subset of some  $(c_0(\Gamma), \mathcal{T}_w)$ .
- (c)  $X$  is homeomorphic to a subset of some  $(c_0(\Gamma), \mathcal{T}_p)$ .
- (d)  $X$  has a  $\sigma$ -point-finite, separating collection of open  $F_\sigma$ -subsets.
- (e)  $X$  has a  $\sigma$ -point-finite,  $F$ -separating collection of open subsets.

All of Theorem 1.4, except (d)  $\leftrightarrow$  (e), is known. For details see §2, where we also give a proof of the equivalence of (b) and (d) (via (c)) which is somewhat more direct than Rosenthal's in [9].

The following remarks explain the significance of the requirements in parts (d) and (e) of Theorem 1.4.

(1.5). The  $F_\sigma$  requirement cannot be omitted from 1.4 (d) (equivalently: " $F$ -separating" cannot be weakened to "separating" in 1.4 (e)). In fact, there exists a compact space with a *countable, separating* open cover which is not an *EC* [10, Remark, p. 183] [11, p. 20]. By contrast, every compact space with a *point-finite, separating* collection of open subsets is an *EC* [10, Proposition 9], and a compact space with a *countable,  $F$ -separating* collection of open subsets must even be metrizable (by 1.6 below).

(1.6). Every compact space with a *point-countable, strongly separating* (see footnote 2) open cover is metrizable [8].

(1.7). It is consistent with *ZFC* that " $\sigma$ -point-finite" cannot be weakened to "point-countable" in 1.4 (d) and (e) [2, Example 5.1].

We conclude this introduction with a question. A positive answer would (in view of Theorem 1.4) provide a significant generalization of Theorem 1.1.

**PROBLEM 1.8.** Suppose  $X$  is a normal space which is homeomorphic to a subset of some  $(c_0(\Gamma), \mathcal{T}_p)$ . Must every image of  $X$  under a perfect map be homeomorphic to a subset of some  $(c_0(\Gamma), \mathcal{T}_p)$ ?

The paper is arranged as follows. Section 2 proves parts of Theorem 1.4. Section 3 contains some lemmas which are needed in the proof Theorem 1.1, §4 proves Theorem 1.1, and §5 considers

additional consequences of this proof. Section 6 deals with a modification of  $EC$ 's.

2. **Proof of Theorem 1.4.** As observed in the introduction, the only part of this theorem which is new is the equivalence  $(d) \leftrightarrow (e)$ . The simple and straightforward proofs for  $(c) \rightarrow (b)$  and  $(c) \leftarrow (d)$  given below are known (although the one for  $(c) \leftrightarrow (d)$  does not seem to have appeared in print), and we include them here for the sake of completeness.

$(a) \rightarrow (b)$ . This is a deep result of Amir and Lindenstrauss [1, Theorem 1].

$(b) \rightarrow (a)$ . Clear.

$(b) \rightarrow (c)$ . Clear, since  $\mathcal{T}_w$  is finer than  $\mathcal{T}_p$ .

$(c) \rightarrow (b)$ . Suppose  $X \subset (c_0(\Gamma), \mathcal{T}_p)$  is compact. Then  $X$  is homeomorphic (via the map  $f \rightarrow (\arctan) \circ f$ ) to a *bounded* compact  $Y \subset (c_0(\Gamma), \mathcal{T}_p)$ , and on such a  $Y$  the topologies  $\mathcal{T}_p$  and  $\mathcal{T}_w$  must agree by a theorem of A. Grothendieck [5, Theorem 5] (applied to the one-point compactification of the discrete space  $\Gamma$ ).

$(c) \rightarrow (d)$ . It will suffice to show that  $c_0(\Gamma)$  has a  $\sigma$ -point-finite, separating collection  $\mathcal{U}$  of open  $F_\sigma$ -subsets. For  $r \neq 0$  and  $\gamma \in \Gamma$ , let  $U_{r,\gamma} = \{f \in c_0(\Gamma) : f(\gamma) > r\}$  if  $r > 0$  and let  $U_{r,\gamma} = \{f \in c_0(\Gamma) : f(\gamma) < r\}$  if  $r < 0$ . Let  $\mathcal{U}_r = \{U_{r,\gamma} : \gamma \in \Gamma\}$ , and let  $\mathcal{U} = \bigcup \{\mathcal{U}_r : r \neq 0, r \text{ rational}\}$ . This  $\mathcal{U}$  has the required properties.

