

Pacific Journal of Mathematics

SOME QUALITATIVE RESULTS ON THE REPRESENTATION THEORY OF Gl_n OVER A p -ADIC FIELD

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The purpose of this paper is to present a coherent approximate picture of the representation theory of GL_n over a p -adic field.

In §2, two very general structural results about representations of GL_n are proved. In §3, certain specific series of representations are constructed and analyzed fairly completely. In §4 the role of these representations in the whole of the representation theory of GL_n , particularly in regard to the Plancherel formula, is discussed.

In writing this I have tried to present an overall picture as directly and simply as possible. I feel this goal is reasonably achieved in §2. In §3, there is an unfortunate amount of technicality in the main construction. Balancing this, however, are three considerations. First, we have stuck to the simplest case sufficiently general to permit the conclusions at the end of §4. For an example of an attempt at a more complete result in a special case, see [12]. Second, the technicalities do reveal the essential features of what so far is the main technique for constructing representations of semisimple p -adic groups. Third, the precision of the results is hopefully some compensation for the effort required to obtain them. In §4, we have been fairly sketchy with the main result. However, we have presented more or less completely the argument in the supercuspidal case, as this is not only much simpler, but reveals the main ideas and the connections with classical notions.

We will now preview the results of §2. We fix for the rest of the paper a non-Archimedean local field F . The ring of integers of F will be written R and π stands for a prime element of R . We put $G = GL_n(F)$ and $K_0 = GL_n(R)$. Then K_0 is a maximal compact subgroup of G , and is open. Furthermore, any compact subgroup of G is conjugate to a subgroup of K_0 . The subgroups $K_\nu = 1 + \pi^\nu M_n(R)$, for $\nu \geq 1$, are open normal subgroups of K_0 and form a neighborhood basis for $1 \in G$.

The question we will be mainly concerned with in §2 is how a representation of G decomposes when restricted to some K_ν . Thus let \hat{K}_ν denote the collection of equivalence classes of irreducible unitary representations of K_ν . Let ρ be an admissible (see [16]) representation of G on a vector space X . Then we may write

$X = \sum_{\delta \in \hat{K}_\ell} X_\delta$, where X_δ is the K_ℓ -invariant subspace of X such that any K_ℓ -irreducible subspace of X_δ defines a representation of the class δ . The X_δ are all finite-dimensional by the definition of admissibility. Of course, some X_δ may be zero. We will call X_δ the isotypic component of X of type δ , and will say any nonzero $x \in X_\delta$ is of K_ℓ -type δ . Thus our goal is to make statements about the K_ℓ -types in X : which types occur, and with what multiplicity. (The multiplicity of the K_ℓ -type δ is $(\dim X_\delta)/(\dim \delta)$, where $\dim \delta$ is the dimension of any representation of class δ .) From these statements, a rough over-all picture of the representation theory of G will hopefully emerge. It is worth remarking here on the difference between the real and the p -adic cases. Knowledge of the decomposition of a representation of a semisimple Lie group on restriction to a maximal compact subgroup says very little about the representation. The case of the pair $Gl_n(\mathbf{R})$, $O(n)$ illustrates this phenomenon especially well. Almost all the representations of $O(n)$ almost always occur in a representation of $Gl_n(\mathbf{R})$. The sub-quotient theorem of Harish-Chandra [9] may be interpreted as follows: given any irreducible representations of $Gl_n(\mathbf{R})$ then one may find finitely many other irreducible representations of $Gl_n(\mathbf{R})$, such that their direct sum decomposes under $O(n)$ in one of finitely many pre-specifiable ways. No analogous statement is even remotely true for p -adic groups, and while $Gl_n(\mathbf{R})$ is extreme among real groups in this sort of behavior, it is not unique—the other real Chevalley groups act similarly. The reason for this difference, of course, is that in our p -adic group, K is open and so in an essential way sees already the most “rigid” aspects of the structure of G . One way of saying this is to note that K is Zariski-dense in G , while the maximal compact subgroup of a semisimple Lie group is a proper algebraic subgroup.

The usefulness of the K_ℓ -decompositions of representations of G has another face. Since the representation theory of K_ℓ does reflect quite well the representation theory of G , it is quite hard in itself; harder than the representation theory of a compact Lie group. It seems unlikely that we will ever have an explicit calculations of all the representations of any K_ℓ for n larger than 4 or 5 at most. Fortunately for us, it is possible to ignore the fine structure of K_ℓ and still arrive at meaningful statements. We may figuratively describe the situation as follows. We divide \hat{K}_ℓ into two sets of representations. Members of one set we refer to as “essential” representations, the other as “inessential” representations. The essential K_ℓ -types may be thought of as controlling the details of structure of the representation theory, while the inessential K_ℓ -types relate to asymptotic behavior of characters and so forth. To help make these suggestions more precise, we state one result which we

will obtain. In the following statement, we do not always take our representation ρ of G to be admissible. If ρ is not necessarily admissible, we take ρ to be smooth in the sense of Harish-Chandra: if ρ acts on X , then X is a direct sum $\sum_{s \in \hat{K}_v} X_s$ of its K_v -isotypic components, but the dimension of a given X_s may be infinite.

THEOREM 1. (a) *Any representation of G contains at least one essential K_v -type.*

(b) *For a given K_v -type δ , there are only finitely many essential K_v -types which can occur with δ in any irreducible representation of G . In particular an irreducible representation of G contains at most finitely many essential K_v -types.*

(c) *The multiplicity of any K_v -type δ in an irreducible representation ρ of G is bounded by a constant depending on δ times the sum of the multiplicities of the essential K_v -types in ρ .*

COROLLARY. *A finitely generated admissible representation of G has a finite composition series.*

REMARK. Casselman [5] has proven this corollary for any reductive group by very different methods.

It follows of course that almost all of the K_v -types occurring in a given ρ are inessential. From this comes their role in the asymptotics. This is a more difficult topic which we will essentially ignore here, except for the very crude result (c) of Theorem 1, and a result on characters which we will now discuss.

In Harish-Chandra's theory of semisimple Lie groups, the characters of representations play a crucial role. It is in terms of characters that the Plancherel theorem is formulated, and until recently the only proof of the existence of discrete series ([8]) produces their characters, not the modules. The point is that the characters seem to provide the most accessible definitive labeling of the representations, besides being the appropriate objects with which to do Fourier analysis. In the p -adic case, the characters seem likely to be of less practical importance because they are harder to compute and more complicated in structure, because other data, such as spherical functions, is easier to come by than in the real case, and because reasonably effective constructions of discrete series exist though there is room for much improvement. However, characters are still important for a complete theory, and it is of interest in any case to compare their behavior in the real and p -adic cases.

Let us recall the basics of character theory. Let $C_c^\infty(G)$ denote the space of locally constant, compactly supported, complex-valued functions on G . A distribution on G is any linear functional on

$C_c^\infty(G)$. Convolution on G defines the structure of associative algebra on $C_c^\infty(G)$. If ρ is a representation of G , then in an elementary fashion ρ may be "integrated" to yield a representation of $C_c^\infty(G)$. If ρ is admissible, it is easy to see that $\rho(f)$ has finite rank for any $f \in C_c^\infty(G)$, so the trace of $\rho(f)$ is defined. The mapping $f \mapsto \text{tr } \rho(f)$ then defines a distribution, denoted θ_ρ , on G . This distribution θ_ρ is called the character of ρ . It has the property of being invariant under inner automorphisms of G . That is, if $g_1, g_2 \in G$, define $\text{Ad } g_1(g_2) = g_1 g_2 g_1^{-1}$. If $f \in C_c^\infty(G)$, define $\text{Ad}^* g_1(f)$ by $\text{Ad}^* g_1(f)(g_2) = f(\text{Ad } g_1^{-1}(g_2))$. Then $\theta_\rho(\text{Ad}^* g(f)) = \theta_\rho(f)$ for all $f \in C_c^\infty(G)$ and $g \in G$.

Now let Δ be a distribution on G and let $U \subseteq G$ be open. Let h be a function on U , locally integrable with respect to Haar measure restricted to U . We say Δ equals h on U if, for any $f \in C_c^\infty(U)$, $\Delta(f) = \int_U f(x)h(x)dx$, dx denoting Haar measure. Recall that G' , the regular set of G , consists of those elements of G which are semi-simple and whose centralizer is a Cartan subgroup. (In terms of our $G = \text{Gl}_n(F)$, G' consists of those elements whose characteristic polynomials have n distinct roots.) G' is an open dense subset of G . Its complement is a proper closed subvariety.

For semisimple Lie groups Harish-Chandra [7] has proved the fundamental theorem that the character of an irreducible representation is a locally integrable function on the whole group which is analytic on the regular set. Of these two properties the first is very difficult to establish, while the second is quite simple. On the other hand, in [9], Harish-Chandra expends some effort to show that the character of a supercuspidal representation of a characteristic zero reductive p -adic group is locally constant on the regular set. (He then goes on to show it is locally integrable on all of G .) Here we will show, for our G (which may be of characteristic p) that the character of an admissible irreducible representation is a locally constant function on G' . In view of the corollary of Theorem 1, this also holds for finitely generated admissible representations.

Our method of establishing the result on characters uses the notion of "partial traces", which we now detail. Let the admissible ρ act on X , and let $X = \sum_{\delta \in \hat{K}_\rho} X_\delta$ be the decomposition of X into K_ρ -types. Let E_δ denote the projection of X onto X_δ . For $f \in C_c^\infty(G)$, define $\theta_{\rho,\delta}(f) = \text{tr}(E_\delta \rho(f) E_\delta)$. Then it is clear that $\theta_\rho(f) = \sum_\delta \theta_{\rho,\delta}(f)$. $\theta_{\rho,\delta}$ is called the δ -component of θ_ρ , or the partial trace of ρ with respect to δ .

$\theta_{\rho,\delta}$ as a distribution on G , is obviously very well-behaved. In fact, for $x \in G$, define $\theta_{\rho,\delta}(x) = \text{tr}(E_\delta \rho(x) E_\delta)$. Then $\theta_{\rho,\delta}$ is obviously a locally constant function on G . Moreover, $\theta_{\rho,\delta}(x)$ is just the distribution $\theta_{\rho,\delta}$ by the identification defined above. That is, for

$f \in C_c^\infty(G)$, $\theta_{\rho,s}(f) = \int_G f(x) \theta_{\rho,s}(x) dx$. Now we may state

THEOREM 2. *Let θ_ρ be the character of the irreducible admissible representation ρ of G . Then θ_ρ is a locally constant function on G' . Also, $\theta_\rho = \sum_s \theta_{\rho,s}$ on G' , the sum being taken as a uniform-on-compacta limit. More precisely, on a given compact set $\omega \subseteq G'$, all but a finite number of the $\theta_{\rho,s}$ vanish identically.*

Theorems 1 and 2 do not convey the coherence of the picture of which they form some of the highlights. The complete picture, though, is somewhat diffuse and hard to describe succinctly. Rather than draw out the introduction interminably, it seems preferable to begin the proofs. We will give more introductory remarks in §§3 and 4. Our last comment: the analysis here will be seen to have a distinctly geometric flavor, and is clearly related to what I have called Kirillov theory (see [14]). Here, however, by looking for sufficiently qualitative results, we are able to avoid the exponential map, which has in the past been essential. It is interesting that, in this highly attenuated form, Kirillov theory applies even to semisimple groups of characteristic p ; these groups are those seemingly farthest from its natural domain.

2. Dual blobs and essential K -types. We will write $M_n(F)$ or more briefly \mathfrak{G} for the $n \times n$ matrices with coefficients in F . Similarly $L = M_n(R)$ is the maximal open compact subring of \mathfrak{G} consisting of matrices with entries in R . For $\nu \in \mathbb{Z}$, we put $L_\nu = \pi^\nu L$.

We regard \mathfrak{G} as the Lie algebra of G . As such we have the adjoint action of G on \mathfrak{G} defined by the formula $\text{Ad } g(m) = gm g^{-1}$. Here $g \in G$, $m \in \mathfrak{G}$, and the product is matrix multiplication. We also have the bracket operation on \mathfrak{G} : $[m_1, m_2] = m_1 m_2 - m_2 m_1$. Let $\text{tr}(\mathfrak{G}/F)$ be the trace on \mathfrak{G} . Let $\langle m_1, m_2 \rangle = \text{tr}(\mathfrak{G}/F)(m_1, m_2)$, for $m_i \in \mathfrak{G}$. Then \langle, \rangle is a symmetric, nondegenerate bilinear form on \mathfrak{G} and it is invariant under $\text{Ad } G$. That is, $\langle \text{Ad } g(m_1), \text{Ad } g(m_2) \rangle = \langle m_1, m_2 \rangle$. Another way of expressing this is $\langle \text{Ad } g(m_1), m_2 \rangle = \langle m_1, \text{Ad } g^{-1}(m_2) \rangle$. Also note $\langle m_1, [m_2, m_3] \rangle = \langle [m_3, m_1], m_2 \rangle$.

Let $\hat{\mathfrak{G}}$ denote the Pontryagin dual group of \mathfrak{G} . It is well-known that $\hat{\mathfrak{G}}$ is isomorphic to \mathfrak{G} and we may define a convenient isomorphism as follows. Let Ω_0 be a nontrivial character of F . For convenience, we assume that largest fractional ideal in F on which Ω_0 is trivial (the conductor of Ω_0) is R itself. For $m_1, m_2 \in \mathfrak{G}$, define $\Omega(m_1)(m_2) = \Omega_0(\langle m_1, m_2 \rangle)$. Then $\Omega(m_1)$ is a character of \mathfrak{G} , and the mapping $\Omega: \mathfrak{G} \rightarrow \hat{\mathfrak{G}}$, given by $m \rightarrow \Omega(m)$ is an isomorphism between

\mathfrak{G} and $\hat{\mathfrak{G}}$.

There is defined on $\hat{\mathfrak{G}}$ an action Ad^* of G , the co-adjoint action, by the formula $\text{Ad}^* g(\psi)(m) = \psi(\text{Ad } g^{-1}(m))$, for $g \in G$, $m \in \mathfrak{G}$, $\psi \in \hat{\mathfrak{G}}$. We note that Ω is equivariant with respect to the actions Ad and Ad^* . That is, $\text{Ad}^* g \circ \Omega = \Omega \circ \text{Ad } g$; for $\text{Ad}^* g(\Omega(m_1))(m_2) = \Omega(m_1)(\text{Ad } g^{-1}(m_2)) = \Omega_0(\langle m_1, \text{Ad } g^{-1}(m_2) \rangle) = \Omega_0(\langle \text{Ad } g(m_1), m_2 \rangle) = \Omega(\text{Ad } g(m_1))(m_2)$.

If $X \subseteq \mathfrak{G}$ is any subset, let $X^* = \{m \in \mathfrak{G}, \Omega(m)(x) = 1 \text{ for all } x \in X\}$. X^* will be a closed subgroup of \mathfrak{G} , and $X \subseteq X^{**}$. If X is a subspace of \mathfrak{G} , then X^* is too, and in fact X^* is just the orthogonal complement of X with respect to \langle, \rangle . If X is an R -module (i.e., closed under scalar multiplication by elements of R), then so is X^* .

By a lattice A in a vector space V over F we understand an open, compact R submodule of V . A will then be a free R -module of degree equal to the dimension of V over F . If $A \subseteq \mathfrak{G}$ is a lattice, then by our normalization of Ω , we see $A^* = \{m \in \mathfrak{G}, \langle m, A \rangle \subseteq R\}$. Thus A^* is a lattice also, and is canonically isomorphic to $\text{Hom}_R(A, R)$. It is easy to demonstrate that $A^{**} = A$, that $(A_1 + A_2)^* = A_1^* \cap A_2^*$, and dually $(A_1 \cap A_2)^* = A_1^* + A_2^*$. Also, if $A_1 \subseteq A_2$, then $(A_2/A_1)^\wedge$ is naturally isomorphic to A_1^*/A_2^* . Finally we note that the L_ν are lattices, and $L_\nu^* = \pi^{-\nu} L_0 = L_{-\nu}$.

If $A \subseteq \mathfrak{G}$ is a lattice, let A^2 be the lattice generated by products $\lambda_1 \lambda_2$, $\lambda_i \in A$. If $A^2 \subseteq \pi A$, we will say A is a small lattice. Note that the L_ν , $\nu \geq 1$ are small lattices, and $L_\nu^* = L_{2\nu}$. If A is a small lattice, then $1 + A \subseteq G$ is an open compact subgroup. We have $1 + L_\nu = K_\nu$ as a special case.

If $A \subseteq \mathfrak{G}$ is a small lattice, then $(1 + A)/(1 + A^2)$ is abelian. A linear character of $1 + A$ which is trivial on $1 + A^2$ will be called a shallow character of $1 + A$. The shallow characters of $1 + A$ clearly form a group, the Pontryagin dual group of $(1 + A)/(1 + A^2)$. The mapping $\lambda \rightarrow 1 + \lambda$ from A to $1 + A$ sends cosets of A^2 in A to cosets of $1 + A^2$ in $1 + A$, and the factored mapping $A/A^2 \rightarrow (1 + A)/(1 + A^2)$ is easily checked to be a homomorphism. Thus we may identify $(A^2)^*/A^* \simeq (A/A^2)^\wedge$ with the shallow character group of $(1 + A)/(1 + A^2)$. If ψ is a shallow character, and $\lambda \in (A^2)^*$ is mapped to ψ by means of the above identifications, we say λ represents ψ , or is a representative of ψ .

If A and N are two small lattices in \mathfrak{G} then $A \cap N$ is clearly a small lattice also. Moreover $(A \cap N)^2 \subseteq A^2 \cap N^2$, so that the restriction of a shallow character of $1 + A$ or $1 + N$ to $1 + (A \cap N)$ is a shallow character of $1 + (A \cap N)$. In this regard, we have the following very simple observation.

LEMMA 2.1. *Let φ and ψ be shallow characters of $1 + A$ and $1 + N$ respectively. Let $\lambda \in (A^2)^*$ and $n \in (N^2)^*$ represent φ and ψ*

respectively. Then φ and ψ agree on $1 + (A \cap N)$ if and only if $\lambda - n \in A^* + N^*$ if and only if $(\lambda + A^*) \cap (n + N^*) \neq \emptyset$.

Proof. By our definitions above, we have $\varphi(1 + x) = \Omega(\lambda)(x)$ for $x \in A$, and $\psi(1 + y) = \Omega(n)(y)$ for $y \in N$. If φ and ψ are to agree on $1 + (A \cap N)$, then $\Omega(\lambda)$ and $\Omega(n)$ agree on $A \cap N$, so $\Omega(\lambda - n)$ is trivial on $A \cap N$, or in other words $\lambda - n \in (A \cap N)^* = A^* + N^*$. If $\lambda - n = \lambda_1 + n_1$ with $\lambda_1 \in A^*$, $n_1 \in N^*$, then $\lambda - \lambda_1 = n + n_1 \in (\lambda + A^*) \cap (n + N^*)$. Going backward is equally easy and proves the lemma.

It is completely clear that the property of being a small lattice in \mathbb{G} is invariant under $\text{Ad } G$. Moreover, if A is a small lattice, and $g \in G$, then $\text{Ad } g(1 + A) = 1 + \text{Ad } g(A)$, and $\text{Ad } g(A^2) = \text{Ad } g(A)^2$. From the second formula, we see that if φ is a shallow character of $1 + A$, then $\text{Ad } g(\varphi)$, defined by $\text{Ad}^* g(\varphi)(\text{Ad } g(x)) = \varphi(x)$ is a shallow character of $\text{Ad}^* g(1 + A)$. Using the fact that Ω is Ad , Ad^* -equivariant, we see that if $\lambda \in (A^2)^*$ represents φ , then

$$\text{Ad } g(\lambda) \in \text{Ad } g((A^2)^*) = ((\text{Ad } g(A))^2)^*$$

represents $\text{Ad}^* g(\varphi)$.

Now we focus attention on the $K_\nu = 1 + L_\nu$. We shall be much concerned with the intertwining properties of representations of the K_ν . We review the notions involved. Let δ_1 and δ_2 be irreducible representations of K_ν and K_μ respectively on vector spaces V_1 and V_2 . By an intertwining distribution on G for δ_1 and δ_2 , we mean a function f from G to $\text{Hom}(V_2, V_1)$ such that $f(k_1 y k_2) = \delta_1(k_1) f(y) \delta_2(k_2)$ for $k_1 \in K_\nu$, $k_2 \in K_\mu$, $y \in G$. We will discuss the function of intertwining distributions later. For the moment, we content ourselves with describing them. The transformation law for f implies f is locally constant. It is also clear that $f(y)$ determines f on the whole double coset $K_\nu y K_\mu$, so that there are at most $\dim \delta_1 \cdot \dim \delta_2$ linearly independent f supported on a given (K_ν, K_μ) double coset. On the other hand, the product of f with the characteristic function of any collection of (K_ν, K_μ) double cosets is again an intertwining distribution for δ_1 and δ_2 . The collection of intertwining distributions for δ_1 and δ_2 forms a linear space under pointwise addition and scalar multiplication.

We will call the dimension $I(\delta_1, \delta_2, y)$ of the space of intertwining distributions supported on $K_\nu y K_\mu$ the intertwining number of δ_1 and δ_2 on $K_\nu y K_\mu$. We also say y intertwines δ_1 and δ_2 $I(\delta_1, \delta_2, y)$ times. If $I(\delta_1, \delta_2, y) > 0$, we say y intertwines δ_1 and δ_2 . If some $y \in G$ intertwines δ_1 and δ_2 , we say δ_1 and δ_2 intertwine.

