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MERCERIAN THEOREMS VIA SPECTRAL THEORY

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Given a regular matrix A, Mercerian theorems are concerned with determining the real or complex values of α for which $\alpha I + (1 - \alpha)A$ is equivalent to convergence. For $\alpha \neq 1$, the problem is equivalent to determining the resolvent set for A, or, determining the spectrum $\sigma(A)$ of A, where $\sigma(A) = \{\lambda \mid A - \lambda I \text{ is not invertible}\}$. This paper treats the problem of determining the spectra of weighted mean methods; i.e., triangular matrices $A = (a_{nk})$ with $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \ge 0$, $\sum_{k=0}^n p_k = P_n$. It is shown that the spectrum of every weighted mean method is contained in the disc $\{\lambda \mid |\lambda - 1/2| \le 1/2\}$ (Theorem 1), and, if $\lim p_n/P_n$ exists,

$$egin{aligned} \sigma(A) &= \{ \lambda \mid | \, \lambda - (2-arepsilon)^{-1} | \ &\leq (1-arepsilon)/(2-arepsilon) \} \cup \{ p_n/P_n \mid p_n/P_n < arepsilon/(2-arepsilon) \} \ , \end{aligned}$$

where $\varepsilon = \lim p_n / P_n$.

Let $\gamma = \underline{\lim} p_n/P_n$, $\delta = \overline{\lim} p_n/P_n$, $S = \{\overline{p_n/P_n \mid n \ge 0}\}$. When $\gamma < \delta$, some examples are provided to indicate the difficulty of determining the spectrum explicitly. It is shown that $\{\lambda \mid |\lambda - (2 - \delta)^{-1}| \le (1 - \delta/(2 - \delta)\} \cup S \subseteq \sigma(A)$ and

$$\sigma(A) \subseteq \{\lambda \mid |\lambda - (2-\gamma)^{-1}| \leq (1-\gamma)/(2-\gamma)\} \cup S$$
.

Theorem 1 is a generalization of the corresponding theorems of: S. Aljancic, L. N. Cakalov, K. Knopp, M. E. Landau, J. Mercer, Y. Okada, W. Sierpinski, and G. Sunouchi.

Using spectral theory we obtain the best possible Mercerian theorems for certain classes of weighted mean methods of summability.

The weighted mean method is a triangular matrix $A = (a_{nk})$ with $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \ge 0$, $n \ge 0$, $P_n = \sum_{k=0}^{n} p_k$ and A is a bounded linear operator on c, the space of convergent sequences.

For $\alpha \neq 0$ we may write $\alpha I + (1 - \alpha)A = \alpha(I + qA)$, where $q = (1 - \alpha)/\alpha$. Mercer's original theorem [9] states the following: Let $\{x_n\}$ be a sequence such that $x_{n+1} - x_n + \mu n^{-1}x_n \rightarrow \lambda$ as $n \rightarrow \infty$. (i) If λ is finite and $\mu > -1$, then $x_{n+1} - x_n$ and $n^{-1}x_n$ both tend to $\lambda/(\mu + 1)$ as $n \rightarrow \infty$. (ii) If λ is infinite and $\mu > -1$, then $n^{-1}x_n \rightarrow \lambda$ and $x_{n+1} - x_n \rightarrow \lambda$ only if $0 \geq \mu > -1$.

Landau [8] showed that, if $\{x_n\}$ is a complex sequence, q a positive integer, then $\lim_{n}(x_n + (q/n)\sum_{k=1}^{n} x_k) = 0$ implies $\lim_{n} x_n = 0$. Sierpinski [14] extended Landau's result to real numbers q > -1 and showed it could not be extended to $q \leq -1$. Sierpinski's result for q > -1 was reproved in [3].

Let $\sum_{n=2}^{\infty} p_n/(p_1 + p_2 + \cdots + p_{n-1})$ be a divergent series of positive terms, $\{x_n\}$ a complex sequence. Okada [10] showed that if q > -1, then $\lim_n (x_n + q(\sum_{k=1}^n p_k x_k / \sum_{k=1}^n p_k)) = l$, l finite, implies $\lim_n x_n = l/(1+q)$. He also verified that the theorem does not hold for $\lim_n \sum_{k=1}^{n-1} p_k/p_n > -(1+q) \ge 0$.

Using a different technique, Knopp [6] reproved Okada's result. Beekman [2] showed that, if A is a conservative triangle with inverse satisfying $a_{nn}^{-1} > 0$, $a_{nk}^{-1} \leq 0$ for n > k, then I + qA is equivalent to convergence for $\operatorname{Re}(q) > -1$.