$(d) \rightarrow (c)$ . Let  $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$  be a separating collection of open  $F_\sigma$ -subsets of  $X$ , with each  $\mathcal{U}_n$  point-finite and with  $\mathcal{U}_n \cap \mathcal{U}_m = \emptyset$  if  $n \neq m$ . For each  $U \in \mathcal{U}_n$ , choose a continuous  $f_U : X \rightarrow [0, 1/n]$  such that  $f_U^{-1}(0) = X - U$ . Let  $\Gamma = \mathcal{U}$ . For each  $x \in X$ , define  $\tilde{x} \in c_0(\Gamma)$  by  $\tilde{x}(U) = f_U(x)$ . Then  $x \rightarrow \tilde{x}$  is a homeomorphism from  $X$  into  $c_0(\Gamma)$ .

$(d) \rightarrow (e)$ . Clear.

$(e) \rightarrow (f)$ . This will be established by our proof of Theorem 1.1.

### 3. Four lemmas.

**LEMMA 3.1** (D. Burke [3, Lemma 4.1]). *Let  $\mathcal{P}$  be a point-finite collection of subsets of  $X$ , let  $E \subset X$  and let  $n \in \mathbb{N}$ . Then there are at most finitely many minimal covers  $\mathcal{F}$  of  $E$  such that  $\mathcal{F} \subset \mathcal{P}$  and  $|\mathcal{F}| = n$ .*

*Proof.* (Included for completeness.) Suppose there were an infinite family  $\Phi$  of such  $\mathcal{F}$ . Pick a maximal  $\mathcal{R} \subset \mathcal{P}$  such that  $\mathcal{R} \subset \mathcal{F}$  for infinitely many  $\mathcal{F} \in \Phi$ , and let  $\Phi' = \{\mathcal{F} \in \Phi : \mathcal{F} \supset \mathcal{R}\}$ . Clearly  $0 \leq |\mathcal{R}| < n$ , so  $\mathcal{R}$  does not cover  $E$  and hence there is some  $y \in E - \bigcup \mathcal{R}$ . Since  $\mathcal{P}$  is point-finite and  $\Phi'$  is infinite, there

is some  $P_0 \in \mathcal{P}$  such that  $y \in P_0$  and  $P_0 \in \mathcal{F}$  for infinitely many  $\mathcal{F} \in \Phi'$ . Since  $y \in P_0$ , we have  $P_0 \notin \mathcal{R}$ . Let  $\mathcal{S} = \mathcal{R} \cup \{P_0\}$ . Then  $\mathcal{R} \subsetneq \mathcal{S}$  and  $\mathcal{S} \subset \mathcal{F}$  for infinitely many  $\mathcal{F} \in \Phi$ , contradicting the maximality of  $\mathcal{R}$ .

**LEMMA 3.2.** *Let  $Y$  be completely regular, let  $\lambda = |Y|$ , and let  $V \subset Y$  be open. Then there is an open cover  $\{V_\alpha: \alpha < \lambda\}$  of  $V$ , and a closed (in  $Y$ ) cover  $\{S_{\alpha,m}: \alpha < \lambda, m \in N\}$  of  $V$ , such that:*

- (a)  $S_{\alpha,m} \subset V_\alpha \subset V$  for all  $\alpha, m$ .
- (b) If  $\alpha > \beta$ , then  $S_{\alpha,m} \cap V_\beta = \emptyset$  for all  $m$ .