There is a way of describing the intertwining number of δ_1 and δ_2 on $K_\nu y K_\mu$, originally due to Mackey [18]. Let C be a compact group, and τ_1 and τ_2 two finite-dimensional representations of C on

U_1 and U_2 . An intertwining operator between τ_2 and τ_1 is a linear map $T: U_2 \rightarrow U_1$ such that $T \cdot \tau_2(c) = \tau_1(c) \cdot T$ for any $c \in C$. The intertwining number of τ_1 and τ_2 is the dimension of the space of intertwining operators between τ_1 and τ_2 . As is well-known [3], if $\tau_1 \simeq \Sigma a_i \sigma_i$ and $\tau_2 \simeq \Sigma b_i \sigma_i$ where the σ_i are irreducible representations of C , then the intertwining number of τ_1 and τ_2 is $\Sigma a_i b_i$. In particular, it is symmetric in τ_1 and τ_2 .

Now consider the double coset $K_\nu y K_\mu$. Put $\text{Ad } y(K_\mu) = y K_\mu y^{-1}$, and let $\text{Ad}^* y(\delta_2)$ be the representation of $\text{Ad } y(K_\mu)$ given by the formula $\text{Ad}^* y(\delta_2)(y k y^{-1}) = \delta_2(k)$ for $k \in K_\mu$. This notation is of course consistent with that we introduced earlier for shallow characters. Let $C = K_\nu \cap \text{Ad } y(K_\mu)$, and let τ_1 and τ_2 be the restrictions to C of δ_1 and $\text{Ad}^* y(\delta_2)$ respectively. Let f be an intertwining distribution for δ_1 and δ_2 supported on $K_\nu y K_\mu$. Then $f(x) \in \text{Hom}(V_2, V_1)$, and if $k_1 y k_2 = y$ with $k_1 \in K_\nu$, $k_2 \in K_\mu$, then the transformation law for f says $\delta_1(k_1) f(y) \delta_2(k_2) = f(y)$. We may write $k_1^{-1} = y k_2 y^{-1} \in C$, and then the above equation becomes $f(y) = \tau_1(k_1) f(y) \tau_2(k_1^{-1})$ so that $f(y)$ is an intertwining operator between τ_2 and τ_1 . Conversely, if T is an intertwining operator between τ_2 and τ_1 , then putting $f(k_1 y k_2) = \delta_1(k_1) \cdot T \cdot \delta_2(k_2)$, we see f is a well-defined function on $K_\nu y K_\mu$, and is an intertwining distribution for δ_1 and δ_2 supported on $K_\nu y K_\mu$. Thus we see that the intertwining number of δ_1 and δ_2 on $K_\nu y K_\mu$ is equal to the intertwining number of δ_1 and $\text{Ad}^* y(\delta_2)$ restricted to $K_\nu \cap \text{Ad } y(K_\mu)$. In particular, if y intertwines δ_1 and δ_2 then the restrictions of δ_1 and $\text{Ad}^* y(\delta_2)$ to $K_\nu \cap \text{Ad } y(K_\mu)$ contain some common irreducible representation of $K_\nu \cap \text{Ad } y(K_\mu)$.

For our immediate purposes, the main significance of intertwining distributions is this very elementary result.

LEMMA 2.2. *Let δ_1 and δ_2 be irreducible representations of K_ν and K_μ respectively. If there is an irreducible representation ρ of G such that δ_1 and δ_2 occur in the restrictions of ρ to K_ν and K_μ respectively, then δ_1 and δ_2 intertwine.*

Proof. Assuming that δ_1 and δ_2 appear in ρ , let V_1 and V_2 be two subspaces of the space of ρ , such that V_1 is invariant and irreducible under K_ν and of the class of δ_1 , and V_2 is invariant and irreducible under K_μ and of the class of δ_2 . Let P_1, P_2 be projections onto V_1, V_2 which respectively commute with the actions of K_ν, K_μ . Since ρ is irreducible, there is $g \in G$ such that $P_1 \rho(g) P_2$ is nonzero. Then $f(k_1 g k_2) = \rho(k_1) P_1 \rho(g) P_2 \rho(k_2)$ is clearly an intertwining distribution between δ_1 and δ_2 .

Now consider by itself the irreducible representation $\delta_1 = \delta$ on K_ν . Let μ be the smallest integer $\geq \nu$ such that K_μ is in the kernel

of δ . Then K_μ will be called the conductor of δ . Set η equal to $\max(\nu, \mu/2)$ or $\max(\nu, (\mu + 1)/2)$ according as μ is even or odd. Then clearly the restriction of δ to K_η is a direct sum of shallow characters of K_η . We associate to δ the collection of cosets of L_η^* in L_μ^* which consist of representatives for the shallow characters of K_η occurring in the restriction of δ to K_η . We denote this subset of \mathfrak{G} by $\beta(\delta)$, and refer to it as the dual blob associated to δ . The main use for the dual blob is to give convenient criteria for intertwining. This next lemma is a simple consequence of the first two lemmas.

LEMMA 2.3. *Let δ_1 and δ_2 be irreducible representations of K_{ν_1} and K_{ν_2} respectively. Then if $g \in G$ intertwines δ_1 and δ_2 , $\beta(\delta_1) \cap \text{Ad } g(\beta(\delta_2)) \neq \emptyset$. It follows that if δ_1 and δ_2 intertwine, $\beta(\delta_1) \cap \text{Ad } G(\beta(\delta_2)) \neq \emptyset$, where $\text{Ad } G(X) = \bigcup_{g \in G} \text{Ad } g(X)$ for $X \subseteq \mathfrak{G}$.*

Proof. We have seen that if g intertwines δ_1 and δ_2 , then the restrictions of δ_1 and $\text{Ad}^* g(\delta_2)$ to $K_{\nu_1} \cap \text{Ad } g(K_{\nu_2})$ must contain a common subrepresentation. Let K_{μ_i} be the conductor of δ_i , and let $\eta_i = \max(\nu_i, [(\mu_i + 1)/2])$, where $[x]$ denotes the greatest integer less than x . Then *a fortiori*, the restrictions of δ_1 and δ_2 to $K_{\eta_1} \cap \text{Ad } g(K_{\eta_2})$ must intertwine. Thus there are shallow characters φ_i of K_{η_i} such that φ_i occurs in the restriction of δ_i to K_{η_i} , and φ_1 and $\text{Ad}^* g(\varphi_2)$ agree on $K_{\eta_1} \cap \text{Ad } g(K_{\eta_2})$. Now applying Lemma 2.1, and the discussion immediately following concerning how representatives of shallow characters transform, the result immediately follows, from the definition of $\beta(\delta_i)$.

In view of this lemma, it is desirable to know something about the geometry of the $\text{Ad } G$ orbits in \mathfrak{G} . In fact, for our present arguments, we do not need to know much, only their very rough shape. To express this, let us introduce the standard ultrametric norm on \mathfrak{G} . Let $|\cdot|_F$ be the usual absolute value of F . That is, if ord_F is the standard valuation on F with $\text{ord}_F(\pi) = 1$, and if $\bar{F} = R/\pi R$ is the residue class field of F , and \bar{F} has q elements, then $|x|_F = q^{-\text{ord}_F(x)}$ for $x \in F$. Now if $m \in \mathfrak{G}$, and $m = \{a_{ij}\}$, $a_{ij} \in F$, put $\|m\| = \max_{i,j} |a_{ij}|_F$. Then $M_n(R) = L_0$ is the "unit ball" of \mathfrak{G} , that is, $L_0 = \{m \in \mathfrak{G}, \|m\| \leq 1\}$. Since $\text{Ad } K_0$ preserves L_0 , $\|\cdot\|$ is $\text{Ad } K_0$ -invariant.

Let \mathfrak{N} denote the set of all nilpotent matrices in \mathfrak{G} . If $X, Y \subseteq \mathfrak{G}$, $X + Y = \{x + y: x \in X, y \in Y\}$. The basic fact about the geometry of $\text{Ad } G$ in which we are interested is that all $\text{Ad } G$ orbits stay close to \mathfrak{N} . Precisely:

LEMMA 2.4. *$\text{Ad } G(L_\nu) \subseteq L_\nu + \mathfrak{N}$. That is, for $g \in G$, $m \in \mathfrak{G}$, $\min_{n \in \mathfrak{N}} \{\|\text{Ad } g(m) - n\|\} \leq \min_{h \in G} \|\text{Ad } h(m)\|$.*

Proof. As is well known, we may write $G = K_0 D^+ K_0$, where D^+ is the semigroup of diagonal matrices with diagonal entries $\{\pi^{l_1}, \pi^{l_2}, \dots, \pi^{l_n}\}$, with $l_i \leq l_j$ for $i \leq j$. We have $\text{Ad } D^+ K_0(L_\nu) = \text{Ad } D^+(L_\nu)$. Now $\text{Ad } D^+$ shrinks subdiagonal matrix entries, leaves diagonal entries alone, and stretches super-diagonal entries. Thus we see $\text{Ad } D^+(L_\nu) \subseteq L_\nu + N^+$, where N^+ is the set of upper triangular nilpotent matrices. Finally $\text{Ad } G(L_\nu) \subseteq \text{Ad } K_0(L_\nu + N^+) \subseteq \text{Ad } K_0(L_\nu) + \text{Ad } K_0(N^+) \subseteq L_\nu + \mathfrak{N}$. The second statement is clearly simply a reformulation of the first. It will be convenient later on.

Thus the shape of $\text{Ad } G$ orbits is determined by \mathfrak{N} . On the other hand, standard linear algebra says the orbit $\text{Ad } G(m)$ is more or less labeled by the eigenvalues of m , so the various Cartan subalgebras of \mathfrak{G} form a system of transversals to the $\text{Ad } G$ orbits. We want to show that, indeed, a given Cartan subalgebra $\mathfrak{A} \subseteq \mathfrak{G}$ is transverse to the $\text{Ad } G$ orbits passing through it in a strong global geometric sense.

As is well known, a Cartan subalgebra \mathfrak{A} of \mathfrak{G} is isomorphic to a direct sum $\bigoplus_i F'_i$ of separable field extensions F'_i of F , with $\sum_i \dim(F'_i/F) = n$. Also the restriction of $\text{tr}(\mathfrak{G}/F)$ to \mathfrak{A} is just the direct sum of the traces of the F'_i . These are nonzero, by a standard result in field theory [20]. We conclude that the form \langle, \rangle is non-degenerate on \mathfrak{A} . Hence we may write $\mathfrak{G} = \mathfrak{A} \oplus \mathfrak{A}^*$. We are concerned with how the points of \mathfrak{N} look in this decomposition. Specifically, for $m \in \mathfrak{G}$, we write $m = m_1 + m_2$ with $m_1 \in \mathfrak{A}$, $m_2 \in \mathfrak{A}^*$.

LEMMA 2.5. *Given \mathfrak{A} , there is a constant $c > 0$ such that for $n \in \mathfrak{N}$, $\|n_2\| \geq c\|n_1\|$.*

Proof. All three of \mathfrak{A} , \mathfrak{A}^* , and \mathfrak{N} are homogeneous closed algebraic subvarieties of \mathfrak{G} . Hence they define closed projective varieties in the projective space of \mathfrak{G} . The statement of the lemma is equivalent to the assertion that the projective varieties defined by \mathfrak{A} and by \mathfrak{N} do not intersect. But that they do not is clear, since \mathfrak{A} consists of semisimple elements, \mathfrak{N} of nilpotent elements.

Having got an idea of the shape of orbits, we return to our study of the intertwining properties of K_ν -types via dual blobs. Take $m \in \mathfrak{G}$ and define $\text{ord}(m)$ by the formula $q^{-\text{ord}(m)} = \|m\|$. Then of course $\text{ord}(m) = \min \text{ord}_F(a_{ij})$ if $m = \{a_{ij}\}$. We have $L_\nu = \{m: \text{ord}(m) \geq \nu\}$. Thus for $\nu \geq 1$, we see m is a representative for a shallow character on K_ν if and only if $-2\nu \leq \text{ord}(m)$, and m does not represent the trivial character of K_ν if and only if $-\nu > \text{ord}(m)$. More precisely, if m represents the shallow character ψ , then if $\text{ord}(m) = -\mu$, K_μ is the conductor of ψ .

Now take a shallow character ψ of K_ν , and consider the set of

all representatives of ψ , that is, $\beta(\psi)$. We will say ψ is inessential if there is $g \in G$ such that $\|\text{Ad } g(m)\| < \|m\|$ for all $m \in \beta(\psi)$. The following alternative description of inessential ψ is very easy, but it seems worthwhile to make it explicit since it is a key link in our reasoning.

LEMMA 2.6. *Let ψ be a shallow character of K_ν , with conductor K_μ . If ψ is inessential then there is $g \in G$ such that $\text{Ad } g(K_\nu) \cap K_\nu$ contains $K_{\mu-1}$ and $\text{Ad}^* g(\psi)$ is trivial on $K_{\mu-1}$. Hence the only representations of K_ν which g intertwines with ψ are shallow characters whose strictly contain K_μ .*

Proof. Since the conductor of ψ is K_μ , we have $\text{ord}(m) = -\mu$ for $m \in \beta(\psi)$. (We exclude the trivial character from consideration. It is easy to see it cannot be inessential.) Choose $g \in G$ such that $\text{ord}(\text{Ad } g(m)) > -\mu$ for every $m \in \beta(\psi)$. Choose γ as small as possible so that $K_\gamma \subseteq K_\nu \cap \text{Ad } g(K_\nu)$. Then $\text{Ad}^* g(\psi)$ agrees with some shallow character φ of K_γ . By Lemma 2.1 and the discussion following it, we have $\beta(\varphi) \cap \text{Ad } g(\beta(\psi)) \neq \emptyset$. Hence $\beta(\varphi)$ contains elements y such that $\text{ord}(y) > -\mu$. If φ is nontrivial, then necessarily $\text{ord}(y) < -\gamma$. In this case $\gamma < \mu - 1$, so $K_{\mu-1} \subseteq \text{Ad } g(K_\nu)$, and φ and $\text{Ad}^* g(\psi)$ agree and are trivial on $K_{\mu-1}$. Thus the lemma holds in this case. If φ is trivial, then $K_{\gamma-1} \not\subseteq \text{Ad } g(K_\nu)$, and $\text{Ad}^* g(\psi)$ is trivial on K_γ . Therefore we may find a nontrivial shallow character of $K_{\gamma-1}$ which agrees with $\text{Ad}^* g(\psi)$ on $K_{\gamma-1} \cap \text{Ad}^* g(K_\nu)$. Following the same line of reasoning as in the first case, we see the lemma holds in this case too, so it is true.

Since $\text{Ad } K_0$ preserves $\|\cdot\|$ and K_ν is normal in K_0 , we see that if ψ is inessential, then $\text{Ad}^* k(\psi)$ is inessential for any $k \in K_0$. Now let δ be an irreducible representation of K_ν and let K_μ be the conductor of δ_1 and put $\eta = \max(\nu, [(\mu + 1)/2])$ as before. Then $\beta(\delta)$ is by definition the union of $\beta(\psi)$ for the shallow characters ψ of K_η occurring in the restriction of δ to K_η . By standard representation theory for finite groups ("Clifford theory", see [3]), if ψ_1 and ψ_2 are two shallow characters of K_η occurring in the restriction of δ , then $\psi_2 = \text{Ad}^* k(\psi_1)$ for some $k \in K_\nu$. We say δ is inessential if one, and hence all, of the ψ occurring in the restriction of δ to K_η is inessential. If δ is not inessential, then we will say δ is essential. This division of \hat{K}_ν into essential and inessential representations will be another aid in the study of intertwining properties of the δ 's. One reason the essential representations are useful is brought out in this next lemma.

LEMMA 2.7. *Let ρ be any representation of G . Then for any*

K_ν , essential K_ν -types occur in ρ . In fact, suppose μ is the smallest integer such that K_μ is a conductor of a K_ν -type occurring in ρ . Then all K_ν -types in ρ of conductor K_μ are essential.

Proof. As usual, put $\eta = \max(\nu, [(\mu + 1)/2])$, and consider the K_η -types occurring in ρ . By hypothesis and choice of η , there is a shallow character ψ of K_η of conductor μ occurring in ρ . Let v be a vector in the space of ρ such that $\rho(k)v = \psi(k)v$ for $k \in K_\eta$. Then for $g \in G$, and $k \in K_\eta \cap \text{Ad } g(K_\eta)$, we have

$$\rho(k)\rho(g)(v) = \rho(g)\rho(g^{-1}kg)(v) = \psi(g^{-1}kg)\rho(g)(v) = \text{Ad}^* g(\psi)(k)\rho(g)(v).$$

Thus under K_η , v must transform according to representations which agree with $\text{Ad}^* g(\psi)$ on $K_\eta \cap \text{Ad}^* g(K_\eta)$, in other words, representations which g intertwines with ψ . But if ψ is inessential, we may find $g \in G$ such that the only representations of K_η which g intertwines with ψ are shallow characters of conductor strictly containing K_μ . By choice of μ , such characters do not occur in ρ . Hence ψ cannot be inessential, and the lemma is proved.

We note that Lemma 2.7 is a slightly sharpened version of statement (a) of Theorem 1. We will prove the rest of Theorem 1 in the next few lemmas. The next step is a geometric criterion on δ which is necessary for $\delta \in \hat{K}_\nu$ to be essential.

LEMMA 2.8. *Let ψ be a shallow character of K_ν and let $m \in \beta(\psi)$ be a representative for ψ . Let K_μ be the conductor of ψ . Then if $\mu \geq \nu + n + 1$, a necessary condition for ψ to be essential is $\|m\| \leq q^n \min_{g \in G} \|\text{Ad } g(m)\|$.*

REMARK. Lemma 2.8 may be sharpened by assuming m belongs to some particular parabolic subgroup of G . The constant bounding m in terms of its "spectral radius" may be made to depend on the parabolic in question, and is smaller for smaller parabolics. For the Borel, the constant is 1.

Proof. By Lemma 2.4 we may write $m = m' + y$ with $\|m'\| = \min_{g \in G} \|\text{Ad } g(m)\|$ and $y \in \mathfrak{N}$. Since $G = K_0 AU$, where $AU = B$ is the Borel subgroup of upper triangular matrices, we may, up to conjugation by K_0 , assume $y \in N^+$, the upper triangular nilpotent matrices. Let g be the diagonal matrix with entries $(\pi^n, \pi^{n-1}, \dots, \pi)$. We have $\beta(\psi) = m' + y + L_{-\nu}$. If $x = \{a_{ij}\}$ then $\text{Ad } g(x) = \{\pi^{j-i} a_{ij}\}$. We conclude $\text{Ad } g(L_{-\nu}) \subseteq L_{-\nu-n}$, $\text{ord}(\text{Ad } g(m')) \geq \text{ord}(m') - n$, and $\text{ord } \text{Ad } g(y) \geq \text{ord}(y) + 1$. Therefore, if

$$\text{ord}(y) + n + 1 \leq \min(-\nu, \text{ord}(m')),$$

we see $\|\text{Ad } g(x)\| < \|x\|$ for every $x \in \beta(\psi)$, so ψ is inessential. Assuming ψ is nontrivial, and $\|m'\| < \|m\|$, as we may without loss of generality, we see $\text{ord}(m) = \text{ord}(y) = -\mu$. Therefore, if we have $-\mu + n + 1 \leq -\nu$ and $\text{ord}(m) + n + 1 \leq \text{ord}(m')$, then ψ is inessential. The conditions of the lemma are direct translations of these.

We have two corollaries of Lemma 2.8 which by Lemma 2.2 are sharpenings of statement (b) of Theorem 1. For purposes of the second corollary, we define the conductor of an irreducible representation ρ of G to be the group K_μ which is minimal with respect to the property of containing the conductors of all K_0 -types occurring in ρ .

COROLLARY 1. *Let $\delta \in \hat{K}_\nu$ have conductor K_μ . Let $\delta' \in \hat{K}_{\nu'}$ be essential and have conductor $K_{\mu'}$. Then in order for δ to intertwine with δ' it is necessary that $\mu' \leq \max(\nu', \mu, n) + n$. In particular δ can intertwine with only finitely many essential $K_{\nu'}$ -types.*

Proof. If δ has conductor K_μ , then $\beta(\delta) \subseteq L_{-\mu}$. In order for δ and δ' to intertwine, therefore, it is by Lemma 2.3 necessary that $\text{Ad } G(\beta(\delta'))$ intersect $L_{-\mu}$. Suppose $\mu' \geq \max(\nu', \mu) + n + 1$, and consider $m \in \beta(\delta')$. By Lemma 2.8, $\max_{g \in G} \text{ord}(\text{Ad } g(m)) \leq -\mu' + n < -\mu$, so $\text{Ad } G(m)$ does not intersect $L_{-\mu}$. Since m was arbitrary in $\beta(\delta')$, the corollary is proved.

COROLLARY 2. *Let ρ be an irreducible representation of G , and let K_μ be the conductor of ρ . Let $\delta' \in \hat{K}_{\nu'}$ be essential, and let $K_{\mu'}$ be the conductor of δ' . Then if δ' occurs in ρ , $\mu' \leq \max(\nu', \mu, n) + n$.*

Proof. By the definition of the conductor of ρ , there exists a K_0 -type δ of conductor K_μ occurring in ρ . If δ' occurs in ρ , δ and δ' intertwine by Lemma 2.2, and then Corollary 1 gives the result.

Since there are only finitely many K_ν -types with given conductor, it is immediate from Corollary 2 that only finitely many essential $K_{\nu'}$ -types occur in ρ .