We determine the spectrum of A, $\sigma(A)$, in every case in which lim p_n/P_n exists (Corollaries 1 and 2). When $\{p_n/P_n\}$ does not converge, in which case A is necessarily regular, the situation seems pathological: Theorems 2 and 3 do give set inclusions for $\sigma(A)$, but, as we show by examples, $\sigma(A)$ can be disconnected and is very difficult to describe explicitly.

Let $B = A - \lambda I$. Our first task is to compute the entries of B^{-1} . Except for Theorem 1, we shall restrict our attention to regular weighted mean methods; i.e., those for which $P_n \to \infty$. For, if P_n tends to a finite limit, then A is compact and $\sigma(A) = \{p_k/P_k : k \ge 0\} \cup \{0\}$. (See, e.g. [13, Theorem 1].)

LEMMA 1. Let A be a weighted mean matrix, $B = A - \lambda I$, λ a scalar such that $b_{nn} \neq 0$ for each n. Then $D = B^{-1}$ has entries

(1) $d_{nn} = \frac{P_n}{p_n - \lambda P_n},$ $d_{nk} = (-1)^{n+k} \frac{\lambda^{n-k-1}p_k}{P_n} \prod_{j=k}^n \frac{P_j}{p_j - \lambda P_j}, \quad k < n.$

Proof. A direct computation verifies d_{nn} and $d_{n,n-1}$. By induction one can show that

$$(2) \qquad \sum_{j=0}^{k} (-1)^{j} \lambda^{j-1} \frac{p_{n-j}}{P_{n-j}} \prod_{i=0}^{j} \frac{P_{n-i}}{p_{n-i} - \lambda P_{n-i}} = (-1)^{k} \lambda^{k} \prod_{j=0}^{k} \frac{P_{n-j}}{P_{n-j} - \lambda P_{n-j}}.$$

With (2), one verifies by induction that (1) is true.

THEOREM 1. Let A be a weighted mean method. Then $\sigma(A) \subseteq \{z \mid |z - 1/2| \leq 1/2\}.$

Proof. Let $\lambda = x + iy$ satisfy $|\lambda - 1/2| > 1/2$. This inequality is equivalent to $\alpha > -1$, where $-1/\lambda = \alpha + i\beta$. Since $\alpha > -1$ and $0 \leq p_j/P_j \leq 1$ for all $j, |1 - p_j/\lambda P_j| \geq |1 + \alpha p_j/P_j = 1 + \alpha p_j/P_j$. For $k < n, |d_{nk}| \leq p_k/|\lambda|^2 P_n \prod_{j=k}^n (1 + \alpha p_j/P_j) = f_{nk}$, say.

Using finite induction we can show, for each 0 < r < n,

$$\sum_{k=0}^{r} f_{nk} = rac{P_r}{|\lambda|^2 P_n (1+lpha) \prod_{j=r+1}^{n} (1+lpha p_j/P_j)}$$

Therefore $\sum_{k=0}^{n} |d_{nn}| \leq |d_{nn}| + \sum_{k=0}^{n-1} f_{nk} = |d_{nn}| + P_{n-1}/|\lambda|^2 P_n$.

$$egin{aligned} &(1+lpha)(1+lpha p_{n}/P_{n}) \leq |p_{n}/P_{n}-\lambda|^{-1}+eta|\lambda|^{-2}(1+lpha)^{-1} \ &\leq eta|\lambda|^{-1}(1+1/|\lambda|(1+lpha)) \ , \end{aligned}$$

where $\beta = 1$ if $\alpha \ge 0$ and $\beta = (1 + \alpha)^{-1}$ if $-1 < \alpha < 0$. Since $d_{nn} \ne 0$ for each *n*, from Problem 32 [16, p. 232], the convergence domain of *D*, (*D*), is equal to *c*, and $\lambda \in \rho(A)$, the resolvent of *A*.

Theorem 1 is a special case of [2, Theorem 1]. Since 0 is not an interior point of $\sigma(A)$, Theorem 1 provides another proof of the fact that every weighted mean method lies in the closure of the maximal group of invertible elements in Δ , the subalgebra of B(c)consisting of triangular matrices. (See [11, p. 287].)

Let $\delta = \overline{\lim}_n p_n / P_n$, $\gamma = \underline{\lim}_n p_n / P_n$.