*Proof.* Let  $\{V_\alpha: \alpha < \lambda\}$  be a cover of  $V$  by open  $F_c$ -subsets of  $Y$ , so  $V_\alpha = \bigcup_{m=1}^\infty T_{\alpha,m}$  with each  $T_{\alpha,m}$  closed in  $Y$ . For each  $\alpha, m$ , let

$$S_{\alpha,m} = T_{\alpha,m} - \bigcup_{\beta < \alpha} V_\beta.$$

It is easy to check that this works.

**LEMMA 3.3.** *Let  $\mathcal{U}$  be an  $F$ -separating open cover of a space  $X$ , and  $A, B$  disjoint compact subsets of  $X$ . Then there is a finite  $\mathcal{E} \subset \mathcal{U}$  which  $F$ -separates any  $x \in A$  from any  $x' \in B$  (in the obvious sense).*

*Proof.* Since  $\mathcal{U}$  is  $F$ -separating and  $A \cap B = \emptyset$ ,

$$\{U \times (X - \bar{U}): U \in \mathcal{U}\} \cup \{(X - \bar{U}) \times U: U \in \mathcal{U}\}$$

covers  $A \times B$ . Since  $A \times B$  is compact, there is a finite  $\mathcal{E} \subset \mathcal{U}$  such that

$$\{U \times (X - \bar{U}): U \in \mathcal{E}\} \cup \{(X - \bar{U}) \times U: U \in \mathcal{E}\}$$

covers  $A \times B$ . This  $\mathcal{E}$  has the required property.<sup>3)</sup>

Henceforth, whenever  $\mathcal{U}$  is a collection of subsets of  $X$  and  $x \in X$ , we will write  $\mathcal{U}(x)$  to denote  $\{U \in \mathcal{U}: x \in U\}$ .

**LEMMA 3.4.** *Let  $A, B$  be disjoint subsets of a set  $X$ , and  $\mathcal{E}$  a finite collection of subsets of  $X$  separating any  $x \in A$  from any  $x' \in B$ . Then some subcollection  $\mathcal{F}$  of  $\mathcal{E}$  covers  $A$  but not  $B$ , or vice versa.*

*Proof.* Suppose not. Then for each  $x \in A$  there is an  $x' \in B$

<sup>3</sup> We are grateful to Roy Olson for pointing out an error in an earlier version of this proof.

such that  $\mathcal{E}(x') \subset \mathcal{E}(x)$ , and vice versa. Hence there is a sequence  $x_n \in X$ , with  $x_n$  alternately in  $A$  and in  $B$ , such that  $\mathcal{E}(x_{n+1}) \subset \mathcal{E}(x_n)$  for all  $n$ . Since  $\mathcal{E}$  is finite,  $\mathcal{E}(x_{n+1}) = \mathcal{E}(x_n)$  for some  $n$ , and that contradicts the separation property of  $\mathcal{E}$ .

4. **Proof of Theorem 1.1.** Let  $f: X \rightarrow Y$  be continuous and onto, with  $X$  an *EC*, and let us show that  $Y$  must be an *EC*. By Theorem 1.4 (all of which except (e)  $\rightarrow$  (d) has been proved in § 2), we may assume that  $X$  satisfies 1.4(e), and we shall conclude that  $Y$  satisfies 1.4(d). That will also establish the implication 1.4(e)  $\rightarrow$  1.4(d).

Let  $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$  be an  $F$ -separating collection of open subsets of  $X$ , with each  $\mathcal{U}_n$  point-finite; we may suppose that  $\mathcal{U}_n \subset \mathcal{U}_{n+1}$  and that  $\mathcal{U}_n$  covers  $X$  for all  $n$ . Let  $\Phi = \{\mathcal{F} \subset \mathcal{U}: \mathcal{F} \text{ finite}\}$ . For each  $\mathcal{F} \in \Phi$ , let

$$V(\mathcal{F}) = \{y \in Y: \mathcal{F} \text{ covers } f^{-1}(y)\},$$

$$M(\mathcal{F}) = \{y \in Y: \mathcal{F} \text{ covers } f^{-1}(y) \text{ minimally}\}.$$

Observe that  $V(\mathcal{F}) = Y - f(X - \bigcup \mathcal{F})$ , so  $V(\mathcal{F})$  is open in  $Y$ , and  $M(\mathcal{F}) = V(\mathcal{F}) - \bigcup \{V(\mathcal{F}'): \mathcal{F}' \subsetneq \mathcal{F}\}$ , so  $M(\mathcal{F})$  is relatively closed in  $V(\mathcal{F})$ .