We will now finish proving Theorem 1. We have left part (c) and the corollary. Take $\delta \in \hat{K}_\nu$ and let K_μ be the conductor of δ . Let ρ be a representation of G . We want to bound the multiplicity of δ in ρ in terms of the multiplicities of the essential K_ν -types occurring in ρ . We will assume that for $\delta' \in \hat{K}_{\nu'}$ of conductor $K_{\mu'}$, with $\mu' < \mu$, we have such a bound. (If μ' is sufficiently small, then δ' is essential by Lemma 2.7, and the desired bound is trivial there, so this assumption amounts to an inductive hypothesis.) Put $\eta = \max(\nu, \lceil (\mu + 1)/2 \rceil)$ and let ψ be a shallow character of K_η occurring in the restriction of δ to K_η . Then the multiplicity of δ

in ρ is no larger than the multiplicity of ψ in ρ . If ψ is essential, then so is δ by definition, and there is nothing to prove. If ψ is inessential, then by Lemma 2.6 we may find $g \in G$ such that $\text{Ad}^* g(\psi)$ intertwines only with shallow characters of K_γ of conductor $K_{\mu'}$ with $\mu' < \mu$. These shallow characters can only appear in the restrictions of K_γ -types δ' such that the conductor of δ' is $K_{\mu'}$ with $\mu' < \mu$. Therefore, the multiplicities of each of these shallow characters of K_γ are bounded by some constant times the sum of the essential K_γ -types in ρ . Since there are only finitely many of these shallow characters, and the multiplicity of ψ is at most the sum of their multiplicities, the desired bound on the multiplicity of ψ , and hence of δ follows. This establishes part (c) of Theorem 1.

To prove the corollary, suppose ρ is admissible and let X be the space of ρ . Let $X = \bigoplus_{\delta \in \hat{K}_0} X_\delta$ be the decomposition of X into K_0 -types. If ρ is finitely generated, there are vectors $\{v_i\}_{i=1}^k$ such that the vectors $\{\rho(g)(v_i)\}$ span X . We may find finitely many K_0 -types $\{\delta_j\}_{j=1}^l$ such that the v_i are contained in the direct sum of the X_{δ_j} . Now exactly the same reasoning as in Lemma 2.2 shows that if δ' occurs in ρ , then δ' must intertwine with one of the δ_j . Since each δ_j can intertwine with only finitely many essential K_0 -types, there are only finitely many essential K_0 -types occurring in ρ . Let X_0 be the direct sum of the X_{δ_j} with δ_j essential. Then X_0 has finite dimension since ρ is admissible. Now let $Y \subseteq X$ be an invariant subspace. Then $Y \cap X_0 \neq \{0\}$ by Lemma 2.7. Suppose among all invariant subspaces Y' , Y minimizes $\dim(Y \cap X_0)$. Then Y must be irreducible. For if $Y_1 \subseteq Y$ is invariant, then $Y_1 \cap X_0 = Y \cap X_0$ by the minimality of $\dim(Y \cap X_0)$. Therefore the quotient representation Y/Y_1 contains no essential K_0 -types and so is trivial. So $Y_1 = Y$. Considering now the quotient representation X/Y , we have

$$\dim(X/Y)_0 < \dim X_0,$$

so by induction, we may assume X/Y has a finite composition series. So, then, does X .

We should point out that the above result may be made more precise in a useful way. Namely, since X_0 is defined as the direct sum of certain K_0 -isotypic components of X , the projection E of X onto X_0 , whose kernel is the sum of the remaining K_0 -types, is the image under ρ of some function in $C_c^\infty(K_0)$. It follows that, if $Z_0 \subseteq X_0$ is a subspace invariant under the operators $E\rho(g)E$, and if Z is the G -invariant subspace of X generated by Z_0 , then $Z \cap X_0 = Z_0$. Clearly $Z \cap X_0 \supseteq Z_0$. On the other hand, suppose for some $z_0 \in Z_0$, some $f \in C_c^\infty(G)$, and some $x_0 \in X_0$, the equation $X_0 = \rho(f)(z_0)$. Then also $x_0 = Ex_0 = E\rho(f)E(z_0)$. Since the isotropy group of any $x_0 \in X_0$

is open in G , we may find constants a_i and elements $g_i \in G$ such that $E\rho(f)E = \sum a_i E\rho(g_i)E$. Hence $x_0 = \sum a_i E\rho(g_i)E(z_0)$, so $x_0 \in Z_0$, as desired. Since conversely, every G -invariant subspace of X must intersect X_0 nontrivially as we have seen, we conclude: there is a one-to-one correspondence between G -invariant subspaces of X and subspaces of X_0 which are invariant under the operators $E\rho(g)E$. We note in particular the length of any composition series for ρ is at most the length of a composition series for the operators $E\rho(g)E$ acting on X_0 . We may trivially majorize this length by the length of a K_0 -composition series for X_0 , or by $\dim X_0$.

We would now like to give some complements to Theorem 1. One concerns series of induced representations, and is an application of the discussion of the previous paragraph. The other concerns the structure of spherical function algebras. We will have to recall some general notions (see [1], [9]) in order to formulate them.

A parabolic subgroup P of G is a group containing a conjugate of B , the Borel subgroup of upper triangular matrices. A nested family of subspaces $\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_k = F^n$ in F^n is called a flag. A parabolic group is describable as a group which preserves some flag in the sense that, if P is the group preserving the flag $\{V_i\}$, then $p\{V_i\} = V_i$ for any $p \in P$ and all i . The unipotent radical of P , denoted U_P , is the set of $u \in P$ such that u acts as the identity on each quotient V_i/V_{i-1} . Then U_P is normal in P , and $P/U_P \simeq \bigoplus_{i=1}^k GL(V_i/V_{i-1})$. We may find a "Levi component", a subgroup $M_P \subseteq P$ such that $P = M_P U_P$ and $M_P \cap U_P = \{1\}$. Then $M_P \simeq \bigoplus_{i=1}^k GL(V_i/V_{i-1})$ also. For each i , we get a map $\det_i: M_P \rightarrow F^\times$ by considering for $m \in M_P$, the determinant of the linear transformation m defines on V_i/V_{i-1} . Then we may combine the \det_i to obtain a homomorphism $\log: M_P \rightarrow Z^k$, where

$$\log m = (\text{ord}_F \det_1(m), \dots, \text{ord}_F \det_k(m)).$$

We extend \log to P by letting it be trivial on U_P . By a quasi-character of P , we mean a homomorphism of P into C^\times , the multiplicative group of complex numbers. We denote by \mathfrak{Q} the collection of quasicharacters φ of P of the form $\varphi = \varphi' \circ \log$, where φ' is a quasicharacter of Z^k . Obviously \mathfrak{Q} is a group, isomorphic to $(C^\times)^k$. We refer to elements of \mathfrak{Q} as principal quasicharacters of P .

Let σ be an irreducible supercuspidal representation of M_P . Supercuspidal means the matrix coefficients of σ are compactly supported modulo the center of M_P . We extend σ to P by letting it be trivial on U_P . We consider the collection $\mathfrak{Q}(\sigma)$ of representations $\sigma_\varphi = \sigma \otimes \varphi$, for $\varphi \in \mathfrak{Q}$. We see $\mathfrak{Q}(\sigma)$ is in an obvious manner a homogeneous space for \mathfrak{Q} , and the isotropy group of σ must be finite, as one sees by looking at the restriction of $\sigma \otimes \varphi$ to the center

of M_p . Hence we may endow $\Omega(\sigma)$ with the structure of complex manifold, isomorphic to the quotient of $(C^X)^k$ by some finite subgroup.

Now we consider the series of representations of G induced from the collection $\Omega(\sigma)$. More precisely, let X be the space of σ , and consider the space $Y = Y(\sigma, \varphi)$ of all locally constant functions f from G to X such that $f(gp) = \sigma(p)^{-1}\varphi(p)^{-1}f(g)$. The natural left action of G on the space of X -valued functions on G preserves the space $Y(\sigma, \varphi)$, and the presentation of G thus defined on $Y(\sigma, \varphi)$ is called the representation induced from $\sigma \otimes \varphi$, and denoted $\text{ind}_P^G(\sigma \otimes \varphi)$, or $\pi(\sigma, \varphi)$. It may be seen easily that for a certain collection of φ , filling out a real subvariety (actually, a real k -dimensional torus) of $\Omega(\sigma)$, the representations $\pi(\sigma, \varphi)$ are unitary with respect to a natural inner product. Furthermore, Harish-Chandra [10] has shown that almost all of these unitarizable representations are irreducible. Here we are concerned with the non-unitarizable representations.

PROPOSITION 2.1. *The collection of representations $\sigma \otimes \varphi$ in $\Omega(\sigma)$ such that $\pi(\sigma, \varphi)$ is reducible form a proper complex analytic subvariety. There is a finite bound on the number of components into which any $\pi(\sigma, \varphi)$ may decompose.*

Proof. We have $G = K_0P$. We see that $f \in Y(\sigma, \varphi)$, by virtue of its transformation law, is determined by its restriction to K_0 . Moreover, since $\varphi \in \Omega$ is trivial on K_0 , the space of functions on K_0 which are restrictions of functions in $Y(\sigma, \varphi)$, is independent of φ . Call this space $Y(\sigma)$. Then $\pi(\sigma, \varphi)$ may be regarded as acting on $Y(\sigma)$. From this point of view, it is not hard to convince oneself that, for fixed $g \in G$, the operator $\pi(\sigma, \varphi)(g)$ varies holomorphically in φ , in the sense that, if E is the projection of $Y(\sigma)$ onto some finite-dimensional subspace, then $E\pi(\sigma, \varphi)(g)E$ is holomorphic in φ in the obvious sense.

Since K_0 acts on $Y(\sigma)$ in the obvious way, by left translation, the restriction of $\pi(\sigma, \varphi)$ to K_0 is independent of φ . Since at least one $\pi(\sigma, \varphi)$ is irreducible, the same reasoning as in the proof of the corollary of Theorem 1 shows that the space $Y(\sigma)_0$, consisting of the span of the essential K_0 -types, is finite-dimensional. It is of course independent of φ . By the discussion after the proof of Corollary 1, the length of a composition series for $\pi(\sigma, \varphi)$ is bounded by $\dim Y(\sigma)_0$, the second statement of the proposition is proved.

Now let E be the projection of $Y(\sigma)$ onto $Y(\sigma)_0$, with K_0 -invariant kernel. Then $E\pi(\sigma, \varphi)(g)E$ is, as a function of φ , a holomorphically varying operator on $Y(\sigma)_0$. The condition that the $E\pi(\sigma, \varphi)(g)E$, for all $g \in G$, have a common proper invariant subspace of $Y(\sigma)_0$ is clearly a holomorphic condition on φ , which is not identically satisfied, since

for at least one φ , $\pi(\sigma, \varphi)$ is irreducible. Hence the set of φ for which there is a proper subspace of $Y(\sigma)_0$ -invariant under $E\pi(\sigma, \varphi)(g)E$ for all $g \in G$ defines a proper analytic subvariety of $\mathfrak{Q}(\sigma)$. But by the discussion following the proof of the corollary to Theorem 1, this is the subvariety for which $\pi(\sigma, \varphi)$ will reduce. This finishes the proposition.

REMARK. The fact that the invariant subspaces of $Y(\sigma)$ under $\pi(\sigma, \varphi)$ may be determined by looking at the operators $E\pi(\sigma, \varphi)E$ on the finite-dimensional space $Y(\sigma)_0$ could be of use in calculating for specific σ when $\pi(\sigma, \varphi)$ reduces and what its components are.

The second complement to Theorem 1 is essentially a reformulation of part (c) of Theorem 1 in terms of spherical function algebras. Again we must recall a few notions. Let δ be an irreducible representation of K_v . Denote by $\mathcal{H}(\delta)$ the space of all compactly supported intertwining operators between δ and itself. An element of $\mathcal{H}(\delta)$ is a compactly supported function from G to $\text{Hom}(V, V)$ (where V is the space of δ) satisfying a certain transformation law. Now the space of all compactly supported functions from G to $\text{Hom}(V, V)$ is an algebra under convolution, and it is easy to see that $\mathcal{H}(\delta)$ is closed under multiplication, and so forms a subalgebra. We refer to $\mathcal{H}(\delta)$ with this algebra structure as the Hecke algebra, or spherical function algebra of δ .

It is well-known that a representation ρ of G gives rise to a representation $\rho(\delta)$ of $\mathcal{H}(\delta)$. We sketch how this happens. Let Y be the space of ρ , and V the space of δ . The convolution algebra $C_c^\infty(G, \text{Hom}(V, V))$ of compactly supported locally constant $\text{Hom}(V, V)$ -valued functions on G is naturally isomorphic to $C_c^\infty(G) \otimes \text{Hom}(V, V)$. Consequently, we get a representation $\tilde{\rho}$ of $C_c^\infty(G, \text{Hom}(V, V))$ on $Y \otimes V$. The explicit formula giving the representation is $\tilde{\rho}(f) = \int_G \rho(g) \otimes f(g) dg$, for $f \in C_c^\infty(G, \text{Hom}(V, V))$. Note that $\tilde{\rho}$ is irreducible if and only if ρ is.

Now let $e_\delta \in C_c^\infty(G, \text{Hom}(V, V))$ be defined by $e_\delta(k) = \delta(k)$ for $k \in K_v$, and $e_\delta = 0$ outside K_v . Then, assuming Haar measure to be normalized so that K_v has measure one, e_δ is an idempotent, belongs to and is the identity of $\mathcal{H}(\delta)$, and $f \rightarrow e_\delta * f * e_\delta$ is a projection of $C_c^\infty(G, \text{Hom}(V, V))$ onto $\mathcal{H}(\delta)$. Thus for $f \in \mathcal{H}(\delta)$, we have $\tilde{\rho}(f) = \tilde{\rho}(e_\delta)\tilde{\rho}(f) = \tilde{\rho}(f)\tilde{\rho}(e_\delta)$. On the other hand, one has $\tilde{\rho}(e_\delta) = \int_{K_v} \rho(k) \otimes \delta(k) dk$. Therefore $(\rho(k) \otimes \delta(k))\tilde{\rho}(e_\delta) = \tilde{\rho}(e_\delta) = \tilde{\rho}(e_\delta)(\rho(k) \otimes \delta(k))$, and one sees that $\tilde{\rho}(e_\delta)$ is the projection of $Y \otimes V$ onto the space E of fixed vectors for the representation $\rho \otimes \delta$ of K_v on $Y \otimes V$. Furthermore $\tilde{\rho}(\mathcal{H}(\delta))$ consists of all operators in $\tilde{\rho}(C_c^\infty(G, \text{Hom}(V, V)))$ which preserve E and annihilate the K_v -stable complement of E . Thus ρ

gives rise essentially to a representation of $\mathcal{H}(\delta)$ on E . If ρ is irreducible, then so will $\mathcal{H}(\delta)$ act irreducibly on E , by a standard argument. But now we may regard V as $(V^*)^*$, which gives $Y \otimes V \simeq \text{Hom}(V^*, Y)$. From this point of view, E is precisely the space of intertwining operators between the representations δ^t and ρ of K_v , where δ^t indicates the representation of K_v on V^* adjoint to δ . In summary: a representation ρ of G gives rise to a representation of $\mathcal{H}(\delta)$, which is irreducible if ρ is, and which is of dimension equal to the multiplicity of δ^t in ρ .

Thus we see the degrees of the representations of the $\mathcal{H}(\delta)$ control the multiplicities with which δ^t can occur in representations of G and conversely. (It is easy to see that enough representations of $\mathcal{H}(\delta)$ arise from representations of G in the above way to separate the points of $\mathcal{H}(\delta)$ — consider $\mathcal{H}(\delta)$ acting on $L^2(G)$.) We will say an algebra A is mildly non-abelian if for some integer s , A has sufficiently many representations of degree at most s to separate the points of A ; that is, no element of A is in the kernel of the representations of A of degree at most s . This is known [4] to imply that all irreducible representations of A have degree at most s , because A then satisfies a certain polynomial identity. In Harish-Chandra's terminology [9], if A is mildly non-abelian, A is s -abelian for some s . With this lengthy recollection finished, we may quickly dispense with our result.

PROPOSITION 2.2. *If $\mathcal{H}(\delta)$ is mildly non-abelian for all essential $\delta \in \hat{K}_v$, then $\mathcal{H}(\delta)$ is mildly non-abelian for all $\delta \in \hat{K}_v$.*

Proof. Suppose $\delta \in \hat{K}_v$. Then by Theorem 1(b), δ^t can occur in an irreducible representation ρ of G with finitely many admissible K_v -types $\{\delta'_i\}_{i=1}^l$ and the multiplicity of δ in ρ is bounded by a constant c times the sum of the multiplicities of the δ'_i . If $\mathcal{H}(\delta'_i)$ is mildly non-abelian for each i (note that δ^t is essential or inessential according as δ is since $\beta(\delta^t) = -\beta(\delta)$), then δ'_i can occur at most some finite constant m_i times in ρ . Thus δ^t can occur at most $c(\sum_{i=1}^l m_i)$ times in ρ , and so the representation of $\mathcal{H}(\delta)$ derived from ρ has dimension at most $c(\sum_{i=1}^l m_i)$. Since as ρ varies, these representations separate $\mathcal{H}(\delta)$, $\mathcal{H}(\delta)$ is indeed mildly non-abelian.

REMARK. By virtue of this proposition, it becomes interesting to know the structure of $\mathcal{H}(\delta)$ for essential δ . In the next section we will show that for many essential δ , $\mathcal{H}(\delta)$ is abelian.

We turn now to the consideration of characters. Let ρ be an irreducible representation of G on a space X . Let $X = \sum_{\delta \in \hat{K}_v} X_\delta$ be

the decomposition of X into K_ν -types. Let E_δ be the K_ν -commuting projection of X onto X_δ . As we said in the introduction, we are concerned with the partial traces $\theta_{\rho,\delta}(g) = \text{tr } E_\delta \rho(g) E_\delta$. The main observation we use in our analysis is the following slight refinement of Lemma 2.2, which we simply state: If $\theta_{\rho,\delta}(g)$ is nonzero, then $E_\delta \rho(g) E_\delta \neq 0$, and hence g intertwines δ with itself. As a corollary to this, suppose $K_\eta \subseteq K_\nu$, and g intertwines no irreducible component of the restriction of δ to K_η with itself. Then $\theta_{\rho,\delta}(g) = 0$. Thus we are led to look further at the intertwining properties of K_ν -types.

Let ψ be a shallow character of some K_η . If the conductor of ψ is $K_{2\eta}$ or $K_{2\eta-1}$, if ψ is inessential, and if $\beta(\psi)$ contains a nilpotent matrix, then we will say ψ is highly inessential. By Lemma 2.8, if the conductor of ψ is $K_{2\eta}$, and if $\beta(\psi) \cap \mathfrak{N} \neq \emptyset$, then in order for ψ to be inessential, hence highly inessential, it is sufficient that η be $\geq n + 1$. Let $\delta \in \hat{K}_\nu$ have conductor K_μ , and put

$$\eta = \max(\nu, [(\mu + 1)/2]).$$

Then $\beta(\delta)$ is the union of $\beta(\psi)$ for certain shallow characters ψ of K_η . If $\mu \geq 2\nu - 1$, then these characters have conductor $K_{2\eta}$ or $K_{2\eta-1}$. Thus we will say, if the conductor of $\delta \in \hat{K}_\nu$ is K_μ with $\mu \geq 2\nu - 1$, that δ is highly inessential if one, and hence all, of the shallow characters ψ of K_η occurring in the restriction of δ to K_η , is highly inessential. Equivalently, if $\nu > n$, $\delta \in \hat{K}_\nu$ is highly inessential if the conductor of δ is K_μ with $\mu \geq 2\nu - 1$ and $\beta(\delta) \cap \mathfrak{N} \neq \emptyset$. With this terminology, we may state two lemmas which together will imply Theorem 2.

LEMMA 2.9. *All but a finite number of K_ν -types appearing in ρ are highly inessential. More precisely, if K_μ is the conductor of ρ , and if the conductor of $\delta \in \hat{K}_\nu$ is K_μ , and δ occurs in ρ , then δ is highly inessential if $\mu > 2 \max(n, \nu, \mu')$.*

Proof. If $\delta \in \hat{K}_\nu$ occurs in ρ , then δ must intertwine with the trivial representation of K_μ , which means $\beta(\delta)$ intersects $\text{Ad } G(L_{-\mu'})$. Thus $\beta(\delta)$ intersects $L_{-\mu'} + \mathfrak{N}$. Now $\beta(\delta)$ is a union of cosets of $L_{-\eta}$, with $\eta = \max(\nu, [(\mu + 1)/2])$. If $\eta > \mu'$, then evidently $\beta(\delta)$ intersects \mathfrak{N} . Since $2\eta \geq \mu$, our hypothesis on μ shows indeed $\eta > \mu'$, and also guarantees that the other criteria given just above for δ to be highly inessential are satisfied.