THEOREM 2. Let A be a regular weighted mean method. Then $\sigma(A) \supseteq \{\lambda \mid |\lambda - (2-\delta)^{-1}| \leq (1-\delta)/(2-\delta)\} \cup S$, where $S = \overline{\{p_n/P_n \mid n \geq 0\}}$.

Proof. Fix λ satisfying $|\lambda - (2 - \delta)^{-1}| < (1 - \delta)/(2 - \delta)$ and $\lambda \neq p_n/P_n$ for any *n*. From (1) we obtain

(3)
$$|d_{nk}| = \frac{p_k}{|\lambda|^2 P_{k-1} \prod_{j=k}^n \left| 1 + \left(1 - \frac{1}{\lambda}\right) \frac{p_j}{P_{j-1}} \right|}.$$

Note that $|1 + (1 - (1/\lambda)p_{n+1}/P_n| \leq 1$ if and only if

$$(1+(1+lpha)p_{{n+1}}/P_{n})^{2}+(eta p_{{n+1}}/P_{n})^{2}\leq 1$$
 ,

where $-1/\lambda = \alpha + i\beta$; i.e.,

$$(4) 2(1+\alpha)p_{n+1}/P_n + ((1+\alpha)^2 + \beta^2)(p_{n+1}/P_n)^2 < 0.$$

For each n such that $p_{n+1} = 0$, (4) is automatically satisfied. For each n such that $p_{n+1} > 0$, (4) is equivalent to

(5)
$$2(1 + \alpha) + ((1 + \alpha)^2 + \beta^2)p_{n+1}/P_n \leq 0$$
.

For (5) to be true for all n sufficiently large, it is sufficient to have δ satisfy

(6)
$$2(1 + \alpha) + ((1 + \alpha)^2 + \beta^2)\delta/(1 - \delta) < 0$$
,

since $p_{n+1}/P_n = p_{n+1}/P_{n+1}(1 - p_{n+1}/P_{n+1})$, which is monotone increasing in p_n/P_n . Inequality (6) is equivalent to $|\lambda - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)$.

Therefore, for all $n \ge N$, using (3),

$$\sum_{k=N}^{n-1} |d_{nk}| \ge rac{1}{|\lambda|^2} \sum_{k=N}^{n-1} rac{p_k}{P_{k-1}} \ge rac{1}{|\lambda|^2} \sum_{k=N}^{n-1} rac{p_k}{P_k}$$
 ,

which diverges by the Abel-Dini theorem [7, p. 290].

If $\lambda = p_n/P_n$ then λ belongs to the spectrum of A. Theorem 2 follows since the spectrum is always closed.

COROLLARY 1. Let A be a regular weighted mean method with $\delta = 0$. Then $\sigma(A) = \{\lambda \mid |\lambda - 1/2| \leq 1/2\}.$

Proof. Combine Theorems 1 and 2, observing that S is already contained in the disc.

Special cases of Corollary 1 for λ real appear in [1], [6], and [10].

THEOREM 3. Let A be a regular weighted mean method with $\gamma > 0$. Then $\sigma(A) \subseteq \{\lambda \mid | \lambda - (2 - \gamma)^{-1}| < (1 - \lambda)/(2 - \gamma)\} \cup S$.

Proof. Let λ be fixed and satisfy $|\lambda - (2 - \gamma)^{-1}| > (1 - \gamma)/(2 - \gamma)$ and $\lambda \neq p_n/P_n$ for any *n*. We shall show that $\lambda \in \rho(A)$, the resolvent of *A*. From Theorem 1 we need consider only those values of λ satisfying $|\lambda - 1/2| \leq 1/2$; i.e., $\alpha < -1$. The value $\alpha = -1$ corresponds to $\lambda = 1$, which we know lies in the spectrum, since $p_0/P_0 = 1$. Therefore we shall assume $\alpha < -1$.

Under the assumption on λ we wish to verify that

$$|1 + (1 - 1/\lambda)p_j/P_{j-1}|$$

is strictly larger than one for all j sufficiently large. To this end, define $f(t) = 1 + 2(1 + \alpha)t + ((1 + \alpha)^2 + \beta^2)t^2$. f has a minimum at $t_0 = -(1 + \alpha)/((1 + \alpha)^2 + \beta^2)$.

The assumption on λ is equivalent to

(7)
$$\gamma(\alpha^2 + \beta^2) + 2\alpha > \gamma - 2$$
.