For each  $\mathcal{F} \in \Phi$ , choose open  $V_{\alpha}(\mathcal{F}) \subset V(\mathcal{F})$  and closed  $S_{\alpha,m}(\mathcal{F}) \subset V_{\alpha}(\mathcal{F})$  ( $\alpha < \lambda = |Y|$ ,  $m \in N$ ) as in Lemma 3.2. Let

$$M_{\alpha,m}(\mathcal{F}) = S_{\alpha,m}(\mathcal{F}) \cap M(\mathcal{F}).$$

Then

$$(4.1) \quad M(\mathcal{F}) = \bigcup \{M_{\alpha,m}(\mathcal{F}): \alpha < \lambda, m \in N\},$$

and each  $M_{\alpha,m}(\mathcal{F})$  is closed in  $Y$ .

For  $\mathcal{F} \in \Phi$ ,  $\alpha < \lambda$ ,  $m \in N$ , we shall define a positive integer  $k = k_{\alpha,m}(\mathcal{F})$  with the following property:

$$(4.2) \quad \text{If } x, x' \in X, \text{ if } \mathcal{U}_k(x) = \mathcal{U}_k(x'), \text{ and if } f(x) \in M_{\alpha,m}(\mathcal{F}),$$

then  $f(x') \in V_{\alpha}(\mathcal{F})$ .

To define  $k$ , let  $A = f^{-1}(M_{\alpha,m}(\mathcal{F}))$ ,  $B = f^{-1}(Y - V_{\alpha}(\mathcal{F}))$ . By Lemma 3.3, there is a finite  $\mathcal{E} \subset \mathcal{U}$  which separates any  $x \in A$  from any  $x' \in B$ . Pick  $k$  so that  $\mathcal{E} \subset \mathcal{U}_k$ ; this  $k$  satisfies (4.2).

Let  $\mathcal{F} \in \Phi$ ,  $\alpha < \lambda$ ,  $m \in N$ , and let  $k$  be as above. Define

$$H_{\alpha,m}(\mathcal{F}) = \bigcup \{\bigcap (\mathcal{U}_k(x)): x \in f^{-1}(M_{\alpha,m}(\mathcal{F}))\},$$

$$K_{\alpha,m}(\mathcal{F}) = Y - f(X - H_{\alpha,m}(\mathcal{F})).$$

Now  $H_{\alpha,m}(\mathcal{F})$  is an open set in  $X$  containing  $f^{-1}(M_{\alpha,m}(\mathcal{F}))$ , so

$K_{\alpha,m}(\mathcal{F})$  is an open set in  $Y$  containing  $M_{\alpha,m}(\mathcal{F})$ . Choose an open  $F_\sigma$ -set  $W_{\alpha,m}(\mathcal{F})$  in  $Y$  such that

$$M_{\alpha,m}(\mathcal{F}) \subset W_{\alpha,m}(\mathcal{F}) \subset (K_{\alpha,m}(\mathcal{F}) \cap V_\alpha(\mathcal{F})).$$

Finally, define

$$\mathcal{W} = \{W_{\alpha,m}(\mathcal{F}): \mathcal{F} \in \Phi, \alpha < \lambda, m \in N\}.$$

We will show that  $\mathcal{W}$  is a  $\sigma$ -point-finite, separating collection of subsets of  $Y$ , thereby completing our proof.