LEMMA 2.10. *Let X be a compact subset of regular elements of a Cartan subgroup A of G . Then there is $\nu > 0$ such that no x in X will intertwine any highly inessential K_ν -type with itself.*

Proof. Let \mathfrak{A} be the Cartan subalgebra of \mathfrak{G} corresponding to A . Write $\mathfrak{G} = \mathfrak{A} \oplus \mathfrak{A}^*$. Since X is compact, we can find an integer $b > 0$ such that $\text{ord}(\text{Ad } x(y)) \geq \text{ord } y - b$ for x in X and y in \mathfrak{G} . Since X consists of regular elements, if we choose b large enough, we will also have $\text{ord}(\text{Ad } x(y) - y) \leq \text{ord } y + b$ for y in \mathfrak{A}^* . Hence, by Lemma 2.5, we can also choose b large enough so that if y is in \mathfrak{N} , then $\text{ord}(\text{Ad } x(y) - y) \leq \text{ord } y + b$. Put $\nu = 2b + 1$. Let δ be a highly inessential K_ν -type of conductor K_μ . Choose y in $\beta(\delta) \cap \mathfrak{N}$. If $\eta = [(\mu + 1)/2]$ (so that, by definition of highly inessential K_ν -type, we have $\eta \geq \nu$), we know that $\beta(\delta) = \text{Ad } K_\nu(y) + L_{-\eta}$. We have $\text{ord}(y) = -\mu$. Thus $\text{Ad } K_\nu(y) \subseteq y + L_{\nu-\mu}$. Hence $\beta(\delta) \subseteq y + L_{\nu-\mu}$. Therefore, for x in X , we have $\text{Ad } x(\beta(\delta)) \subseteq \text{Ad } x(y) + L_{\nu-\mu-b}$. If x intertwines δ with itself then $\beta(\delta)$ intersects $\text{Ad } x(\beta(\delta))$, which means $\text{Ad } x(y) - y$ is in $L_{\nu-\mu-b}$. But by choice of b ,

$$\text{ord}(\text{Ad } x(y) - y) \leq -\mu + b.$$

Whence $-\mu + b \geq \nu - \mu - b$, or $\nu \leq 2b$, which is false. Thus the lemma is proved.

It is standard that if $X \subseteq \mathfrak{A}$ is a compact open set of regular elements, then $\text{Ad } K_\nu(X)$ is an open set in G . Therefore if $\omega \subseteq G'$ is a compact set of regular elements, we can find a finite number of Cartan subgroups A_i and compact sets X_i contained in $A_i \cap G'$, such that $\omega \subseteq \text{Ad } K_\nu(\bigcup_i X_i)$.

From Lemmas 2.9 and 2.10, and the discussion preceding the definition of highly inessential K_ν -type, we see that for any irreducible admissible ρ , we have $\theta_{\rho,\delta}(x) = 0$ for x in $\bigcup_i X_i$, for all but a finite number of K_ν -types δ . Since each $\theta_{\rho,\delta}$ is invariant under conjugation by K_ν , it follows that $\theta_{\rho,\delta} \equiv 0$ on ω for all but a finite number of δ , so Theorem 2 is proved.

3. Fourier analysis of sufficiently regular K -types. In the previous section we distinguished certain classes of K_ν -types and tried to illustrate the different roles they played in harmonic analysis on G . In particular we saw that the essential K_ν -types served to partition the representations of G into relatively small sets, with a given K_ν -type occurring only in a few representations of G . This section expresses this fact in a more quantitative way. We will select a certain set of essential K_ν -types and describe in some detail the representations of G in which these K_ν -types occur. Actually, in our construction, we will not deal directly with the K_ν 's, but with certain subgroups constructed from the K_ν 's. The groups we construct have a more convenient shape for computational purposes than do the K_ν 's. The substantive differences are slight. Specifically, we will construct a certain representation, denoted by δ' , of a certain

open compact subgroup J' . We will give an explicit parametrization for those representations of G in which δ' occurs, and we will give explicit realizations of these representations as induced representations. We will show that the unitary representations in which δ' occurs are all irreducible. We will show δ' occurs at most once in any representation of G . On the basis of these results, we will give the Fourier analysis of the algebra $\mathcal{H}(\delta'')$, where δ'' is the representation of J' contragredient to δ' . We will show $\mathcal{H}(\delta'')$ is abelian and isomorphic to an affine algebra. We will compute explicitly the δ'' -spherical functions and the Fourier transform on $\mathcal{H}(\delta'')$. From this we will compute the Plancherel measure of the representations in which δ' occurs. The unitary representations of G in which δ' occurs will be parametrized by a certain finite number of real tori (e.g., products of circles), and the Plancherel measure on these tori will be ordinary Lebesgue measure suitably normalized.

The considerations of this section are “local” in the sense that they concern first the construction of a single representation δ' , then a study of the properties of δ' in relation to harmonic analysis on G . In the next section we will deal with more “global” aspects of representation theory for G . We will show that in a certain sense, the δ' of this section account for “most” of the Fourier analysis of G . Of course, in another and very important sense, it is precisely the representations for which we cannot account that are most interesting. We will discuss this and similar problems in the next section.

We want to emphasize what will no doubt be apparent: the considerations of this section represent only the next stage in precision beyond §2. They are still quite crude. Nevertheless the arguments are noticeably fussier than in §2. The crudity is partly to ease the exposition, but also partly from present lack of understanding. In various favorable cases (see [12] and [13] for examples) refinements of the techniques used here lead to quite complete results on certain series of representations. From the point of view implicit in these methods, it is an important problem to extend the list of favorable cases.

We have seen that, roughly speaking, the dual blobs of essential K_λ -types lie along Cartan subalgebras of \mathfrak{G} , whereas inessential K_λ -types lie along \mathfrak{N} . Therefore we fix a Cartan subalgebra \mathfrak{A} of \mathfrak{G} and look at those K_λ -types intersecting \mathfrak{A} . By imposing a geometrical condition on \mathfrak{A} and by looking at a certain subset of \mathfrak{A} , consisting elements which are “sufficiently regular” in a certain sense, we will arrive at a set of representations we can analyze. In the next section, we will show our conditions are not too restrictive. Before going into the details of the conditions, which are somewhat techni-

cal, let us describe their value. One may roughly say that if one could analyze $\mathcal{H}(\delta)$ for most or all K_ν -types δ , the Fourier analysis of $L^2(G)$ would follow. Thus given $\delta \in \hat{K}_\nu$, one might try to find out the structure of $\mathcal{H}(\delta)$. As a first step, one could try to determine which (K_ν, K_ν) double cosets support nontrivial elements of $\mathcal{H}(\delta)$. As an approximation to this, guided by Lemma 2.3, we could try to find those $g \in G$ such that $\text{Ad } g(\beta(\delta)) \cap \beta(\delta)$ is nonempty. Clearly if $\text{Ad } g$ fixes some point in $\beta(\delta)$, then indeed $\text{Ad } g(\beta(\delta)) \cap \beta(\delta) \neq \emptyset$. Thus, making another approximation, we could try to determine the isotropy groups of the points of $\beta(\delta)$. The point of our condition on \mathfrak{A} and of the notion of sufficient regularity is that they make this latter problem very easy and simultaneously make it essentially equivalent to the original problem. We now plunge into the details.

Let A be the Cartan subgroup of G corresponding to \mathfrak{A} . In A there is a unique maximal compact subgroup A_0 , and $A/A_0 \simeq \mathbb{Z}^r$ for an appropriate integer r , called the split rank of A . Also in A is A_s , the maximal split subtorus of A . We have $A_s \simeq (F^\times)^r$ and $A/(A_0 A_s)$ is finite. Let M be the centralizer of A_s in G . It is a standard fact that M is the Levi component of a certain finite number of parabolic subgroups, which are in one-to-one correspondence with orderings of the roots of A_s . (See [1].) Let P be one of these parabolics, and let \bar{P} be the "opposite" parabolic to P — that is, \bar{P} corresponds to the ordering of the roots of A_s opposite to the ordering defining P . We have $P = M \cdot U_P$, where U_P is the unipotent radical of P , i.e., the maximal normal unipotent subgroup of P . Likewise $\bar{P} = M \cdot U_{\bar{P}}$, where $U_{\bar{P}}$ is the "opposite" of U_P . Let \mathcal{P} , \mathfrak{M} , \mathcal{U} , and $\bar{\mathcal{U}}$ be the Lie algebras of P , M , U_P , and $U_{\bar{P}}$ respectively. Then $\mathfrak{G} = \mathfrak{M} \oplus \mathcal{U} \oplus \bar{\mathcal{U}}$, and all three spaces are invariant by $\text{Ad } A$. Let M_ν , P_ν , etc., be the intersections of M , P , etc., with K_ν . Let \mathfrak{M}_ν , \mathcal{P}_ν , etc., be the intersections of \mathfrak{M} , \mathcal{P} , etc., with L_ν . Then we assume that for $\nu \geq 1$, $L_\nu = \bar{\mathcal{U}}_\nu \oplus \mathfrak{M}_\nu \oplus \mathcal{U}_\nu$, or what is equivalent, since it is easily seen that $M_\nu = 1 + \mathfrak{M}_\nu$ and so forth, that $K_\nu = (U_{\bar{P}})_\nu \cdot M_\nu \cdot (U_P)_\nu$. It is easy to see by setting up coordinates that any conjugacy class of Cartan in \mathfrak{G} contains an \mathfrak{A} for which these decompositions hold. Harish-Chandra [10] has shown similar decompositions are obtainable in any semisimple p -adic group.

Now we say what conditions an element of \mathfrak{A} must satisfy in order to be sufficiently regular. Recall \mathfrak{A}' denotes the set of regular elements of \mathfrak{A} . The group $\mathcal{N} \subseteq G$ consisting of those $g \in G$ such that $\text{Ad } g(\mathfrak{A}) = \mathfrak{A}$ clearly contains A , and it is well-known [1] that $\mathcal{N}/A = W$ is a finite group, called the Weyl group of \mathfrak{A} . Since $\text{Ad } A$ leaves \mathfrak{A} pointwise fixed, W may be regarded as a group of linear transformations on \mathfrak{A} . It is known that for $a \in \mathfrak{A}'$ we have

$w(a) \neq a$ for any $w \in W$. Since W is finite we can choose a constant γ_0 so that $\|w(a)\| \leq \gamma_0 \|a\|$ for all $a \in \mathfrak{A}$, $w \in W$.

Let $\{x_i\}_{i=1}^k$ be a set of coset representatives for $A_0 A_i$ in A . Since A_0 is compact, we see that if the constant γ_0 of the preceding paragraph is taken large enough then we will also have $\|\text{Ad } x_i a_0(m)\| \leq \gamma_0 \|m\|$ for $i = 1, \dots, k$, and every $a_0 \in A_0$, $m \in \mathfrak{G}$. Put $c_0 = \lceil \log_q \gamma_0 \rceil + 1$.

Write $\mathfrak{G} = \mathfrak{A} \oplus \mathfrak{A}^*$. Given $m \in \mathfrak{G}$, write $m = x + y$ with $x \in \mathfrak{A}$, $y \in \mathfrak{A}^*$. We have by the ultra-metric inequality that $\|m\| \leq \max(\|x\|, \|y\|)$. On the other hand, we may find a positive constant γ_1 such that $\|m\| \geq \gamma_1 \max(\|x\|, \|y\|)$. Put $c_1 = \lceil -\log_q \gamma_1 \rceil + 1$.

Recall for $m_1, m_2 \in \mathfrak{G}$ the formulas $\text{ad } m_2(m_1) = [m_1, m_2] = m_1 m_2 - m_2 m_1$. If $a \in \mathfrak{A}$, then \mathfrak{A} and \mathfrak{A}^* are invariant under $\text{ad } a$. If $a \in \mathfrak{A}'$, then $\mathfrak{A} = \ker \text{ad } (a)$ and $\text{ad } (a)$ is nondegenerate on \mathfrak{A}^* . For $\varepsilon > 0$, we define $\mathfrak{A}'(\varepsilon)$ to be the set of $a \in \mathfrak{A}$ satisfying the following two conditions. First, $\|\text{ad } (a)(y)\| \geq \varepsilon \|a\| \|y\|$ for all $y \in \mathfrak{A}^*$. Second, $\|w(a) - a\| \geq \varepsilon \|a\|$ for all $w \in W$. Clearly $\mathfrak{A}'(\varepsilon)$ is an open and closed subset of \mathfrak{A}' , and it is not hard to see that $\mathfrak{A}' = \bigcup_{\varepsilon > 0} \mathfrak{A}'(\varepsilon)$.

Now take $a \in \mathfrak{A}$, and suppose $\text{ord } (a) = -\mu$, with $\mu > 0$. For the purposes of this paper, we will say a is sufficiently regular if for some $\varepsilon > 0$, $a \in \mathfrak{A}'(\varepsilon)$, and $\mu \geq 6(e + c_0 + c_1 + 1)$, where $e = \lceil -\log_q \varepsilon \rceil + 1$. It is not hard to see that the set of sufficiently regular elements of \mathfrak{A} is quite large. Indeed, for any $x \in \mathfrak{A}'$, rx is sufficiently regular for all $r \in F$ with $|r|$ sufficiently large.

Let us now derive some properties of sufficiently regular elements. It is well-known that the conjugacy classes of elements in an open set in \mathfrak{A}' fill an open set in \mathfrak{G} . Our first lemma is a quantitative expression of this fact. Compare [12], Lemma 6.

LEMMA 3.1. *Take $a \in \mathfrak{A}'$, and suppose for $b \in \mathfrak{A}^*$, $\| [a, b] \| \geq \gamma \|b\|$ for some constant $\gamma > 0$. Put $c = \lceil -\log_q \gamma \rceil + 1$, and let c_1 be as in the definition of sufficient regularity. Then for any $l > 0$,*

$$\text{Ad } K_{c_1+l}(a + \mathfrak{A}_{c+c_1+l}) \supseteq a + L_{c+2c_1+l}.$$

REMARK. Note that $\gamma_1 \leq 1$, so that $c_1 \geq 1$. Also note that if $x \in \mathfrak{A}$, and $\text{ord } (x) \geq c + c_1 + 1$, then $\|x\| < \gamma$, and so $\| [x, b] \| < \gamma \|b\|$ for any $b \in \mathfrak{A}^*$. Thus $\| [a + x, b] \| \geq \gamma \|b\|$ and in particular $a + x \in \mathfrak{A}'$.

Proof. Consider $m \in \mathfrak{G}$ of the form $m = x + y$, with $x \in \mathfrak{A}$, $y \in \mathfrak{A}^*$, and $\text{ord } (x)$ and $\text{ord } (y) \geq c + c_1 + l$. By the hypothesis on a , we can find $z \in \mathfrak{A}^*$ such that $[a, z] = y$ and $\|z\| \leq \gamma^{-1} \|y\|$, or $\text{ord } (z) \geq \text{ord } (y) - c \geq c_1 + l$. We compute

$$\begin{aligned}
\text{Ad}(1+z)(a+m) &= (1+z)(a+x+y)(1+z)^{-1} \\
&= a+x+y+([z, a]+[z, m])(1-z(1+z)^{-1}) \\
&= a+x-yz(1+z)^{-1}+[z, m](1+z)^{-1} \\
&= a+x+m',
\end{aligned}$$

where

$$m' = ([z, m] - yz)(1+z)^{-1}.$$

Hence $\text{ord}(m') \geq \min(\text{ord}(x), \text{ord}(y)) + \text{ord}(y) - c \geq \text{ord}(y) + c_1 + l$. Thus if $m' = x' + y'$ with $x' \in \mathfrak{A}$, $y' \in \mathfrak{A}^*$, since

$$\|m'\| \geq \gamma_1 \max(\|x'\|, \|y'\|),$$

we have $\text{ord}(x') \geq \text{ord}(m') - c_1 \geq c + c_1 + l$; and similarly $\text{ord}(y') \geq \text{ord}(m') - c_1 > \text{ord}(y)$. It follows that we may successively move $a+m$ closer and closer to \mathfrak{A} , so that it is eventually conjugated into an element of $a + \mathfrak{A}_{c+c_1+l}$. Moreover, we note that, again by the relation $\|m\| \geq \gamma_1 \max(\|x\|, \|y\|)$, we may, with x and y in L_{c+c_1+l} , express any $m \in L_{c+2c_1+l}$. Finally we observe that to do our conjugating we only required elements of the form $1+z$ with $\text{ord}(z) \geq c_1 + l$. This concludes the lemma.

We may now state the main properties of sufficiently regular elements.

LEMMA 3.2. *Suppose $a \in \mathfrak{A}$ is sufficiently regular, and choose $\varepsilon > 0$ such that $a \in \mathfrak{A}'(\varepsilon)$, and $\mu = -\text{ord}(a) \geq 6(e + c_0 + c_1 + 1)$, where these constants are all as in the definition of sufficient regularity. Then a has the following properties. Put $\eta = [(\mu + 1)/2]$ and $\nu = [(\mu + 2)/3]$.*

- (i) $\|\text{ad}(a)(y)\| \geq \varepsilon q^\nu(y)$ for $y \in \mathfrak{A}^*$. In other words $\text{ad}(a)(\mathfrak{A}_s^*) \supseteq \mathfrak{A}_{s+\varepsilon-\mu}^*$.
- (ii) $\text{Ad } K_\nu(a + \mathfrak{A}_{-\eta-c_0-c_1}) \supseteq a + L_{-\eta-c_0}$.
- (iii) For any nontrivial $w \in W$, $w(a + \mathfrak{A}_{-\eta-c_0-c_1})$ is disjoint from $a + \mathfrak{A}_{-\eta-c_0-c_1}$.
- (iv) Every element of $a + \mathfrak{A}_{-\eta-c_0-c_1}$ is regular.

Proof. Statement (i) is a trivial calculation from the definition of $\mathfrak{A}'(\varepsilon)$. From Lemma 3.1, using $c = e - \mu$, as derived from (i), for (ii) we see we must have $\nu \leq c_1 + l$, where l is determined by the equation $-\eta - c_0 = e - \mu + 2c_1 + l$. Substituting in this the inequality for ν , we see we need $\mu - \eta - \nu \geq e + c_0 + c_1$, and this is satisfied if $\mu \geq 6(e + c_0 + c_1 + 1)$.

For (iii), we note that for any nontrivial $w \in W$, we have $\text{ord}(w(a) - a) \leq -\mu + e$, by part of the definition of $\mathfrak{A}'(\varepsilon)$. On the

other hand, if $x \in \mathfrak{U}_{-\gamma-c_0-c_1}$, then $\text{ord}(w(x)) \geq -\gamma - 2c_0 - c_1$ by part of the original definition of c_0 . But $-\gamma - 2c_0 - c_1 \geq -\mu + 3(e + c_0 + c_1) - 2c_0 - c_1 > -\mu + e$. Hence $\text{ord}(w(a + x) - a) \leq -\mu + \varepsilon < -\gamma - c_0 - c_1$, so $w(a + \mathfrak{U}_{-\gamma-c_0-c_1})$ is indeed disjoint from $a + \mathfrak{U}_{-\gamma-c_0-c_1}$, and (iii) is proved. Finally, (iv) is an immediate consequence of the remark following the statement of Lemma 3.1. This finishes the lemma.

We now turn to harmonic analysis. We fix for the rest of this section a sufficiently regular element $\bar{a} \in \mathfrak{U}$. Let $\text{ord}(\bar{a}) = -\mu$ with $\mu > 0$ and let $\eta = [(\mu + 1)/2]$ as usual. Then \bar{a} represents a certain shallow character ψ of K_η . The conductor of ψ will be K_μ , and the dual blob of ψ will be $\bar{a} + L_{-\eta}$. In this section ψ will always denote this particular shallow character of K_η .

Let $\{x_i\}_{i=1}^k$ be the representatives for $A_0 A_s$ in A used in the definition of sufficient regularity. Put $\nu = [(\mu + 2)/3]$. Let $J_\nu = \bigcap_{i=1}^k \bigcap_{a_0 \in A_0} \text{Ad } x_i a_0(K_\nu)$. Then $K_\nu \supseteq J_\nu \supseteq K_{\nu+c_0} \supseteq K_\eta$. Moreover, since $K_\nu = (U_{\bar{P}})_\nu \cdot M_\nu \cdot (U_P)_\nu$, and $U_{\bar{P}}$, M , and U_P are normalized by A , we see that $J_\nu = (J_\nu \cap U_{\bar{P}}) \cdot (J_\nu \cap M) \cdot (J_\nu \cap U_P)$. By construction J_ν is normalized by A_0 , and $J_\nu \cap M$ is even normalized by A since A_s is central in M . Also note

$$\begin{aligned} \text{Ad } J_\nu(\bar{a} + \mathfrak{U}_{-\gamma-c_0-c_1}) &= \bigcap_{i=1}^k \bigcap_{a_0 \in A_0} \text{Ad } (x_i a_0)(\text{Ad } K_\nu(\bar{a} + \mathfrak{U}_{-\gamma-c_0-c_1})) \\ &\supseteq \bigcap_{i=1}^k \bigcap_{a_0 \in A_0} \text{Ad } (x_i a_0)(\bar{a} + L_{-\eta-c_0}) \supseteq \bar{a} + L_{-\eta}. \end{aligned}$$

Let δ be a representation of J_ν lying over ψ . That is, the restriction of δ to K_η contains ψ . From the inclusions given above, we can deduce the support of $\mathcal{H}(\delta)$.