Therefore

$$rac{\gamma}{2(1-\gamma)} > rac{-(1+lpha)}{(1+lpha)^2+eta^2} = t_{\scriptscriptstyle 0}$$

and f is monotone increasing for all $t > \gamma/2(1-\gamma)$.

Let $\varepsilon > 0$ and small. $f((\gamma/(1-\gamma)) - \varepsilon) = f(\gamma/(1-\gamma)) - 2 \in g(\varepsilon)$, where $g(\varepsilon) = 1 + \alpha + ((1+\alpha)^2 + \beta^2)(\gamma/(1-\gamma) - \varepsilon/2)$. $g(\varepsilon) > 0$ for small ε , since f is monotone increasing for $t > \gamma/2(1-\gamma)$.

We shall now show that $f(\gamma/(1-\gamma)) > 1$. From the hypothesis on λ and (6),

$$lpha^{2}+eta^{2}+rac{2lpha}{\gamma}>rac{\gamma-2}{\gamma}$$
 ,

which is equivalent to

$$\left|\frac{1}{1-\gamma}-\frac{\gamma}{\lambda(1-\lambda)}\right|>1.$$

But $1/(1 - \gamma) = 1 + \gamma/(1 - \gamma)$, so we have

$$(f(\gamma/(1-\gamma)) = |1 + (1-1/\lambda)\gamma/(1-\gamma)|^2 > 1$$
 .

Now choose $\varepsilon > 0$ and so small that $f(\gamma/(1-\gamma)-\varepsilon) = f(\gamma/(1-\gamma)) - 2\varepsilon g(\varepsilon) = m^2 > 1$. Then, by the definition of γ there exists an N such that n > N implies $p_{n+1}/P_n > \gamma/(1-\gamma) - \varepsilon$, so that $f(p_n/P_{n-1}) > f(\gamma/(1-\gamma)-\varepsilon) = m^2$.

Using (3), $|d_{nk}|/|d_{n+1,k}| = (f(p_{n+1}/P_n)) > m^2 > 1$ for all $n \ge N$. Therefore $|d_{nk}|$ is monotone decreasing in n for each $k, n \ge N$, so that D has bounded columns. Thus, to show that D has finite norm it is sufficient to show that $|d_{nn}|$ is bounded, and that $\sum_{k=N}^{n-1} |d_{nk}|$ is bounded.

Recall that p_n/P_{n-1} is monotone increasing in p_n/P_n . For the ε we are using, we can enlarge N, if necessary, to ensure that $p_n/P_{n-1} < \delta/(1-\delta) + 1$ for $n \ge N$.

From (3),

$$\sum_{k=N}^{n-1} \left| \left| d_{nk}
ight| \leq rac{1}{\left| \lambda
ight|^2} \left(rac{\delta}{1-\delta} + 1
ight) \sum_{k=N}^{n-1} \left(\prod_{j=k}^n \left| 1 + \left(1 - rac{1}{\lambda}
ight) rac{p_j}{P_{j-1}}
ight|^{-1}
ight) \\ \leq rac{1}{\left| \lambda
ight|^2} \left(rac{\delta}{1-\delta} + 1
ight) \sum_{k=N}^{n-1} m^{-n+k-1} < H ext{ ,}$$

where H is independent of n.

$$\begin{split} |d_{nn}| &= \frac{P_n}{|p_n - \lambda P_n|} = \frac{P_n}{|\lambda| |P_n - p_n/\lambda|} = \frac{P_n}{|\lambda| |P_{n-1} + (1 - 1/\lambda)p_n|} \\ &= \frac{P_n/P_{n-1}}{|\lambda| |1 + (1 - 1/\lambda)p_n/P_{n-1}|} = \frac{(1 + p_n/P_{n-1})}{|\lambda| |1 + (1 - 1/\lambda)p_n/P_{n-1}|} \\ &< \frac{1 + \delta/(1 - \delta) + 1}{|\lambda| m} \,. \end{split}$$

Therefore D has finite norm. From [16, loc. cit.], (D) = c and $\lambda \in \rho(A)$.

COROLLARY 2. Let A be a regular weighted mean method with $\lim_{n} p_{n}/P_{n} = \gamma > 0$. Then $\sigma(A) = \{\lambda \mid \mid \lambda - (2 - \gamma)^{-1} \mid \leq (1 - \gamma)/(2 - \gamma)\} \cup E$, where $E = \{p_{n}/P_{n} \mid p_{n}/P_{n} < \gamma/(2 - \gamma)\}$.

Proof. Combine Theorems 2 and 3 and note that $S \setminus E$ is already contained in the disc, and E is a finite set.