(A)  $\mathcal{W}$  is separating on  $Y$ : Let  $y \neq y'$  be in  $Y$ . By Lemmas 3.3 and 3.4, there is a finite  $\mathcal{F} \subset \mathcal{U}$  which covers  $f^{-1}(y)$  but not  $f^{-1}(y')$ , or vice versa (we won't repeat this phrase). We may assume that  $\mathcal{F}$  is a minimal cover of  $f^{-1}(y)$ . Hence  $y \in M(\mathcal{F})$  and  $y' \notin V(\mathcal{F})$ . By (4.1), there are  $\alpha < \lambda$  and  $m \in N$  so that  $y \in M_{\alpha,m}(\mathcal{F})$ . Hence  $y \in W_{\alpha,m}(\mathcal{F})$  and  $y' \notin W_{\alpha,m}(\mathcal{F})$ .

(B)  $\mathcal{W}$  is  $\sigma$ -point-finite: For  $m, n, k \in N$ , let

$$\begin{aligned} \mathcal{W}_{m,n,k} &= \{W_{\alpha,m}(\mathcal{F}): \mathcal{F} \in \Phi, \mathcal{F} \subset \mathcal{U}_n, |\mathcal{F}| \leq n, \\ &\quad k_{\alpha,m}(\mathcal{F}) = k, \alpha < \lambda\}. \end{aligned}$$

Then  $\mathcal{W} = \bigcup_{m,n,k=1}^{\infty} \mathcal{W}_{m,n,k}$ , so we need only show that each  $\mathcal{W}_{m,n,k}$  is point-finite.

Suppose some  $\mathcal{W}_{m,n,k}$  were not point-finite. Then there are distinct  $W_{\alpha_j,m}(\mathcal{F}_j)$  ( $j \in N$ ) in  $\mathcal{W}_{m,n,k}$ , all containing the same  $y^* \in Y$ . Fix some  $x^* \in f^{-1}(y^*)$ . Then  $x^* \in H_{\alpha_j,m}(\mathcal{F}_j)$ , so there are  $x_j \in f^{-1}(M_{\alpha_j,m}(\mathcal{F}_j))$  such that  $x^* \in \bigcap \mathcal{U}_k(x_j)$  for all  $j$ . Hence  $\mathcal{U}_k(x_j) \subset \mathcal{U}_k(x^*)$  for all  $j$ ; since  $\mathcal{U}_k(x^*)$  is finite, we may assume, by passing to a subsequence, that  $\mathcal{U}_k(x_i) = \mathcal{U}_k(x_j)$  for all  $i, j$ . By (4.2), it follows that  $f(x_i) \in V_{\alpha_j}(\mathcal{F}_j)$  for all  $i, j$ . Let  $y_i = f(x_i)$ . Then

$$(4.3) \quad y_i \in M_{\alpha_i,m}(\mathcal{F}_i) \cap V_{\alpha_j}(\mathcal{F}_j) \quad (i, j \in N).$$

We now distinguish two cases to obtain our contradiction.

*Case 1.* There are  $i \neq j$  with  $\mathcal{F}_i = \mathcal{F}_j$ : Then  $\alpha_i \neq \alpha_j$ ; suppose  $\alpha_i > \alpha_j$ : Hence (4.3) implies that (writing  $\mathcal{F}$  for  $\mathcal{F}_i = \mathcal{F}_j$ )

$$y_i \in S_{\alpha_i,m}(\mathcal{F}) \cap V_{\alpha_j}(\mathcal{F}),$$

contradicting condition (b) of Lemma 3.2.

*Case 2.* The  $\mathcal{F}_j$  are all distinct: Let

$$E = \bigcup_{j=1}^{\infty} f^{-1}(y_j).$$

Since  $y_i \in V_{\alpha_j}(\mathcal{F}_j) \subset V(\mathcal{F}_j)$  for all  $i, j$ , each  $\mathcal{F}_j$  convers  $E$ . Since  $y_j \in M_{\alpha_j}(\mathcal{F}_j) \subset M(\mathcal{F}_j)$  for all  $j$ , each  $\mathcal{F}_j$  is a *minimal* cover of  $f^{-1}(y_j)$  and thus also of  $E$ . Hence  $E$  has infinitely many minimal covers  $\mathcal{F}_j$  with  $\mathcal{F}_j \subset \mathcal{U}_n$  and  $|\mathcal{F}_j| \leq n$ , contradicting Lemma 3.1.

That completes the proof.