LEMMA 3.3. *The only (J_ν, J_ν) double cosets which support non-zero elements of $\mathcal{H}(\delta)$ are of the form $J_\nu b J_\nu$ with $b \in A$.*

Proof. If $g \in G$ intertwines δ with itself then g certainly intertwines the restriction of δ on K_η with itself. Thus for some $k_1, k_2 \in J_\nu$, we have that $\text{Ad}^* k_1(\psi)$ and $\text{Ad}^* g(\text{Ad}^* k_2(\psi))$ agree on $K_\eta \cap \text{Ad } g(K_\eta)$. (Recall δ restricted to K_η will consist of a direct sum of shallow characters of K_η of the form $\text{Ad}^* k(\psi)$ with $k \in J_\nu$.) If we apply Lemma 2.3, we see that for some $x_1, x_2 \in L_{-\eta}$, we have $\text{Ad } k_1(\bar{a} + x_1) = \text{Ad } g(\text{Ad } k_2(\bar{a} + x_2))$. Now by the inclusion given above, we have $\bar{a} + x_i = \text{Ad } h_i(a_i)$ for $i = 1, 2$, and $h_i \in J_\nu$, $a_i \in \bar{a} + \mathfrak{U}_{-\gamma-c_0-c_1}$. Thus $\text{Ad } k_1 h_1(a_1) = \text{Ad } g(\text{Ad } k_2 h_2(a_2))$, or $\text{Ad } (h_1^{-1} k_1^{-1} g k_2 h_2)(a_2) = a_1$. Since a_1 and a_2 are regular in \mathfrak{U} by statement (iv) of Lemma 3.2, we see $\text{Ad } (h_1^{-1} k_1^{-1} g h_2 h_2)(\mathfrak{U}) = \mathfrak{U}$, and so $h_1^{-1} k_1^{-1} g k_2 h_2$ defines an element w of W . Since $w(a_2) = a_1$, w does not transform $\bar{a} + \mathfrak{U}_{-\gamma-c_0-c_1}$ out of itself, so w must be the identity. Hence $h_1^{-1} k_1^{-1} g k_2 h_2 \in A$, or $g \in J_\nu A J_\nu$, as

was to be proved.

As this point it seems worthwhile to point out that if A is minisotropic, which in our case means A is the multiplicative group of a field extension of F of degree n , then we have just constructed a series of supercuspidal representations of G associated to A . Indeed, in the case when A is minisotropic, A_* consists of constant matrices and $M = G$. Hence J_ν is normalized by A , and $A \cdot J_\nu$ is a subgroup of G , compact modulo the center of G .

PROPOSITION 3.1. *Let δ' be any representation of $A \cdot J_\nu$ lying above δ on J_ν . Then δ' induces an irreducible supercuspidal representation of G .*

Proof. Indeed, by Lemma 3.3, the only $(A \cdot J_\nu, A \cdot J_\nu)$ double coset which supports an intertwining distribution for δ' is $A \cdot J_\nu$ itself. In these circumstances it is known [18] and easy to see that the representation induced to G from δ' is irreducible.

For general A , the supercuspidal representations of Proposition 3.1 must be replaced by series of representations induced from P . The construction of these series requires close examination of the structure of δ . Define J_μ analogously to J_ν . That is

$$J_\mu = \bigcap_{i=1}^k \bigcap_{a_0 \in A_0} \text{Ad } x_i a_0 (K_\mu).$$

Define $J_{2\nu}$ similarly. By the analogous statements for K_ν , $K_{2\nu}$, and K_μ , we see $J_\nu/J_{2\nu}$ is abelian, and $J_{2\nu}/J_\mu$ is central in J_ν/J_μ . Thus J_ν/J_μ is a two-step nilpotent group. Since $J_\mu \subseteq \ker \delta$, δ is a representation of a two-step nilpotent group. Also, the restriction of δ to $J_{2\nu}$ is some multiple of ψ restricted to $J_{2\nu}$. Since $\mathcal{U}^* \supseteq \mathfrak{M}^* = \mathcal{U} \oplus \overline{\mathcal{U}}$, both $U_P \cap J_{2\nu}$ and $U_{\overline{P}} \cap J_{2\nu}$ are contained in the kernel of ψ , hence of δ . Put $\Gamma_1 = M \cap J_\nu$ and $\Gamma_2 = (U_P \cap J_\nu) \cdot (U_{\overline{P}} \cap J_\nu) \cdot J_{2\nu}$. Then $\Gamma_1 \cdot \Gamma_2 = J_\nu$. Also Γ_1 normalizes $U_P \cap J_\nu$ and $U_{\overline{P}} \cap J_\nu$, and centralizes them modulo the kernel of δ . Thus Γ_1 and Γ_2 commute modulo the kernel of δ , and $\Gamma_1 \cap \Gamma_2 = J_{2\nu} \cap M$ is central modulo $\ker \delta$. It follows that the restriction of δ to Γ_i is a multiple of some irreducible representation δ_i . Moreover δ is the direct image of the outer tensor product of the δ_i , in the following sense. If δ acts on V and δ_i acts on V_i , then $V \simeq V_1 \otimes V_2$ in such a way that the following diagram commutes.

$$\begin{array}{ccc} \Gamma_1 \times \Gamma_2 & \xrightarrow{\alpha} & J_\nu \\ \delta_1 \downarrow \delta_2 & & \downarrow \delta \\ \text{Hom}(V_1) \otimes \text{Hom}(V_2) & \simeq & \text{Hom}(V). \end{array}$$

The upper horizontal map α is the product of the inclusions. That is, if $x \in \Gamma_1 \subseteq J_\nu$ and $y \in \Gamma_2 \subseteq J_\nu$, then $\alpha(x, y) = xy$. Although α is not a homomorphism, it is a homomorphism modulo the kernel of δ .

Under the action of $\text{Ad } A$, it is obvious that the above situation is stable in the following sense. For any $b \in A$, we have $\text{Ad } b(\Gamma_1) = \Gamma_1$, and the restriction of $\text{Ad}^* b(\delta)$ to Γ_1 is a multiple of $\text{Ad}^* b(\delta_1)$. Similarly $\text{Ad } b(\Gamma_2) \cap J_\nu = \text{Ad } b(\Gamma_2) \cap \Gamma_2$, and the restriction of $\text{Ad}^* b(\delta)$ to $\text{Ad } b(\Gamma_2)$ is a multiple of $\text{Ad}^* b(\delta_2)$. Finally, $\Gamma_1 \cap \Gamma_2$ is normalized by b , and δ and $\text{Ad}^* b(\delta)$ agree on $\Gamma_1 \cap \Gamma_2$. These facts immediately lead to the conclusion that the intertwining number of δ and $\text{Ad}^* b(\delta)$ on $J_\nu \cap \text{Ad } b(J_\nu)$ is equal to the product of the intertwining numbers of δ_i and $\text{Ad}^* b(\delta_i)$ on Γ_i and $\text{Ad } b(\Gamma_i)$. Since $\text{Ad } b(\Gamma_1) = \Gamma_1$, the intertwining number of δ_1 and $\text{Ad}^* b(\delta_1)$ on Γ_1 is one or zero according as δ_1 and $\text{Ad}^* b(\delta_1)$ are equivalent or not. Let us now look more closely at δ_2 and $\text{Ad}^* b(\delta_2)$.

We recall some facts about computing intertwining numbers in terms of characters. Let C be a compact group, and σ_1 and σ_2 two distinct irreducible representations of C . Let $\chi(\sigma_i)$ denote the character of σ_i . Let dx be Haar measure, and let C have total measure $m(C)$. Then the Schur orthogonality relations for C say

$$\int_C \chi(\sigma_1)(x) \overline{\chi(\sigma_2)(x)} dx = 0$$

and

$$\int_C \chi(\sigma_i)(x) \overline{\chi(\sigma_i)(x)} dx = m(C).$$

Here $\overline{}$ indicates complex conjugation. From this it is immediate that if τ_1 and τ_2 are any finite-dimensional representations of C , with characters $\chi(\tau_i)$, then

$$\int_C \chi(\tau_1)(x) \overline{\chi(\tau_2)(x)} dx = m(C) I(\tau_1, \tau_2)$$

where $I(\tau_1, \tau_2)$ is the intertwining number between τ_1 and τ_2 . For as we have seen, if $\tau_1 = \Sigma a_i \sigma_i$ and $\tau_2 = \Sigma b_i \sigma_i$ with $a_i, b_i \in \mathbb{Z}$ where the σ_i are distinct and irreducible, then $I(\tau_1, \tau_2) = \Sigma a_i b_i$, while $\chi(\tau_1) = \Sigma a_i \chi(\sigma_i)$ and $\chi(\tau_2) = \Sigma b_i \chi(\sigma_i)$, so that

$$\begin{aligned} \int_C \chi(\tau_1)(x) \overline{\chi(\tau_2)(x)} dx &= \int_C (\Sigma a_i \chi(\sigma_i)(x)) \overline{(\Sigma b_j \chi(\sigma_j)(x))} dx \\ &= \Sigma a_i b_j \int_C \chi(\sigma_i)(x) \overline{\chi(\sigma_j)(x)} dx = m(C) (\Sigma a_i b_i). \end{aligned}$$

If now C_1 and C_2 are open compact subgroups of G , and τ_i is an irreducible representation of C_i , let $I(\tau_1, \tau_2, g)$ be the number of

times g intertwines τ_1 and τ_2 . We have seen $I(\tau_1, \tau_2, g)$ is equal to the intertwining number of the restrictions of τ_1 and $\text{Ad}^* g(\tau_2)$ to $C_1 \cap \text{Ad } g(C_2)$. Let $\chi(\tau_i)$ now denote the function on G which equals $\chi(\tau_i)$ on C_i and vanishes off C_i . The formula $\chi(\text{Ad}^* g(\tau_i))(x) = \chi(\tau_i)(g^{-1}xg)$ is immediate from the definitions. Therefore we have the formula

$$(1) \quad I(\tau_1, \tau_2, g) = m(C_1 \cap \text{Ad } g(C_2))^{-1} \int_{C_1 \cap \text{Ad } g(C_2)} \chi(\tau_1)(x) \overline{\chi(\tau_2)(g^{-1}xg)} dx.$$

Here now dx is the restriction to $C_1 \cap \text{Ad } g(C_2)$ of some fixed Haar measure on G .

Now we apply the above remarks to δ_2 . Let $\mathcal{A} \subseteq \Gamma_2$ be the inverse image in Γ_2 of the center of $\Gamma_2/\ker \delta_2$. The restriction of δ_2 to \mathcal{A} will then be a multiple of a character $\varphi(\delta_2)$. Of course $\varphi(\delta_2)$ agrees with ψ on J_{2v} . (Note $\mathcal{A} \supseteq J_{2v}$.) Since $\Gamma_2/\ker \delta_2$ is two-step nilpotent, it is known and easy to verify that $\chi(\delta_2)$, the character of δ_2 , is supported on \mathcal{A} . There it is given by the formula $\chi(\delta_2) = (\dim \delta_2) \varphi(\delta_2)$. Thus if $b \in \mathcal{A}$, then from (1) we may calculate

$$\begin{aligned} I(\delta_2, \delta_2, b) &= m(\Gamma_2 \cap \text{Ad } b(\Gamma_2))^{-1} \int_{\Gamma_2 \cap \text{Ad } b(\Gamma_2)} \chi(\delta_2)(x) \overline{\chi(\delta_2)(b^{-1}xb)} dx \\ &= m(\Gamma_2 \cap \text{Ad } b(\Gamma_2))^{-1} (\dim \delta_2)^2 \int_{\mathcal{A} \cap \text{Ad } b(\mathcal{A})} \varphi(\delta_2)(x) \overline{\varphi(\delta_2)(b^{-1}xb)} dx. \end{aligned}$$

The last integral is either 0 or 1 times $m(\mathcal{A} \cap \text{Ad } b(\mathcal{A}))$. Another elementary fact about two-step nilpotent groups is that $(\dim \delta_2)^2 = \#(\Gamma_2/\mathcal{A})$, the index of \mathcal{A} in Γ_2 . Thus $I(\delta_2, \delta_2, b)$ is either 0 or 1 times the quantity

$$\begin{aligned} &m(\Gamma_2 \cap \text{Ad } b(\Gamma_2))^{-1} m(\mathcal{A} \cap \text{Ad } b(\mathcal{A})) \#(\Gamma_2/\mathcal{A}) \\ &= \#(\Gamma_2 \cap \text{Ad } b(\Gamma_2)) / (\mathcal{A} \cap \text{Ad } b(\mathcal{A}))^{-1} \#(\Gamma_2/\mathcal{A}). \end{aligned}$$

We want to show $I(\delta_2, \delta_2, b)$ is either 0 or 1; hence we will try to show the two indexes above are equal.

Write $U_P \cap \Gamma_2 = 1 + Y$ where Y is a lattice in \mathcal{U} . Similarly write $U_{\bar{P}} \cap \Gamma_2 = 1 + \bar{Y}$. Consider the function B_ψ on $\Gamma_2 \times \Gamma_2$ given by the formula $B_\psi(x, z) = \psi(xzx^{-1}z^{-1})$ for $x, z \in \Gamma_2$. Simple checking shows that for fixed x , $B_\psi(x, \cdot)$ is a linear character on Γ_2 ; similarly for fixed z , $B_\psi(\cdot, z)$ is a linear character on Γ_2 . Moreover $B_\psi(x, z) = \overline{B_\psi(z, x)}$. Thus B_ψ is a T -valued antisymmetric "bilinear form" on Γ_2 . Moreover, since δ is a multiple of ψ on J_{2v} , which contains the commutator subgroup of Γ_2 , it follows from the definition of \mathcal{A} , that $B_\psi(x, \cdot)$ is the trivial character of Γ_2 if and only if $x \in \mathcal{A}$. That is, \mathcal{A} is the "radical" of B_ψ . Let $x = 1 + y_1 + \bar{y}_1$, $z = 1 + y_2 + \bar{y}_2$ with $y_i \in Y$, $\bar{y}_i \in \bar{Y}$. Then an easy computation shows $B_\psi(x, z) = \psi(1 + [y_1 + \bar{y}_1, y_2 + \bar{y}_2]) = \Omega(\bar{\alpha})([y_1 + \bar{y}_1, y_2 + \bar{y}_2])$. Thus let us define

the antisymmetric, T -valued bilinear form \tilde{B} on $\mathcal{U} \oplus \overline{\mathcal{U}}$ by $\tilde{B}(u, v) = \Omega(\bar{a})([u, v]) = \Omega_0(\langle \bar{a}, [u, v] \rangle)$, for $u, v \in \mathcal{U} \oplus \overline{\mathcal{U}}$. Here Ω_0 is the "basic character" of F used identifying \mathfrak{G} and $\bar{\mathfrak{G}}$. Thus if $u, v \in Y \oplus \bar{Y}$, $\tilde{B}(u, v) = B_\psi(1 + u, 1 + v)$.

For a given lattice $A \subseteq \mathcal{U} \oplus \overline{\mathcal{U}}$, let $\tilde{A} = \{u \in \mathcal{U} \oplus \overline{\mathcal{U}}, \tilde{B}(u, v) = 1 \text{ for all } v \in A\}$. Since, $\mathcal{U} \oplus \overline{\mathcal{U}} \subseteq \mathfrak{M}^* \subseteq \mathfrak{U}^*$, $\text{ad } a$ is nondegenerate on $\mathcal{U} \oplus \overline{\mathcal{U}}$. Also, \mathcal{U} and $\overline{\mathcal{U}}$ are paired nondegenerately against each other by \langle, \rangle (see [10]). Therefore, rewriting $\langle \bar{a}, [u, v] \rangle = \langle [\bar{a}, u], v \rangle$, we see this is a nondegenerate bilinear form on $\mathcal{U} \oplus \overline{\mathcal{U}}$. Since \tilde{A} is obviously an R -module, and since $\ker \Omega_0 = R$, we see an alternative description of \tilde{A} is $\tilde{A} = \{u \in \mathcal{U} \oplus \overline{\mathcal{U}}, \langle a, [u, v] \rangle \subseteq R \text{ for all } v \in A\}$. It follows that \tilde{A} is also a lattice, and the usual duality relations hold: $(\tilde{A})^\sim = A$, $(A_1 + A_2)^\sim = \tilde{A}_1 \cap \tilde{A}_2$, and $(A_1 \cap A_2)^\sim = \tilde{A}_1 + \tilde{A}_2$. Now put $A = Y \oplus \bar{Y}$. From the relation between \tilde{B} and B_ψ on A , we conclude $A \cap (1 + A) = 1 + (A \cap \tilde{A})$. Since $\Gamma_2 = (1 + A) \cdot M_{2\nu}$, we see $^*(\Gamma_2/A) = ^*(A/A \cap \tilde{A})$.

Now take $b \in A$. Since $\bar{a}, \psi, \mathcal{U} \oplus \overline{\mathcal{U}}$ and \tilde{B} are $\text{Ad } b$ invariant, we see that if we repeat the same analysis for $\text{Ad}^* b(\delta_2)$ and $\text{Ad } b(\Gamma_2)$, we will get $^*(\text{Ad } b(\Gamma_2)/\text{Ad } b(A)) = ^*(\text{Ad } b(A)/\text{Ad } b(A \cap \tilde{A}))$, and $\text{Ad } b(A)^\sim = \text{Ad } b(\tilde{A})$. Combining the two situations, we get

$$^*((\Gamma_2 \cap \text{Ad } b(\Gamma_2))/(\text{Ad } b(A) \cap \text{Ad } b(A))) = ^*((A \cap \text{Ad } b(A))/(\text{Ad } b(A) \cap \text{Ad } b(A \cap \tilde{A}))) .$$

Now I claim that actually $\tilde{A} \subseteq A$. If this is so, then the desired equality of indexes holds, as we calculate, putting $\text{Ad } b(A) = N$. We have

$$\begin{aligned} ^*((A \cap N)/(\tilde{A} \cap \tilde{N})) &= ^*(A/A \cap N)^{-1} ^*(A/\tilde{A} \cap \tilde{N}) \\ &= ^*(A/A \cap N)^{-1} ^*(A/\tilde{A}) ^*(\tilde{A}/\tilde{A} \cap \tilde{N}) . \end{aligned}$$

Similarly $^*((A \cap N)/(\tilde{A} \cap \tilde{N})) = ^*(N/A \cap N)^{-1} ^*(N/\tilde{N}) ^*(\tilde{N}/\tilde{A} \cap \tilde{N})$. Multiplying these equations gives

$$\begin{aligned} ^*((A \cap N)/(\tilde{A} \cap \tilde{N}))^2 &= ^*((A + N)/(A \cap N))^{-1} ^*(A/\tilde{A}) ^*(N/\tilde{N}) ^*((\tilde{A} + \tilde{N})/(\tilde{A} \cap \tilde{N})) . \end{aligned}$$

Now the two outer factors cancel by duality. Since $N = \text{Ad } b(A)$, we have $^*(A/\tilde{A}) = ^*(N/\tilde{N})$. Hence taking square roots gives

$$^*((A \cap N)/(\tilde{A} \cap \tilde{N})) = ^*(A/\tilde{A}) .$$

Translating this into terms involving Γ_2 gives the equality of indexes we sought.

REMARK. The above equality of indexes is related to the representation theory of Heisenberg groups, and may be proved in those terms.

It remains to show $\tilde{\Lambda} \subseteq \Lambda$. We recall that by virtue of a string of definitions, $\Lambda = \bigcap_{i=1}^k \bigcap_{a_0 \in A_0} \text{Ad}(x_i a_0)(\mathcal{U}_\nu \oplus \overline{\mathcal{U}}_\nu)$. Here the $\{x_i\}_{i=1}^k$ are as in the definition of sufficient regularity and $\mathcal{U}_\nu = \mathcal{U} \cap L_\nu$, $\overline{\mathcal{U}}_\nu = \overline{\mathcal{U}} \cap L_\nu$ as usual. Therefore $\tilde{\Lambda}$ is the lattice of $\mathcal{U} \oplus \overline{\mathcal{U}}$ generated by the lattices $\text{Ad}(x_i a_0)((\mathcal{U} \oplus \overline{\mathcal{U}})_\nu)$ where $(\mathcal{U} \oplus \overline{\mathcal{U}})_\nu = \mathcal{U}_\nu \oplus \overline{\mathcal{U}}_\nu$. Let us calculate $(\mathcal{U} \oplus \overline{\mathcal{U}})_\nu$. Because of the expression of \tilde{B} in terms of \langle, \rangle , we have $(\mathcal{U} \oplus \overline{\mathcal{U}})_\nu = ((\text{ad } \bar{a}((\mathcal{U} \oplus \overline{\mathcal{U}})_\nu))^* \cap (\mathcal{U} \oplus \overline{\mathcal{U}}))_\nu$. Since $\mathcal{U} \oplus \overline{\mathcal{U}} \subseteq \mathfrak{A}^*$, we may apply statement (i) of Lemma 3.2, and conclude $\text{ad } \bar{a}(\mathcal{U} \oplus \overline{\mathcal{U}})_\nu \supseteq (\mathcal{U} \oplus \overline{\mathcal{U}})_{\nu+e-\mu}$. Thus $(\mathcal{U} \oplus \overline{\mathcal{U}})_\nu \subseteq (\mathcal{U} \oplus \overline{\mathcal{U}})_{\mu-\nu-e}$. Therefore we get $\tilde{\Lambda} \subseteq (\mathcal{U} \oplus \overline{\mathcal{U}})_{\mu-\nu-e-c_0}$. On the other hand, the expression for Λ given above implies $\Lambda \supseteq (\mathcal{U} \oplus \overline{\mathcal{U}})_{\nu-c_0}$. Therefore if $\nu - c_0 < \mu - \nu - e - c_0$, we are done. But $\mu - 2\nu \geq 2(e + c_0 + c_1)$, by part of the definition of sufficient regularity, so we are indeed done.

We record one result of the foregoing discussion.