We now obtain a necessary and sufficient condition for a weighted mean method to be equivalent to convergence.

THEOREM 4. Let A be a regular weighted mean method. Then (A) = c if and only if $\theta = \underline{\lim}_n p_{n+1}/P_n > 0$.

Proof. $\theta > 0$ implies $p_{n+1}/P_n \ge \theta/2$ for all *n* sufficiently large. For each $np_{n+1}/P_{n+1} = (p_{n+1}/P_n)/(1 + p_{n+1}/P_n)$. Note that f(y) = y/(1+y) is monotone increasing in *y*, so that, for all $n \ge N$, $p_{n+1}/P_{n+1} \ge \theta/(2+\theta)$, and the diagonal entries of *A* are nonzero for $n \ge N$. If $a_{nn} = 0$ for any n < N, replace the zero by 1. The new matrix *B* has the same convergence domain as *A*. For $n \ge N$, the nonzero terms of B^{-1} are $b_{nn}^{-1} = P_n/p_n$, $b_{n,n-1}^{-1} = -P_{n-1}/p_n$.

Suppose $a_{kk} = 0$ for some k < N. Then $p_k = 0$, $b_{kk} = 1$ and $b_{nk} = 0$ for n > k. Thus $b_{kk}^{-1} = 1$, $b_{k+1,k}^{-1} = 0$ and, by induction, $b_{nk}^{-1} = 0$ for n > k.

Therefore $||B^{-1}|| = \sup_n [P_{n-1}/p_n + P_n/p_n] \le \sup_n 2P_n/p_n \le 2(2+\theta)/\theta < \infty$. By [16], (B) = c. Thus (A) = c.

Suppose $\theta = 0$. Then there exists a subsequence $\{n_k\}$ of n such that $\lim_k p_{n_k+1}/P_{n_k} = 0$.

Case I. $p_n = 0$ for at most a finite number of values of n. Let B be the matrix A with each zero diagonal entry replaced by 1. Then (B)=(A). Since $p_{n+1}/P_{n+1}=(p_{n+1}/P_n)/(1+p_{n+1}/P_n)$, $\lim_k P_{n_k}/p_{n_k}=0$. Therefore $||B^{-1}|| \ge \sup_k |b_{n_k,n_k}^{-1}| = +\infty$, and $(B) \neq c$.

Case II. $p_n = 0$ for an infinite number of values of n. Let $\{n_k\}$ denote this set. Define a sequence $\{x_n\}$ by $x_{n_k} = 1$, $x_k = 0$ otherwise. Then Ax = 0, and $(A) \neq c$.

The special case of this theorem for $0 < p_n \leq 1$ appears in [4]. A special case of the sufficiency of this theorem appears in [5, p. 59].

We now consider the pathology which may arise when $\gamma < \delta$.

With $p_0 = 1$, $p_n \ge 0$ for n > 0, $c_n = p_n/P_n$, then, as in [12, pp. 163-4], one can show that $p_n = c_n \prod_{j=1}^n (1-c_j)^{-1}$, $c_0 = 1$, $0 \le c_n < 1$ for n > 0, and $P_n \to \infty$ is equivalent to $\sum_{n=0}^{\infty} c_n = \infty$.

For any sequence $s = \{s_n\}$ define $u_n = \sum_{k=0}^n p_k s_k / P_n$. Then $u_n - (1 - c_n)u_{n-1} = c_n s_n$. Let

$$(8) t_n = u_n - \lambda s_n.$$

For each $c_n \neq 0$,

(9)
$$t_n = \lambda (1 - c_n) u_{n-1} / c_n + (1 - \lambda / c_n) u_n .$$

Now for the examples. Let p, q be real numbers satisfying $1 . Define <math>\{c_n\}$ by $c_0 = 1$, $c_{2n} = 1/p$, $c_{2n-1} = 1/q$, n > 0. Using (8) and (9), $t_0 = (1 - \lambda)u_0$, $t_{2n} = (p - 1)\lambda u_{2n-1} + (1 - p\lambda)u_{2n}$, and $t_{2n+1} = (q - 1)\lambda u_{2n} + (1 - q\lambda)u_{2n+1}$. Therefore t = Bu, where $b_{00} = 1$, $b_{2n,2n} = 1 - p\lambda$, $b_{2n-1,2n-1} = 1 - q\lambda$, $b_{2n,2n-1} = (q - 1)\lambda$, $b_{2n-1,2n-2} = (p - 1)\lambda$, n > 0, $b_{nk} = 0$ otherwise. From Theorem 4, (A) = c.