**5. Consequences of the proof of Theorem 1.1.** The first consequence, which has already been observed in §4, is that  $1.4(e) \rightarrow 1.4(d)$ . This implication follows from the proof of Theorem 1.1 in the special case where  $X = Y$  and  $f = \text{id}_X$ , and, not surprisingly, the proof becomes somewhat simpler in this case: The only  $\mathcal{F} \subset \mathcal{U}$  which now concern us are singletons, and if  $\mathcal{F} = \{U\}$  then  $V(\mathcal{F}) = M(\mathcal{F}) = U$ . That  $\mathcal{W}$  is separating is now trivial, as is Case 2 of the proof that  $\mathcal{W}$  is  $\sigma$ -point-finite; hence Lemmas 3.4 and 3.1 are not needed here. The general structure of the proof is retained, however, and Lemmas 3.2 and 3.3 are apparently still required.

Our proof that  $1.4(e) \rightarrow 1.4(d)$  reveals that this implication can be split into two stages, thereby obtaining yet another necessary and sufficient condition for a compact space  $X$  to be an *EC*, as follows:

(5.1)  *$X$  has a  $\sigma$ -point-finite collection  $\mathcal{U}$  of open subsets such that, if  $A, B$  are disjoint, closed subsets of  $X$ , then some finite  $\mathcal{E} \subset \mathcal{U}$  separates any  $x \in A$  from any  $x' \in B$ .*

In fact,  $1.4(e) \rightarrow (5.1)$  by Lemma 3.3, and our proof of Theorem 1.1 shows that  $(5.1) \rightarrow 1.4(d)$ .

We conclude this section by recording one more consequence of our proof of Theorem 1.1: Every open subset of an *EC* is the union of a  $\sigma$ -point-finite collection of open  $F_\sigma$ 's.

**6. A modification of Eberlein compacts.** Let  $\Sigma(\Gamma)$  denote the space of real-valued functions on the set  $\Gamma$  which vanish at all but countably many  $\gamma \in \Gamma$ , and let  $\mathcal{T}_p$  be the topology of pointwise convergence on this space. Observe that  $\Sigma(\Gamma) \supset c_0(\Gamma)$ , and that  $(\Sigma(\Gamma), \mathcal{T}_p)$  is just the  $\Sigma$ -product (in the sense of H.H. Corson [4]) of  $|\Gamma|$ -many real lines.

The proof of the following result is essentially the same as the proof of the equivalence of (c), (d) and (e) in Theorem 1.4, with the reference to Theorem 1.1 replaced by an analogous reference to Theorem 6.2 below.

**THEOREM 6.1.** *The following properties of a compact space  $X$  are equivalent.*



- (a)  $X$  is homeomorphic to a subset of some  $(\Sigma(I), \mathcal{T}_r)$ .
- (b)  $X$  has a point-countable, separating collection of open  $F_\sigma$ -subsets.
- (c)  $X$  has a point-countable,  $F$ -separating collection of open subsets.

Let us call a compact space satisfying the equivalent conditions of Theorem 6.1 a *Corson compact* (a term suggested by their relation to Corson's  $\Sigma$ -products). Observe that every Eberlein compact is a Corson compact and that, by Remark 1.7, it is consistent with *ZFC* that the two concepts are distinct.

Analogously to Theorem 1.1, we now have:

**THEOREM 6.2.** *Every continuous image of a Corson compact is again a Corson compact.*

The proof of Theorem 6.2 is virtually the same as that of Theorem 1.1, except that Lemma 3.1 must be replaced by the following result.

**LEMMA 6.3** (A. S. Miščenko [7]). *Let  $\mathcal{P}$  be a point-countable collection of subsets of  $X$ , and let  $E \subset X$ . Then there are at most countably many minimal finite covers  $\mathcal{F}$  of  $E$  with  $\mathcal{F} \subset \mathcal{P}$ .*

*Added in Proof.* The second author has shown that the answer to Problem 1.8 is "yes" if  $X$  is a closed subset of  $(c_0(I), \mathcal{T}_r)$ .

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