LEMMA 3.4. *Notations as above. The coset $J_b J_\nu$, for $b \in A$, supports at most one nontrivial intertwining distribution for δ . That is $I(\delta, \delta, b) \leq 1$.*

Now we want to show how to choose δ_2 so that $I(\delta_2, \delta_2, b) = 1$ for all $b \in A$. Consider $\Gamma_2'' = (J_\nu \cap U_P) \cdot J_{2\nu}$. Modulo the kernel of ψ , $(J_\nu \cap U_P)$ is normal in Γ_2'' . Since ψ is trivial on $U_P \cap K_{2\nu}$, we may extend ψ to a character ψ'' on Γ_2'' , by letting ψ'' be trivial on $(J_\nu \cap U_P)$. Let Γ_2' be the annihilator of Γ_2'' with respect to the form B_ψ on Γ_2 introduced above. That is $\Gamma_2' = \{x \in \Gamma_2 : B_\psi(x, y) = 1 \text{ for all } y \in \Gamma_2''\}$. Since ψ extends to a character of Γ_2'' , we see Γ_2' contains Γ_2'' . Since $\Gamma_2 = \Gamma_2' \cdot (\Gamma_2 \cap U_{\bar{P}})$, we may write $\Gamma_2' = \Gamma_2'' \cdot (\Gamma_2' \cap U_{\bar{P}})$. Now B_ψ is trivial on $\Gamma_2 \cap U_{\bar{P}}$, so $\Gamma_2' \cap U_{\bar{P}}$ annihilates itself as well as Γ_2'' . Hence Γ_2' is commutative modulo $\ker \psi$. Therefore on Γ_2' there is a unique character ψ' which is trivial on $\Gamma_2' \cap U_{\bar{P}}$ and agrees with ψ'' on Γ_2'' . Now it follows from elementary facts about representations of two-step nilpotent groups that ψ' induces an irreducible representation δ_2 of Γ_2 , and any other irreducible representation of Γ_2 lying above ψ on $J_{2\nu}$ is of the form $\delta_2 \otimes \varphi$, where φ is a linear character of $\Gamma_2/J_{2\nu}$. From now on, we will let δ_2 stand for the representation of Γ_2 induced from ψ' on Γ_2' .

We want to express the intertwining properties of δ_2 in terms of those of ψ' . To do this, we must once again recall some facts about intertwining numbers. (See [18].)

Again let C_1 and C_2 be open compact subgroups of G , and let τ_1 and τ_2 be irreducible representations of C_1 and C_2 respectively. Suppose $C'_i \subseteq C_i$ are open subgroups and suppose the τ_i are induced

from representations σ_i of C'_i . We have the Frobenius formula for induced characters: $\chi(\tau_i)(x) = \sum_j \chi(\sigma_i)(y_{ij}xy_{ij}^{-1})$. Here $\chi(\tau_i)$ is the function on G which is the character of τ_i on C_i and zero off C_i , and $\chi(\sigma_i)$ is defined analogously. The $\{y_{ij}\}$ are a set of coset representatives for C'_i in C_i . That is, $C_i = \bigcup_j C'_i y_{ij}$, the union being disjoint. If we plug this into formula (1) for $I(\tau_i, \tau_2, g)$, $g \in G$, we obtain

$$\begin{aligned} m(C_1 \cap \text{Ad } g(C_2))I(\tau_1, \tau_2, g) &= \int_{C_1 \cap \text{Ad } g(C_2)} \left(\sum_j \chi(\sigma_1)(y_{1j}xy_{1j}^{-1}) \right) \overline{\left(\sum_k \chi(\sigma_2)(y_{2k}g^{-1}xgy_{2k}^{-1}) \right)} dx \\ &= \int_{C_1 \cap \text{Ad } g(C_2)} \chi(\sigma_1)(x) \overline{\left(\sum_{j,k} \chi(\sigma_2)(g_{jk}^{-1}xg_{jk}) \right)} dx \quad (\text{where } g_{jk} = y_{1j}gy_{2k}^{-1}) \\ &= \sum_{j,k} \int_{C'_1 \cap \text{Ad } g_{jk}(C'_2)} \chi(\sigma_1)(x) \overline{\chi(\sigma_2)(g_{jk}^{-1}xg_{jk})} dx \\ &= \sum_{j,k} m(C'_1 \cap \text{Ad } g_{jk}(C'_2))I(\sigma_1, \sigma_2, g_{jk}). \end{aligned}$$

Taking first $C'_2 = C_2$, then $C_1 = C'_1$, one may reduce this in stages to

$$(2) \quad I(\tau_1, \tau_2, g) = \sum_{h_i} I(\sigma_1, \sigma_2, h_i),$$

where $C_1 g C_2 = \bigcup_i C'_1 h_i C'_2$, and the union is disjoint. (The reduction is very simple, when, as in our case, C'_i is normal in C_i .)

Apply this to δ_2 and ψ' . For any $b \in A$, we know $I(\delta_2, \delta_2, b) \leq 1$. On the other hand, ψ' is trivial on $\Gamma'_2 \cap U_P$ and on $\Gamma'_2 \cap U_{\bar{P}}$, and equals ψ on $\Gamma'_2 \cap M$. Since ψ on $\Gamma'_2 \cap M$ is $\text{Ad}^* A$ -invariant, we conclude $I(\psi', \psi', b) = 1$. Then equation (2) shows $I(\delta_2, \delta_2, b) = 1$, and $I(\psi', \psi', y) = 0$ for any double coset $\Gamma'_2 y \Gamma'_2 \neq \Gamma'_2 b \Gamma'_2$ for some $b \in A$.

Now consider $\Gamma_3 = \Gamma_1 \cdot \Gamma'_2$, and let δ_3 be any irreducible representation of Γ_3 whose restriction to Γ'_2 is a multiple of ψ' . Then it is obvious that δ_3 induced up to J_ν is an irreducible representation whose restriction to Γ_2 is δ_2 . We see from this that δ_3 restricted to Γ_1 is an irreducible representation δ_1 of Γ_1 . Summarizing the discussion since Proposition 3.1, we find we have established the following result.

LEMMA 3.5. *The only (Γ_3, Γ_3) double cosets which support non-zero intertwining distributions for δ_3 are those which contain representatives in A . Moreover if $b \in A$, $I(\delta_3, \delta_3, b) \leq 1$, and is equal to one or zero according as $\text{Ad}^* b(\delta_1)$ is equivalent to δ_1 on Γ_1 or not. Furthermore δ_3 and $\text{Ad}^* b(\delta_3)$ restricted to $\Gamma_3 \cap \text{Ad } b(\Gamma_3)$ are both irreducible.*

Now we look at δ_1 and put it in good shape. Then the con-

struction of our induced series of representations will be a simple matter. Let X denote the set $(1 + \mathfrak{U}^*) \cap \Gamma_1$. Then $(A \cap \Gamma_1) \cdot X = (1 + \mathfrak{U}_\nu) \cdot X \supseteq 1 + \mathfrak{U}_\nu + (\mathfrak{M}_{\nu+c_0} \cap \mathfrak{U}^*) \supseteq 1 + \mathfrak{M}_{\nu+c_0+c_1} \supseteq 1 + \mathfrak{M}_\gamma = M_\gamma$. Since $(A \cap \Gamma_1) \cdot X \subseteq \Gamma_1 \subseteq M_\nu$, and $M_\nu = 1 + \mathfrak{M}_\nu$, and $\mathfrak{M}_\nu^2 \subseteq \mathfrak{M}_{2\nu} \subseteq \mathfrak{M}_\gamma$, it follows that $(A \cap \Gamma_1) \cdot X = \Gamma'_1$ is a group, and a normal subgroup of Γ_1 . Since A normalizes Γ_1 and $\text{Ad } A$ preserves $\mathfrak{M} \cap \mathfrak{U}^*$, we see A normalizes Γ'_1 . I claim the representation δ_1 of Γ_1 may be induced from Γ'_1 . In fact, more is true. Let δ'_1 be an irreducible component of δ_1 restricted to Γ'_1 , and assume the restriction of δ'_1 to M_γ contains the restriction of ψ to M_γ . Such a δ'_1 clearly exists. We notice that in the proof of Lemma 3.1, the only elements used for conjugating $x \in a + L_{c+2c_1+l}$ into $a + \mathfrak{U}_{c+c_1+l}$ were of the form $1 + z$ with $z \in \mathfrak{U}_{c_1+l}^*$. It follows from this and the discussion before Lemma 3.3, that $\text{Ad } \Gamma'_1(\bar{a} + \mathfrak{U}_{-\gamma-c_1}) \supseteq \bar{a} + \mathfrak{M}_{-\gamma}$. Here \bar{a} is, as it has been, our fixed element of \mathfrak{U} , a representative of ψ on K_γ . Now the proof of Lemma 3.3 applies to Γ' and shows that the only (Γ'_1, Γ'_1) double cosets in M which intertwine δ'_1 with itself are of the form $\Gamma'_1 b \Gamma'_1$ with $b \in A$. Since $A \cap \Gamma_1 \subseteq \Gamma'_1$, it follows that in particular δ'_1 induces an irreducible representation of Γ_1 , and since δ'_1 is contained in δ_1 , this irreducible representation must be δ_1 .

So now forget δ_1 and look at δ'_1 . Let A_1 be the subgroup of A such that $\text{Ad}^* a(\delta'_1) = \delta'_1$ if $a \in A_1$. I claim $A_1 = A$ and δ'_1 may be extended to a representation of $A \cdot \Gamma'_1$. To see this, let B_ψ now be defined on $\Gamma'_1 \times \Gamma'_1$ by the formula $B_\psi(h_1, h_2) = \psi(h_1 h_2 h_1^{-1} h_2^{-1})$. As with the previous B_ψ , one verifies easily this B_ψ defines an antisymmetric bi-additive T -valued form on Γ'_1 . Let \mathcal{A} be the radical of B_ψ , e.g., $\mathcal{A} = \{h \in \Gamma'_1 : B_\psi(h, \Gamma'_1) = 1\}$. Since one easily calculates for $1 + x, 1 + y \in \Gamma'_1$ that $B_\psi(1 + x, 1 + y) = \Omega_0(\langle \bar{a}, [x, y] \rangle) = \Omega_0(\langle [\bar{a}, x], y \rangle)$, we see $A \cap \Gamma'_1 \subseteq \mathcal{A}$, for if $x \in \mathfrak{U}$, then $[\bar{a}, x] = 0$. I claim $\mathcal{A} \subseteq (A \cap \Gamma'_1) \cdot M_\gamma$. In fact, we see this would be implied by the inclusion $\mathcal{A} \cap X \subseteq M_\gamma$. In view of the form of B_ψ , precisely the same calculation that in the discussion of δ_2 showed $\tilde{\mathcal{A}} \subseteq \mathcal{A}$, shows in this situation that indeed $\mathcal{A} \cap X \subseteq M_\gamma$.

From the elementary theory of two-step nilpotent groups, the restriction of δ'_1 to \mathcal{A} is a multiple of some linear character χ of \mathcal{A} , and χ determines \mathcal{A} . Since δ'_1 lies over ψ on M_γ , the restriction of χ to $\mathcal{A} \cap M_\gamma$ must agree with ψ . Since $(A \cap \Gamma'_1) \cdot (\mathcal{A} \cap M_\gamma) = \mathcal{A}$, we see χ , hence δ'_1 , is determined by the restriction of χ to $A \cap \Gamma'_1$. Since $\text{Ad}^* b$ clearly leaves χ on $A \cap \Gamma'_1$ unchanged, it also leaves δ'_1 unchanged, so $A_1 = A$ as claimed. Since δ'_1 is $\text{Ad}^* A$ -invariant, $\ker \delta'_1$ is normalized by A . Again by the representation theory of two-step groups, we have $\ker \delta'_1 \subseteq \mathcal{A}$, and so $\ker \delta'_1 = \ker \chi$. It follows that $X \cap \mathcal{A} \subseteq \ker \delta'_1$. Therefore $X/X \cap \mathcal{A}$ forms a set of coset representatives for \mathcal{A} in Γ'_1 , and since A normalizes X , these represen-

tatives have the property that, if $\text{Ad } A$ centralizes them modulo A , it centralizes them modulo $\ker \delta'_1$. Now the theory of Weil's representation for finite groups (see [12], [15], [24]) tells us the following fact: there is a well-defined one-to-one correspondence, involving no arbitrary choice of basepoint, between the extensions of χ from $A \cap \Gamma'_1$ to A , and the extensions of δ'_1 from Γ'_1 to $A \cdot \Gamma'_1$. For a given character $\tilde{\varphi}$ of A extending χ , let $\tau(\tilde{\varphi})$ denote the corresponding extension of δ'_1 to $A \cdot \Gamma'_1$ given by Weil's representation. If we fix a character $\tilde{\varphi}_0$ of A , then the relation $\tau(\tilde{\varphi}_0 \tilde{\varphi}) = \tau(\tilde{\varphi}_0) \otimes \tilde{\varphi}$ holds for any character of $A/A \cap \Gamma'_1$. We may also use this formula to define $\tau(\tilde{\varphi}_0 \tilde{\varphi})$ for an arbitrary quasicharacter $\tilde{\varphi}$ of $A/A \cap \Gamma'_1$. Let us note for completeness' sake that an arbitrary representation of $A \cdot \Gamma'_1$ lying above ψ on M_η has the form $\tau(\tilde{\varphi}_0) \otimes \tilde{\varphi}$ where $\tilde{\varphi}$ is now any character (or quasicharacter) of $A/A \cap M_\eta$, extended to $A \cdot \Gamma'_1$ by letting it be trivial on $X \cdot M_\eta$.

Now define $J' = \Gamma'_1 \cdot \Gamma'_2$, and let δ' be the representation of J' whose restriction to Γ'_2 is a multiple of ψ' and whose restriction to Γ'_1 is δ'_1 . (We note that these conditions do indeed define an irreducible representation of J' .) It will be our business from here to the end of the section to analyze the representations of G containing the J' -type δ' . First we will construct them. To this end let $\mathcal{J} = A \cdot \Gamma'_1 \cdot U_P$. We consider the set $\tilde{A}(\delta')$ of all quasicharacters $\tilde{\varphi}$ of A which agree with χ on $A \cap \Gamma'_1 = A \cap J'$, and we denote by $\tau'(\tilde{\varphi})$ the representation of \mathcal{J} which is trivial on U_P , and which agrees with $\tau(\tilde{\varphi})$ on $A \cdot \Gamma'_1$.

Although we have thus far spoken of intertwining only in the context of compact open subgroups of G , we can extend this terminology easily to noncompact, nonopen subgroups. Specifically, if H_1, H_2 are any closed subgroups of G , and if σ_i are finite-dimensional representations of the H_i , then for $g \in G$, we define $I(\sigma_1, \sigma_2, g)$ to be the intertwining number of the restrictions of σ_1 and $\text{Ad}^* g(\sigma_2)$ to $H_1 \cap \text{Ad } g(H_2)$. There is an interpretation of $I(\sigma_1, \sigma_2, g)$ in terms of intertwining distributions between σ_1 and σ_2 , where these are also defined in analogy with the case of H_i open and compact. We say g intertwines σ_1 and σ_2 if $I(\sigma_1, \sigma_2, g) \neq 0$. We say the σ_i intertwine if some $g \in G$ intertwines them. Applying these concepts to J' and δ' , \mathcal{J} and $\tau'(\tilde{\varphi})$ we get the following result.

LEMMA 3.6. (a) *The only $g \in G$ which intertwine δ' with itself belong to $J'AJ'$. For every $b \in A$, $I(\delta', \delta', b) = 1$.*

(b) *The only (J', \mathcal{J}) double coset which supports an intertwining distribution for δ' and $\tau'(\tilde{\varphi})$ is $J' \mathcal{J}$ itself. We have $I(\delta', \tau'(\tilde{\varphi}), 1) = 1$.*

(c) *The only $(\mathcal{J}, \mathcal{J})$ double coset which supports an inter-*

twining distribution for $\tau'(\tilde{\varphi})$ with itself is \mathcal{J} itself.

(d) If $\tilde{\varphi}_1 \neq \tilde{\varphi}_2$, $\tau'(\tilde{\varphi}_1)$ and $\tau'(\tilde{\varphi}_2)$ do not intertwine.

Proof. The statements that $I(\delta', \delta', b) = 1$ and $I(\delta', \tau'(\tilde{\varphi}), 1) = 1$ are clear from the construction of δ' and $\tau'(\tilde{\varphi})$. Just observe that $\tau'(\tilde{\varphi})$ and δ' are irreducible and equivalent on Γ'_1 , and that A normalizes Γ'_1 and leaves δ'_1 -invariant there, and use Lemma 3.5, plus the construction of δ'_1 from δ_1 . The fact that only double cosets of the form $J'bJ'$ with $b \in A$ support intertwining distributions for δ' with itself is almost immediate from Lemmas 3.4 and 3.5. We leave the details to the reader. Now consider the intertwining properties of $\tau'(\tilde{\varphi})$. The group $M_\gamma \cdot U_P = 1 + (\mathfrak{M}_\gamma \oplus \mathcal{U})$ is a subgroup of \mathcal{J} . The function $\tilde{\psi}(1+x) = \Omega(\bar{a})(x)$ can easily be verified to be a linear character on $M_\gamma \cdot U_P$, trivial on $M_\mu \cdot U_P$. Of course $\tilde{\psi}$ coincides with ψ on $(M_\gamma \cdot U_P) \cap K_\gamma$. Precisely the same reasoning as used in Lemmas 2.1 to 2.3 tells us the following two facts. First if $g \in G$, then g intertwines $\tilde{\psi}$ with itself if and only if $\bar{a} + (\mathfrak{M}_\gamma \oplus \mathcal{U})^*$ intersects $\text{Ad } g(\bar{a} + (\mathfrak{M}_\gamma \oplus \mathcal{U})^*)$. Second, g intertwines ψ and $\tilde{\psi}$ if and only if $\bar{a} + L_{-\gamma}$ and $\text{Ad } g(\bar{a} + (\mathfrak{M}_\gamma \oplus \mathcal{U})^*)$ intersect. But if g intertwines $\tau'(\tilde{\varphi})$ with itself, or $\tau'(\tilde{\varphi}_1)$ with $\tau'(\tilde{\varphi}_2)$, or δ' with $\tau'(\tilde{\varphi})$, then some element of $\mathcal{J}g\mathcal{J}$ or $J'g\mathcal{J}$ must intertwine $\tilde{\psi}$ with itself or ψ with $\tilde{\psi}$. A simple calculation gives $(\mathfrak{M}_\gamma \oplus \mathcal{U})^* = \mathfrak{M}_{-\gamma} \oplus \mathcal{U}$. In the discussion since Lemma 3.5 we have seen that $\text{Ad } \Gamma'_1(\bar{a} + \mathfrak{A}_{-\gamma-c_0-c_1}) \supseteq \bar{a} + \mathfrak{M}_{-\gamma}$. Since $\text{ad } \bar{a}$ is nonsingular on \mathcal{U} , we see $\text{Ad } U_P(\bar{a}) = \bar{a} + \mathcal{U}$. Combining these facts gives $\text{Ad } \mathcal{J}(\bar{a} + \mathfrak{A}_{-\gamma-c_0-c_1}) \supseteq \bar{a} + (\mathfrak{M}_{-\gamma} \oplus \mathcal{U})$. Now exactly the same proof as used in Lemma 3.3 shows g must belong to $\mathcal{J}A\mathcal{J} = \mathcal{J}$ to intertwine $\tilde{\psi}$ with itself, and must belong to $J'A\mathcal{J}$ to intertwine ψ with $\tilde{\psi}$. From this the lemma follows immediately.

We are now in a position to construct the series of induced representations attached to δ' . For each $\tilde{\varphi}$ in $\tilde{A}(\delta')$, we let $\pi(\delta', \tilde{\varphi}) = \pi(\tilde{\varphi})$ be the induced representation $\pi(\tilde{\varphi}) = \text{ind}_{\mathcal{J}}^G \tau'(\tilde{\varphi})$.

PROPOSITION 3.2. *Those of the $\pi(\tilde{\varphi})$ which are unitary are irreducible. Those of the $\pi(\tilde{\varphi})$ which are irreducible are pairwise inequivalent.*

Proof. Using Lemma 3.6, this result is immediate from Bruhat [2].

At this point I would like to digress briefly and comment on the relationship of this result to Harish-Chandra's philosophy of cusp forms, and more particularly to his theory of the Eisenstein integral on p -adic groups [10]. First let us give an alternate description of the $\pi(\tilde{\varphi})$ in terms of series induced from $P = M \cdot U_P$. Recall the map

$\log: P \rightarrow Z'$ as defined in §2. In the present situation, r is the split rank of A and $\log(A)$ has finite index in Z' . Recall that \mathcal{Q} was the group of quasicharacters of P which factored through \log . By restriction, \mathcal{Q} is mapped into the group of quasicharacters of A . The kernel of this restriction map is finite, of order equal to the index of $\log(A)$ in Z . Since $(\ker \log) \cap A = A_0$, the image of \mathcal{Q} under the restriction map is the group of quasicharacters of A/A_0 . If $\varphi \in \mathcal{Q}$, let $\tilde{\varphi}$ be its restriction to A . We see that if $\tilde{\varphi}_0 \in \tilde{A}(\delta')$, then $\tilde{\varphi}_0 \tilde{\varphi} \in \tilde{A}(\delta')$ also for $\varphi \in \mathcal{Q}$. Thus \mathcal{Q} acts on $\tilde{A}(\delta')$. Under this action $\tilde{A}(\delta')$ clearly breaks up into finitely many orbits, each orbit being a set of quasicharacters with fixed restriction to A_0 . Let $\{\tilde{\varphi}_i\}_{i=1}^l$, with $l = \#(A_0/A \cap J')$, be a set of unitary characters of A , one from each orbit of \mathcal{Q} in $\tilde{A}(\delta')$. Then every element of $\tilde{A}(\delta')$ may be written in the form $\tilde{\varphi}_i \tilde{\varphi}$, with $\varphi \in \mathcal{Q}$ and $\tilde{\varphi}_i$ uniquely determined. Define $\sigma_i = \text{ind}_{\mathcal{P}}^P \tau'(\tilde{\varphi}_i)$. Then the series of representations $\mathcal{Q}(\sigma_i) = \{\sigma_i \otimes \varphi \text{ for } \varphi \in \mathcal{Q}\}$, as defined in §2, is the same set as $\{\text{ind}_{\mathcal{P}}^P (\tau'(\tilde{\varphi}_i) \otimes \tilde{\varphi})\} = \{\text{ind}_{\mathcal{P}}^P \tau'(\tilde{\varphi}_i \tilde{\varphi})\}$. Thus $\bigcup_{i=1}^l \mathcal{Q}(\sigma_i) = \{\text{ind}_{\mathcal{P}}^P \tau'(\tilde{\varphi}'): \tilde{\varphi}' \in \tilde{A}(\delta')\}$. By transitivity of induction we have $\text{ind}_{\mathcal{P}}^G \sigma_i \otimes \varphi = \pi(\tilde{\varphi}_i \tilde{\varphi})$. By the same reasoning as in Proposition 3.1, we see that the σ_i are all irreducible unitary supercuspidal representations of P . Therefore we conclude that the $\pi(\tilde{\varphi}')$, for $\tilde{\varphi}' \in \tilde{A}(\delta')$, are equivalent to the representations of finitely many series of representations of G induced from supercuspidal representations of P . Thus this construction of the $\pi(\tilde{\varphi}')$'s falls within the scope of the philosophy of cusp forms. In this connection let us also note that Harish-Chandra has formulated a version of Bruhat's theory which applies to representations induced from supercuspidal representations of parabolic subgroups. It would follow from Lemma 3.5 that the $\sigma_i \otimes \varphi$ would always be unramified in Harish-Chandra's sense, so that his results would also imply Proposition 3.2.