Suppose $\lambda \neq \{1/p, 1/q, 1\}$, and let $E = B^{-1}$. If $||E|| < \infty$, then from [16, loc. cit.] E is conservative and (B) = c. Therefore $t \in c \Rightarrow$ $u \in c \Rightarrow s \in c$ and $(A - \lambda I) = c$, which implies $\lambda \notin \sigma(A)$. Conversely, if $\lambda \notin \sigma(A)$, then $(A - \lambda I) = c$, so that $t \in c \Rightarrow s \in c \Rightarrow u \in c \Rightarrow E$ is conservative $\Rightarrow ||E|| < \infty$. We have shown that, if $\lambda \neq \{1/p, 1/q, 1\}$ then $\lambda \notin \sigma(A)$ if and only if $||E|| < \infty$.

To compute the norm of E, observe that $b_{nn}e_{nk} + b_{n,n-1}e_{n-1,k} = 0$ for k < n, so that $e_{nk} = -b_{n,n-1}e_{n-1,k}/b_{nn}$.

Thus $e_{2n,k} = -(p-1)\lambda e_{2n-1,k}/(1-p\lambda)$, k < 2n, $n = 1, 2, \dots$, and $e_{2n+1,k} = -(q-1)\lambda e_{2n,k}/(1-q\lambda)$. Let $R_n = \sum_{k=0}^n |e_{nk}|$. For $n \ge 1$,

(10)

$$R_{2n} = \sum_{k=0}^{2n-1} |e_{2n,k}| + |e_{2n,2n}|$$

$$= \frac{(p-1)|\lambda|}{|1-p\lambda|} \sum_{k=0}^{2n-1} |e_{2n-1,k}| + \frac{1}{|1-p\lambda|}$$

$$= \frac{1}{|1-p\lambda|} [(p-1)|\lambda| R_{2n-1} + 1],$$

and, for $n \ge 0$,

(11)
$$R_{2n+1} = \frac{1}{|1-q\lambda|} [(q-1)|\lambda|R_{2n}+1].$$

Substituting (11) into (10) we have

$$R_{2n+2} = rac{(p-1)(q-1)|\lambda|^2}{|1-p\lambda|\,|1-q\lambda|}R_{2n} + rac{(p-1)|\lambda|}{|1-p\lambda|\,|1-q\lambda|} + rac{1}{|1-p\lambda|}\,,$$

and

$$R_{2n+1} = rac{(p-1)(q-1)|\lambda|^2}{|1-p\lambda| \, |1-q\lambda|} R_{2n-1} + rac{(q-1)|\lambda|}{|1-p\lambda| \, |1-q\lambda|} + rac{1}{|1-q\lambda|} \; .$$

Let $\{\sigma_n\}$ be defined by $\sigma_{n+1} = a\sigma_n + b$, where a and b are fixed positive constants. Then

$$rac{\sigma_{n+1}}{a^{n+1}}-rac{\sigma_n}{a^n}=rac{b}{a^{n+1}}$$
 ,

so that

$$rac{\sigma_{n+1}}{a^{n+1}} - rac{\sigma_0}{a^0} = rac{b}{a} rac{(1-a^{-n-1})}{(1-a^{-1})}$$
 ,

or $\sigma_{n+1} - \sigma_0 a^{n+1} = b(a^{n+1} - 1)/(a - 1)$. For 0 < a < 1, $\{\sigma_n\}$ is bounded, and, for $a \ge 1$, $\{\sigma_n\}$ is unbounded. Therefore

$$egin{aligned} \sigma(A) &= \{\lambda | ~|| E || = \infty \} \cup \{1/p, 1/q, 1\} \ &= \{\lambda | (p-1)(q-1) \,|\, \lambda |^2 \geqq |1-p\lambda| \,|1-q\lambda| \} \,, \end{aligned}$$

since 1/p, 1/q and 1 already belong to those values of λ for which $||E|| = \infty$.

For p = 2, q = 3, $\partial\sigma(A)$ is an oval with *x*-intercepts of 1/4, 1. For p = 2, q = 8, the boundary consists of a pair of ovals which are tangent at $x = (10 - \sqrt{8})/23$. For p = 3, q = 13, $\sigma(A)$ is contained in two disjoint ovals. The left oval has *x*-intercepts at 1/15, 1/9, and the right oval has *x*-intercepts at 1/7, 1.

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