With the above description of the $\pi(\tilde{\varphi}')$, we may identify the $\tilde{\varphi}'$ which yield unitary representations. Indeed, for $y \in P$, define $d_P(y) = |\det(\text{Ad } y_{1 \times r})|$. Then it is well-known [9] that if σ is a unitary representation of P , $\text{ind}_{\mathcal{P}}^G \sigma \otimes d_P^{-1/2}$ is a unitary representation of G . Therefore we conclude that $\pi(\tilde{\varphi}' d_P^{-1/2})$ is unitary if $\tilde{\varphi}'$ is unitary. In a while, we will see the converse is also true.

We end this digression with some further remarks on the irreducibility of the $\pi(\tilde{\varphi}')$. What the techniques of Bruhat and Harish-Chandra show is that $\pi(\tilde{\varphi}')$ allows no intertwining mappings with itself, and that there are no intertwining mappings from $\pi(\tilde{\varphi}_1')$ to $\pi(\tilde{\varphi}_2')$ if $\tilde{\varphi}_1' \neq \tilde{\varphi}_2'$. These results are true for arbitrary $\tilde{\varphi}', \tilde{\varphi}_1', \tilde{\varphi}_2' \in \tilde{A}(\delta')$. However, it is only when $\pi(\tilde{\varphi}')$ is unitary, or when the $\pi(\tilde{\varphi}_i')$ are irreducible that one may deduce irreducibility or inequivalence directly from the lack of intertwining operators. Nevertheless, it

is to be expected that $\pi(\tilde{\varphi}')$ irreducible for all $\tilde{\varphi}'$. We will not attempt a proof here, but we will comment on the methods available for such a proof. They appear to be at least three. One is that used by Wallach [23] in his study of reducibility of nonunitary principal series for semisimple Lie groups. This method uses cyclic vectors. As we will see shortly, the J' -type δ' occurs with multiplicity one in each $\pi(\tilde{\varphi}')$. It seems it would not be hard to demonstrate that vectors of type δ' in $\pi(\tilde{\varphi}')$ are cyclic. The second possible method is that of c -functions. We will give explicit formulas for the δ'^* -spherical functions. These formulas will probably imply the c -functions Harish-Chandra [10] attaches to the series $\mathcal{O}(\sigma_i)$ will be constant, or perhaps exponentials. When Harish-Chandra's theory is completed, this should in turn imply irreducibility for all $\tilde{\varphi}' \in \tilde{A}(\delta')$. The third possible method which would necessitate taking $\bar{\alpha}$ to be very nondegenerate would involve an analysis of all essential J' -types which might possibly occur in the $\pi(\tilde{\varphi}')$. Execution of any of these three methods of attack would yield interesting additional information on the $\pi(\tilde{\varphi}')$.

We now return to our main concern, the Fourier analysis of $\mathcal{H}(\delta'')$.

LEMMA 3.7. *The representation δ' of J' occurs exactly once in the restriction of $\pi(\tilde{\varphi}')$ to J' , for any $\tilde{\varphi}' \in \tilde{A}(\delta')$.*

Proof. This lemma follows from Lemma 3.6 together with some general considerations [18] concerning the quantitative relation between intertwining and multiplicities, which we now review.

Suppose C is a compact open subgroup of G , and choose $\delta \in \tilde{C}$. Let H be some closed subgroup of G , and τ an admissible representation of H . Let $\rho = \text{ind}_H^G \tau$ be the representation of G induced from τ on H . Explicitly, if V is the space of τ , then X , the space of ρ , is the space of all locally constant functions f from G to V , compactly supported modulo H and satisfying the transformation law $f(gh) = \tau(h)^{-1}f(g)$ for all $g \in G$, $h \in H$. The action ρ of G on X is given by $\rho(g)f(x) = f(g^{-1}x)$ for $g, x \in G$. We want to consider the restriction of ρ to C , and in particular, to compute the multiplicity of δ in ρ . The first observation is that if $f \in X$ and α is any right H -invariant complex-valued function, then $\alpha f \in X$. In particular we could let α be the characteristic function of a (C, H) double coset. Then we see that the subspace X_g of X consisting of functions supported on the CgH is invariant by $\rho(C)$, and that X is the direct sum of the spaces X_g .

Consider the action of C on X_g . By virtue of its transformation law, $f \in X_g$ is determined by its restriction to Cg . The function f'

on C defined by $f'(x) = f(xg)$ for $x \in C$ is easily seen to satisfy the transformation law $f'(xy) = \text{Ad}^* g(\tau)(y)^{-1} f'(x)$ for $y \in C \cap \text{Ad } g(H)$. Conversely given $f': C \rightarrow V$ satisfying this transformation law, define f on CgH by $f(xgh) = \tau(h)^{-1} f'(x)$. Then $f \in X_g$, and the correspondences $f \leftrightarrow f'$ are mutually inverse. In this manner one concludes that the representation of C defined on X_g is equivalent to $\text{ind}_B^G \text{Ad}^* g(\tau)$, where $B = C \cap \text{Ad } g(H)$. Since C is compact, we may apply Frobenius reciprocity to conclude that the multiplicity of δ in $\text{ind}_B^G \text{Ad}^* g(\tau)$ is equal to the intertwining number $I(\delta, \tau, g)$ of the restrictions to B of δ and $\text{Ad}^* g(\tau)$. Note that this will be finite by admissibility of τ and openness of B in $\text{Ad } g(H)$. Therefore the total multiplicity of δ in ρ is the sum of $I(\delta, \tau, g)$ over all (C, H) double cosets.

Now taking $C = J'$, $\delta = \delta'$, $H = \mathcal{J}$, and $\tau = \tau'(\tilde{\varphi})$ for any $\tilde{\varphi} \in \tilde{A}(\delta')$, we see that Lemma 3.6 implies Lemma 3.7.

Recall from the discussion in §2 that the occurrence α times of δ' in a representation of G leads to a representation of $\mathcal{H}(\delta'')$ of degree α . Therefore, attached to each $\pi(\tilde{\varphi})$ we have a one-dimensional representation of $\mathcal{H}(\delta'')$, that is, a homomorphism from $\mathcal{H}(\delta'')$ to \mathbb{C} . We will now explicitly calculate the spherical functions attached to these representations. This will yield the Fourier decomposition of $\mathcal{H}(\delta'')$ and will allow computation of the Plancherel measure of the $\{\pi(\tilde{\varphi})\}$. For the usual principal series, this analysis was done in [12]. Again we must begin with recollections of the general context of our discussion.

If Z is the space of δ' , then Z^* , the complex vector space dual of Z , is the space of δ'' . If $T \in \text{Hom}(Z)$, let $T^b \in \text{Hom}(Z^*)$ be the transformation adjoint to T , and likewise if $S \in \text{Hom}(Z^*)$, $S^b \in \text{Hom}(Z)$ is the adjoint of S . Remember $\mathcal{H}(\delta'')$ is the space of all compactly supported functions $f: G \rightarrow \text{Hom}(Z^*)$ such that for $g \in G$, x_1 and $x_2 \in J'$, the formula $f(x_1 g x_2) = \delta''(x_1) f(g) \delta''(x_2)$. Since $\delta''(x_i) = (\delta'(x_i)^{-1})^b$, we see that, defining $f^b(g) = (f(g))^b$, we have

$$f^b(x_1 g x_2) = \delta'^{-1}(x_2) f^b(g) \delta'(x_1)^{-1}$$

for $f \in \mathcal{H}(\delta'')$. By a δ'' -spherical function we mean an intertwining distribution Φ for δ'' with itself, such that there is a homomorphism $\lambda: \mathcal{H}(\delta'') \rightarrow \mathbb{C}$ such that for any $f \in \mathcal{H}(\delta'')$, $f * \Phi = \lambda(f)\Phi$. Here $f * \Phi$ indicates the convolution of f with Φ . There is a well-known method (see [6]) to construct from $\pi(\tilde{\varphi})$ a δ'' -spherical function whose associated homomorphism of $\mathcal{H}(\delta'')$ is the same as the homomorphism defined by $\pi(\tilde{\varphi})$. We will now perform that construction.

We remember $\pi(\tilde{\varphi}) = \text{ind}_{\mathcal{J}}^G \tau'(\tilde{\varphi})$ for $\tilde{\varphi} \in \tilde{A}(\delta')$. For convenience, we suppose $\tau'(\tilde{\varphi})$ acts on the space Z of δ' in such a way that

$\delta'(x) = \tau'(\tilde{\varphi})(x)$ for $x \in J' \cap \mathcal{J}$. Then the space X of $\pi(\tilde{\varphi})$ is the space of locally constant functions $\alpha: G \rightarrow Z$ such that $\alpha(gy) = \tau'(\tilde{\varphi})(y)^{-1}\alpha(g)$ for $g \in G$, $y \in \mathcal{J}$, and such that α has compact support modulo \mathcal{J} . To construct our spherical function, we consider $X \otimes Z^*$. We see $X \otimes Z^*$ is the space of locally constant functions $\alpha: G \rightarrow \text{Hom}(V)$ such that $f(gy) = \tau'(\tilde{\varphi})(y^{-1})f(g)$, for $y \in \mathcal{J}$ and $g \in G$, and which are compactly supported modulo \mathcal{J} . Let $X^b = \{\alpha^b: \alpha \in X \otimes Z^*\}$. Then if $\beta \in X^b$, we find for $g \in G$, $y \in \mathcal{J}$, that $\beta(gy) = \beta(g)(\tau'(\tilde{\varphi})(y^{-1}))^b = \beta(g)\tau'(\tilde{\varphi})^t(y)$. The action $\pi(\tilde{\varphi})$ of G on X was given by $\pi(\tilde{\varphi})(g)\alpha(u) = \alpha(g^{-1}u)$ for $g, u \in G$, and $\alpha \in X$. Transferring this action to $X \otimes Z^*$ by letting G act trivially on Z^* , and thence to X^b by letting the isomorphism $\alpha \rightarrow \alpha^b$ for $\alpha \in X \otimes Z^*$ be an intertwining map, we still find $\pi(\tilde{\varphi})(g)\beta(u) = \beta(g^{-1}u)$ for $g, u \in G$, and $\beta \in X^b$.

Denote by $X(\delta')$ that subspace of the functions $\beta \in X^b$ such that $\pi(\tilde{\varphi})(x)\beta = \delta'^t(x)\beta$ for $x \in J'$. Then $\beta^b \in X \otimes Z^*$ is an intertwining map between δ' and the restriction of $\pi(\tilde{\varphi})$ to J' . Thus Lemma 3.7 shows $X(\delta')$ is one-dimensional. Alternatively, by virtue of the transformation law defining X^b , any $\beta \in X(\delta')$ is an intertwining distribution between δ'^t and $\tau'(\tilde{\varphi})^t$, so Lemma 3.6, which applies as well to δ'^t as it does to δ' , shows $\dim X(\delta') = 1$. In any case, we see that if $\beta_0: G \rightarrow \text{Hom}(Z^*)$ is defined to be zero of $fJ' \setminus \mathcal{J}$ and on $J' \setminus \mathcal{J}$ to be given by the formula $\beta_0(xy) = \delta'^t(x)\tau'(\tilde{\varphi})^t(y)$ for $x \in J'$, $y \in \mathcal{J}$, then β_0 is a well-defined nonzero function in $X(\delta')$. Since $X(\delta')$ is one-dimensional, and since, as is easily checked, left convolution by $f \in \mathcal{H}(\delta'^t)$ preserves $X(\delta')$, it follows that for any $f \in \mathcal{H}(\delta'^t)$, $f * \beta_0 = \lambda(f)\beta_0$ for some complex number $\lambda(f)$. Then $f \rightarrow \lambda(f)$ is by definition the homomorphism of $\mathcal{H}(\delta'^t)$ associated to $\pi(\tilde{\varphi})$. There is a very simple way to produce from β_0 an intertwining distribution for δ'^t with itself. Namely, define

$$\Phi(\tilde{\varphi})(g) = \Phi(g) = \int_{J'} \beta_0(gx) \delta'^t(x^{-1}) dx, \quad \text{for } g \in G.$$

Since right convolution commutes with left convolution, we still have $\Phi(xg) = \delta''(x)\Phi(g)$ for $x \in J'$, $g \in G$, and we still have $f * \Phi = \lambda(f)\Phi$ for $f \in \mathcal{H}(\delta'^t)$. Moreover, if $y \in J'$, we compute

$$\begin{aligned} \Phi(gy) &= \int_{J'} \beta_0(gyx) \delta'^t(x^{-1}) dx = \int_{J'} \beta_0(gx) \delta'^t((y^{-1}x)^{-1}) dx \\ &= \left(\int_{J'} \beta_0(gx) \delta'^t(x^{-1}) dx \right) \delta'^t(y) = \Phi(g) \delta'^t(y). \end{aligned}$$

Hence Φ is a δ'^t -spherical function with associated homomorphism λ .

Let us compute Φ explicitly. By Lemma 3.6, Φ is supported on $J'A'J'$, and because of the transformation law Φ satisfies, it will be enough to compute $\Phi(b)$ for $b \in A$. We fix Haar measures dg , $d\bar{u}$,

dm and du on G , $U_{\bar{P}}$, M , and U_P respectively, such that the measures of K_0 , $U_{\bar{P}} \cap J'$, and $M \cap K_0$ are one, and so that $dg = d\bar{u}dmdu$. All integrations below are taken with respect to the restrictions of these measures to the sets appearing in the integrations. We get

$$\begin{aligned}\Phi(b) &= \int_{J'} \beta_0(by) \delta'^t(y^{-1}) dy \\ &= \int_{(J' \cap U_{\bar{P}})} dy_1 \left(\int_{J' \cap P} \beta_0(by_1 y_2) \delta'^t(y_1^{-1}) \delta'^t(y_2^{-1}) dy_2 \right) \\ &= \alpha \int_{J' \cap U_{\bar{P}}} \beta_0(by_1) dy_1,\end{aligned}$$

where $\alpha = m(J' \cap P)$ is the measure of $J' \cap P$, since δ'^t is trivial on $J' \cap U_{\bar{P}}$ and δ'^t and $\tau'(\tilde{\varphi})^t$ agree on $J' \cap P$. Write $J' \cap U_{\bar{P}} = C$. Continuing the calculation we get

$$\int_C \beta_0(by_1) dy_1 = \int_C \beta_0(by_1 b^{-1}b) dy_1 = m(C \cap \text{Ad } b^{-1}(C)) \beta_0(b)$$

since for $u \in U_{\bar{P}}$, $\beta_0(ub)$ is equal to $\beta_0(b)$ or zero according as u is or is not in J' . Since $\beta_0(b) = \tau'(\tilde{\varphi})^t(b)$, we finally get

$$(3) \quad \Phi(b) = m(J' \cap P) m(C \cap \text{Ad } b^{-1}(C)) \tau'(\tilde{\varphi})^t(b).$$

Now we may compute the Fourier transform on $\mathcal{H}(\delta'^t)$, sometimes called the spherical (Fourier) transform. If $\Phi(\tilde{\varphi})$ denotes the spherical function associated to $\pi(\tilde{\varphi})$ for $\tilde{\varphi} \in \tilde{A}(\delta')$, then for $f \in \mathcal{H}(\delta')$ we define a function \hat{f} on $\tilde{A}(\delta')$ by the formula $f * \Phi(\tilde{\varphi}) = \hat{f}(\tilde{\varphi}) \Phi(\tilde{\varphi})$. Since $\tilde{A}(\delta')$ has the structure of a finite union of complex affine varieties, each isomorphic to $(C^*)^r$, there is defined on $\tilde{A}(\delta')$ a preferred ring of functions, the affine functions. Denote this ring $\mathcal{P}(\tilde{A}(\delta'))$. We see $\mathcal{P}(\tilde{A}(\delta'))$ contains the functions \hat{b} gotten by evaluating $\tilde{\varphi} \in \tilde{A}(\delta')$ at b — that is, $\hat{b}(\tilde{\varphi}) = \tilde{\varphi}(b)$. In fact as b ranges over A , the \hat{b} form a basis for $\mathcal{P}(\tilde{A}(\delta'))$.

PROPOSITION 3.3. (a) For $f \in \mathcal{H}(\delta'^t)$, $\hat{f} \in \mathcal{P}(\tilde{A}(\delta'))$.

(b) The resulting map $\hat{\cdot}: \mathcal{H}(\delta'^t) \rightarrow \mathcal{P}(\tilde{A}(\delta'))$ is an isomorphism of algebras.

(c) In particular, $\mathcal{H}(\delta'^t)$ is abelian, and isomorphic to an affine algebra.

(d) The homomorphisms $f \rightarrow \hat{f}(\tilde{\varphi})$ for $\tilde{\varphi} \in \tilde{A}(\delta')$ are the totality of complex homomorphisms of $\mathcal{H}(\delta'^t)$.

(e) If for $b \in A$, $f_b \in \mathcal{H}(\delta'^t)$ is the function which is supported on $J'bJ'$ and given there by $f_b(x_1 b x_2) = \delta'^t(x_1) \tau'(\varphi_0)^t(b) \delta'^t(x_2)$, where $x_i \in J'$ and $\varphi_0 \in \tilde{A}(\delta')$ is a fixed unitary character, then

$$(4) \quad \hat{f}_b = \alpha(b) \hat{b},$$

where the constant $\alpha(b)$ is given by

$$(5) \quad \alpha(b) = m(C \cap \text{Ad } b(C))m(J'bJ')\tilde{\varphi}_0^{-1}(b).$$

Proof. Clearly (e) implies (a) and (b), since the f_b 's form a basis for $\mathcal{H}(\delta'')$. But (c) follows from (b) and (d) follows from (b) and (c) and Hilbert's Nullstellensatz. So we need to prove (e) and to do this we see it is enough to compute $f_b * \Phi(\tilde{\varphi})(1)$. We get

$$\begin{aligned} f_b * \Phi(\tilde{\varphi})(1) &= \int_G f_b(g)\Phi(\tilde{\varphi})(g^{-1})dg = \int_{J' \times J'} f_b(g)\Phi(\tilde{\varphi})(g^{-1})dg \\ &= \alpha_1 \alpha_2 \int_{J' \times J'} (\delta''(x_1)\tau'(\tilde{\varphi}_0)'(b)\delta''(x_2))(\delta''(x_2^{-1})\tau'(\tilde{\varphi})'(b^{-1})\delta''(x_1^{-1}))dx_1dx_2 \\ &= \alpha_1 \alpha_2 \alpha_3 \tilde{\varphi}\tilde{\varphi}_0^{-1}(b), \end{aligned}$$

where the constants α_i are given by: $\alpha_1 = m(J' \cap P)m(C \cap \text{Ad } b(C))$, and $\alpha_2 = m(J'bJ')m(J')^{-2}$, and $\alpha_3 = m(J')^2$. Combining these constants and plugging them into the definition of \hat{f}_b , using $\Phi(\tilde{\varphi})(1) = m(J' \cap P)$, we get $\hat{f}_b(\tilde{\varphi}) = m(C \cap \text{Ad } b(C))m(J'bJ')\tilde{\varphi}_0^{-1}(b)\tilde{\varphi}(b)$. This equation is by inspection equivalent to (4) and (5).

COROLLARY. *Any irreducible representation of G containing the J' -type δ' is isomorphic to a sub-quotient of some $\pi(\tilde{\varphi})$. In particular δ' can occur at most once in any representation of G .*

REMARK. When the $\pi(\tilde{\varphi})$'s are shown all to be irreducible, then this sub-quotient result may be replaced by an isomorphism result.

Proof. The reasoning that leads from Proposition 3.3 to this corollary is standard. In fact, if the irreducible representation ρ contains δ' , then the representation of $\mathcal{H}(\delta'')$ associated to ρ produces some $\Phi(\tilde{\varphi})$ as a matrix coefficient of ρ , by (d) of the proposition. Then $\Phi(\tilde{\varphi})$ generates under left translations by G a space Y of functions, and G acts on Y by a representation equivalent to ρ . See Harish-Chandra [10] for details. But we may map the space X of $\pi(\sigma_i, \varphi)$ into Y by right convolution with the identity of $\mathcal{H}(\delta'')$, in the same way as we constructed the $\Phi(\tilde{\varphi})$'s, and this mapping is clearly an intertwining operator for the left action of G on X and Y .

In order to establish "Fourier inversion" for $\mathcal{H}(\delta'')$ and compute the Plancherel measure on $\tilde{A}(\delta')$, we must establish an inner product on $\mathcal{H}(\delta'')$. Here we use the fact that δ' and $\tau'(\tilde{\varphi}_0)$ are actually unitary representations. Thus choose a $\tau(\tilde{\varphi}_0)^*$ -invariant Hermitian inner product on Z^* , the space of δ'' . For $T \in \text{Hom}(Z^*)$, let T^* denote the adjoint of T with respect to this inner product. If

$f: G \rightarrow \text{Hom}(Z^*)$ is a function, then the function f^* is defined by $f^*(g) = (f(g^{-1}))^*$ for $g \in G$. We say T or f is self-adjoint if $T = T^*$ or $f = f^*$. We note that since δ'' will be a unitary representation with respect to our inner product, $\mathcal{H}(\delta'')$ is self-adjoint in the sense that if $f \in \mathcal{H}(\delta'')$, then also $f^* \in \mathcal{H}(\delta'')$.

If $T_1, T_2 \in \text{Hom}(Z^*)$, then $\text{tr}(T_1 T_2^*)$ is the usual Hilbert-Schmidt inner product of T_1 and T_2 . We use this to define an inner product on $\mathcal{H}(\bar{\delta})$. If $f_1, f_2 \in \mathcal{H}(\bar{\delta})$, then their inner product is given by

$$(f_1, f_2) = \text{tr}((f_1 * f_2^*)(1)) = \int_G \text{tr}(f_1(g)(f_2(g))^*) dg.$$

If f_b is as in Proposition 3.3, part (e), we compute

$$(f_b, f_b) = (\dim \delta'^t) m(J' b J')$$

since $f_b(x)$ is unitary on $J' b J'$, and zero off $J' b J'$. We also note that for b_1, b_2 not congruent modulo $A \cap J'$, f_{b_1} and f_{b_2} are orthogonal, so the f_b 's form an orthogonal basis for $\mathcal{H}(\delta'')$ with respect to $(,)$.

Certain of the $\pi(\tilde{\varphi})$ are unitary. Let $\mathcal{U}(\tilde{A}(\delta'))$ denote the set of $\tilde{\varphi}$ for which $\pi(\tilde{\varphi})$ is unitary. If we identify $\tilde{\varphi}$ and $\pi(\tilde{\varphi})$ for the present, then the Plancherel measure on $\tilde{A}(\delta')$ is by definition a measure $d\tilde{\varphi}$, supported on $\mathcal{U}(\tilde{A}(\delta'))$, such that with respect to it, the spherical Fourier transform becomes a unitary map. That is, the equation

$$(6) \quad (f, f) = \int_{\mathcal{U}(\tilde{A}(\delta'))} \hat{f}(\tilde{\varphi}) \overline{\hat{f}(\tilde{\varphi})} d\tilde{\varphi}$$

should hold for all $f \in \mathcal{H}(\delta'^t)$. The Plancherel measure is known to exist uniquely [4]. We will now determine $\mathcal{U}(\tilde{A}(\delta'))$, and then determine the Plancherel measure.

If $\pi(\tilde{\varphi})$ is unitary, then $\Phi(\tilde{\varphi})$ must be self-adjoint. From formula (3), we deduce the necessary and sufficient condition for $\Phi(\tilde{\varphi})$ to be self-adjoint is

$$(7) \quad m(C \cap \text{Ad } b^{-1}(C)) \tau(\tilde{\varphi})^t(b) = m(C \cap \text{Ad } b(C)) (\tau(\tilde{\varphi})^t(b^{-1}))^*.$$

Suppose $\Phi(\tilde{\varphi}_i)$ are self-adjoint for $i = 1, 2$. Then dividing equation (7) for $\tilde{\varphi}_1$ by equation (7) for $\tilde{\varphi}_2$, we get $\tilde{\varphi}_2 \tilde{\varphi}_1^{-1}(b) = \overline{(\tilde{\varphi}_2^{-1} \tilde{\varphi}_1)}(b)$. That is, $\tilde{\varphi}_2 \tilde{\varphi}_1^{-1}$ must be unitary. But we already know, from the digression following Proposition 3.2 that if $\tilde{\varphi} \in \tilde{A}(\delta')$ is unitary, then $\pi(\tilde{\varphi} d_P^{-1/2})$ is unitary. Thus we conclude $\mathcal{U}(\tilde{A}(\delta)) = \{\tilde{\varphi} d_P^{-1/2}: \tilde{\varphi} \in \tilde{A}(\delta'), \tilde{\varphi} \text{ unitary}\}$. Thus as one would expect, no ‘‘complementary series’’ occur among the $\pi(\tilde{\varphi})$. Also note if $\tilde{\varphi}$ is unitary, then $(\tau(\tilde{\varphi})^t(b^{-1}))^* = \tau(\tilde{\varphi})^t(b)$. Plugging this in (7), we get the relation

$$(8) \quad m(C \cap \text{Ad } b^{-1}(C)) d_P(b)^{1/2} = m(C \cap \text{Ad } b(C)) d_P(b)^{-1/2}.$$

The unitary characters in $\tilde{A}(\delta')$ fill up $^*(A_0/A \cap J')$ real r -dimensional tori. We see $\mathcal{Z}(\tilde{A}(\delta'))$ is simply the translate of this set of tori by $d_P^{-1/2}$. Thus $\mathcal{Z}(\tilde{A}(\delta'))$ also is the disjoint union of $^*(A_0/A \cap J')$ real r -dimensional tori. We may now state the final result of the section.

PROPOSITION 3.4. *The Plancherel measure on any component torus of $\mathcal{Z}(\tilde{A}(\delta'))$ is ordinary Lebesgue measure, normalized so that the total measure of the torus is equal to*

$$\gamma = ^*(A_0/A \cap J')^{-1} m(J)^{-1} \dim \delta'^t.$$

Thus the total Plancherel measure of the representations $\{\pi(\tilde{\varphi})\}$ is equal to $\gamma' = ^(A_0/A \cap J)\gamma = m(J)^{-1} \dim \delta'^t$.*

Proof. We note that if the Plancherel measure $d\tilde{\varphi}$ is as stated, then for b_1 and b_2 in A , not congruent modulo $A \cap J'$, part (e) of Proposition 3.3 plus the usual orthogonality relations for characters of abelian groups shows that \hat{f}_{b_1} and \hat{f}_{b_2} are orthogonal in

$$L^2(\mathcal{Z}(\tilde{A}(\delta')), d\tilde{\varphi}).$$

Since \hat{f}_{b_1} and \hat{f}_{b_2} are orthogonal in $\mathcal{H}(\delta'^t)$ as we have already remarked, we see if we verify (6) for $f = f_b$, the proposition will follow.

From (4), and our knowledge of which $\tilde{\varphi}$ belong to $\mathcal{Z}(\tilde{A}(\delta'))$, we see that on $\mathcal{Z}(\tilde{A}(\delta'))$, $\hat{f}_b \hat{\bar{f}}_b$ has the constant value $\alpha(b) \overline{\alpha(b)} d_P^{-1}(b)$, with $\alpha(b)$ as in (5). Using the known value of (f_b, f_b) , we may translate (6) into the relation

$$(\dim \delta'^t) m(J' b J') = \gamma' m(C \cap \text{Ad } b(C))^2 m(J' b J')^2 d_P^{-1}(b).$$

Now recall $J' = C \cdot (M \cap J') \cdot D$, where $C = J' \cap U_{\bar{P}}$ as before and $D = J' \cap U_P$. We see $m(J' b J') = m(J')^2 m(J' \cap \text{Ad } b(J'))^{-1}$. Since b normalizes $M \cap J'$, and since by our normalization of Haar measures $dg = d\bar{u} dm dn$, we see

$$m(J' \cap \text{Ad } b(J')) = m(C \cap \text{Ad } b(C)) m(D \cap \text{Ad } b(D)) m(J' \cap M).$$

If $C = 1 + \bar{Y}$ and $D = 1 + Y$, then $A = Y \oplus \bar{Y}$ is, by the construction of $\Gamma'_2 \subseteq J'$, a self-dual lattice in $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ with respect to the form \tilde{B} used in the discussion preceding Lemma 3.4. Since $\text{Ad } b$ preserves \tilde{B} and $m(C) = 1$ by agreement, it follows that

$$m(D \cap \text{Ad } b(D)) = m(D) m(C \cap \text{Ad } b^{-1}(C)).$$

Therefore we finally get

$$m(J' \cap \text{Ad } b(J')) = m(J')m(C \cap \text{Ad } b(C))m(C \cap \text{Ad } b^{-1}(C)).$$

Putting this back into our equation for γ' and simplifying, we get $\dim \delta'^t = \gamma' m(C \cap \text{Ad } b(C))m(C \cap \text{Ad } b^{-1}(C))^{-1} m(J') d_{\bar{P}^{-1}}(b)$. Now using (8), this becomes simply $\dim \delta'^t = \gamma' m(J')$, and the proposition is proved.

4. 99(44/100)% of the Plancherel formula. In the previous section we constructed a single representation δ' of a single compact subgroup J' , and essentially performed an analysis of $\text{ind}_{K_0}^G \delta'$. In this section we will consider many δ' 's and J' 's simultaneously. We will try to get a picture of how they interact, try to see how much of the harmonic analysis of $L^2(G)$ they account for, and try to see the nature of the portion unaccounted for and some of the difficulties to be overcome in obtaining it.

If we take $f \in C_c^\infty(G)$, we can decompose it into its Fourier components with respect to the action of K_0 on G by right translations. In this way we get the easy decomposition

$$C_c^\infty(G) \simeq \sum_{\delta \in \hat{K}_0} (\dim \delta) \text{ind}_{K_0}^G \delta.$$

Thus if we could perform for arbitrary δ the kind of analysis carried out in §3, we would in some sense have done the harmonic analysis of G . In this form, however, the problem is both too hard and the wrong problem. It is too hard because, as we said in the introduction, it seems unlikely that we will even enumerate all $\delta \in \hat{K}_0$, let alone decompose $\text{ind}_{K_0}^G \delta$. It is the wrong problem because it gives a decomposition of $C_c^\infty(G)$ into left-invariant subspaces, rather than bi-invariant subspaces. Upon right translation the above decomposition will not be preserved. There will be a great deal of mixing. This mixing will be precisely expressed by the intertwining of the δ 's. As a simpler, more appropriate, and vaguer problem, one might ask if there is a subset $S \subseteq \hat{K}_0$ which has the following properties:

(i) The subspace of $C_c^\infty(G)$ consisting of functions whose Fourier coefficients under right translations by K_0 lie in S generates all of $C_c^\infty(G)$ under right translations by G .

(ii) For $\delta \in S$, $\text{ind}_{K_0}^G \delta$ is analyzable.

(iii) There is only a small amount of mixing between different elements of S .

All three conditions are subject to interpretation of course. In (i) we can specify in what sense (algebraically, uniformly-on-compacta limits, in L^2 , etc.) $C_c^\infty(G)$ is to be generated. Condition (ii) depends on the degree of explicitness you require. Small is clearly the key word in (iii). As a minimal requirement, we should demand that

any element of S intertwine with at most finitely many others. If this is all we ask, then Sections 1 and 2 show that the essential K_0 -types more or less satisfy Conditions (i) and (iii). On the other hand, for the rough results of those sections we needed only a very rough decomposition of \hat{K}_0 , and on closer inspections one sees many superfluous representations of K_0 have been labeled essential by that classification. Thus it seems unlikely that the essential K_0 -types will satisfy (ii).

On the other hand, in §3 we found some representations which satisfy (ii) extremely well. We will see in this section they also satisfy (iii) very well. They do not satisfy (i), but we shall see that in a certain sense, they fail to do so only by terms of lower order. I would hope that the set of representations of §3 and this section could be enlarged suitably so as to satisfy (i) while still satisfying (ii) and (iii).

Conditions (i) and (iii) together say that elements of S mix only weakly with each other, but very strongly with K_0 -types not in S . In order to discuss these conditions intelligently therefore, we should have some quantitative measure of the amount of mixing between two K_0 -types. We will now develop such a measure. To do this, we will have to discuss the Plancherel measure. Since G is known [11] [21] to be type I (in fact $C_c^\infty(G)$ acts on irreducible modules by finite rank operators) this entails no difficulty.

Let \hat{G} denote the dual space of G , that is, the space of equivalence classes of unitary representations of G . There is a standard topology and compatible Borel structure on \hat{G} for which \hat{G} becomes a reasonable Borel space. For a nice G such as ours, \hat{G} , or many parts of it, should even be endowable with the structure of real analytic space.

Then it is known [4] that there is unique measure $d\rho$ on \hat{G} such that for $f \in C_c^\infty(G)$

$$(1) \quad f(1) = \int_{\hat{G}} \theta_\rho(f) d\rho.$$

The measure $d\rho$ is called the Plancherel measure. For our purposes, there is a somewhat more suggestive way of writing (1). For $f \in C_c^\infty(G)$, define f^* by the recipe $f^*(g) = \overline{f(g^{-1})}$, $g \in G$. Then we note that $(f_1, f_2) = \int_G f_1(g) \overline{f_2(g)} dg = \int_G f_1(g) f_2^*(g^{-1}) dg = f_1 * f_2^*(1)$. Hence we may rewrite (1) as

$$(2) \quad (f_1, f_2) = \int_{\hat{G}} \theta_\rho(f_1 * f_2^*) d\rho = \int_{\hat{G}} \text{tr}(\rho(f_1) \rho(f_2^*)) d\rho.$$

Thus the usual inner product in $L^2(G)$ is decomposed into an integral

over the Hilbert-Schmidt inner products in the spectrum of G .

Now suppose $C \subseteq G$ is an open compact subgroup, and take some $\delta \in \hat{C}$. For each $\rho \in \hat{G}$, let $\omega(\rho, \delta)$ be the multiplicity with which δ occurs in the restriction of ρ to C . Then $\omega(\rho, \delta)$ is a positive, integer-valued, tolerably well-behaved (semi-continuous) function on \hat{G} . Let $\chi(\delta)$ be that function on G which is equal to the character of δ on C and is zero off C . Let $m(C)$ denote the measure of C with respect to the Haar measure on G . Then $E(\delta) = m(C)^{-1}(\dim \delta)\overline{\chi(\delta)}$ is an idempotent in $C_c^\infty(G)$, and for each $\rho \in \hat{G}$, $\rho(E(\delta))$ is projection onto the δ -isotypic component of ρ restricted to C . Hence $\theta_\rho(E(\delta)) = (\dim \delta)\omega(\rho, \delta)$. Thus in this case, equation (1) specializes to

$$(3) \quad m(C)^{-1}(\dim \delta) = \int_{\hat{G}} \omega(\rho, \delta) d\rho.$$

Compare this with Proposition 3.4. Also, note that if we integrate $\omega(\rho, \delta)$ only over some subset of \hat{G} (for example, the discrete series) we get an inequality. Compare [9], Chapter 1.

Now suppose C_1 and C_2 are two open compact subgroups of G , and take $\delta_i \in \hat{C}_i$. Then we define $\mathcal{J}(\delta_1, \delta_2)$, the interaction of δ_1 and δ_2 , by the formula

$$(4) \quad \mathcal{J}(\delta_1, \delta_2) = \int_{\hat{G}} \omega(\rho, \delta_1)\omega(\rho, \delta_2) d\rho.$$

It is fairly clear intuitively that $\mathcal{J}(\delta_1, \delta_2)$ will measure how often δ_1 and δ_2 appear in the same representation of G . It is a sort of weighted intertwining number for δ_1 and δ_2 . It also can be of use. Suppose that we know the function $\omega(\rho, \delta_1)$ very well. Then $\mathcal{J}(\delta_1, \delta_2)$ gives us some knowledge of $\omega(\rho, \delta_2)$. Let $\hat{G}(\delta_1)$ be the support of $\omega(\rho, \delta_1)$ — that is, $\hat{G}(\delta_1)$ is the closure of the set in \hat{G} where $\omega(\rho, \delta_1)$ is nonzero. Then a weak form of Frobenius reciprocity says that it is essentially the representations $\rho \in \hat{G}(\delta_1)$ that occur in $\text{ind}_{C_1}^G \delta_1$. Suppose for instance that $\omega(\rho, \delta_1)$ is always zero or one. Then $\mathcal{J}(\delta_1, \delta_2)$ gives an estimate of what portion of information about the spectrum of $\mathcal{H}(\delta_2^*)$, (where δ_2^* is the representation of C_2 contra-gradient to δ_2) we could expect to derive from decomposing $\text{ind}_{C_1}^G \delta_1$ into C_2 -types. To be extreme, if we had $\mathcal{J}(\delta_1, \delta_2) = m(C_2)^{-1}(\dim \delta_2)$, then we should be able to perform the complete (L^2 !) Fourier analysis of $\mathcal{H}(\delta_2)$ from knowledge of $\text{ind}_{C_1}^G \delta_1$. If on the other hand $\mathcal{J}(\delta_1, \delta_2) = 0$, then $\text{ind}_{C_i}^G \delta_i$ for $i = 1, 2$ give essentially irredundant information about harmonic analysis on G . We may also state the ideal form of Conditions (i) and (ii) in terms of interactions. Is there a set $S \in \hat{K}_0$ such that: (i) for $\delta_1, \delta_2 \in S$, $\mathcal{J}(\delta_1, \delta_2) = 0$; (ii) for $\delta \in S$, $\omega(\rho, \delta) \leq 1$; for any $\delta' \in \hat{K}_0$, $\sum_{\delta \in S} \mathcal{J}(\delta, \delta') = m(K_0)^{-1} \dim \delta'$? To this extreme

form of the question the answer is no. It is no already for Gl_2 ; the trouble comes from the special representation. However, it is conceivable that a slightly weaker form of the question would have an affirmative answer.

Our first item of business will be to see how in certain cases to compute $\mathcal{J}(\delta_1, \delta_2)$. Specifically, we shall consider the case where one of the δ 's is an arbitrary shallow character and the other is a shallow character related to the representations we considered in §3. For finite groups, the main tool for computing $\mathcal{J}(\delta_1, \delta_2)$ is the Frobenius formula for induced characters. Although we must tread more lightly here, the idea is the same. Our main goal (see Theorem 4.1) will be an interesting, useful, and suggestive expression for $\mathcal{J}(\delta_1, \delta_2)$ in terms of the geometry of dual blobs. To those familiar with Kirillov theory [14] this will come as no surprise, although again it is interesting to see these principles working so far from their apparent home turf. We remark in this regard that considerably more precise results are obtainable in characteristic zero.

The general computation we have in mind is somewhat laborious. Before treating it in all its technicality, we will illustrate the main ideas of the argument by dealing first with the supercuspidal case. As in the constructions of §3, this case is much simpler than the general case. Also, it gives us the opportunity to do some general calculations of related interest. Specializing these computations step by step will lead us to the desired formulas.

First we give an integral formula for the actual (unweighted) intertwining number of two finite-dimensional representations. Take an open compact subgroup C of G and a representation $\delta \in \hat{C}$. Let H be another closed subgroup of G , not necessarily compact, and let τ be a finite-dimensional representation of H . (We could also take for τ an admissible representation whose character is a locally integrable function on H .) Following the discussion in Lemma 3.7, we see that the total multiplicity $I(\delta, \tau)$ of δ in $\text{ind}_H^G \tau$ is given by the sum over the (C, H) double cosets of the intertwining numbers $I(\delta, \tau, g)$, for $g \in G$, of δ with $\text{ind}_{C_g}^G \text{Ad}^* g\tau$, where $C_g = C \cap \text{Ad } g(H)$. Although it has not been our custom, we could also compute $I(\delta, \tau, g)$ as the multiplicity of $\text{Ad } g^{-1}(\delta)$ in $\text{ind}_{\text{Ad } g^{-1}(C) \cap H}^{\text{Ad } g^{-1}(G)} \tau$. Let us fix a Haar measure dg on C and a right Haar measure $d_r h$ on H . Let $\chi(\delta)$ be the function on G which equals the character of δ on C and is zero off C . Let $\chi(\tau)$ on H be the character of τ . Then the parallel in the present situation of formula (1) of §3 is

$$(5) \quad I(\delta, \tau, g) = m_H(\text{Ad } g^{-1}(C) \cap H)^{-1} \int_H \chi(\delta)(ghg^{-1}) \overline{\chi(\tau)(h)} d_r h.$$

Here $m_H(X)$ indicates the measure of the set $X \subseteq H$ with respect to

the measure $d_r h$. Note that this expression is independent of $g \in CgH$ even though H may be nonunimodular. Summing (5) over (C, H) double cosets we obtain:

$$(6) \quad I(\delta, \tau) = \sum_{g \in C \backslash G/H} m_H(\text{Ad } g^{-1}(C) \cap H)^{-1} \int_H \chi(\delta)(ghg^{-1}) \overline{\chi(\tau)(h)} d_r h.$$

As a special case we can take $H = \Pi \cdot H'$, where H' is an open compact subgroup of G and Π is the subgroup of the center of G generated by a prime element $\pi \in F$. Then we can take $d_r h$ to be the restriction of dg to H . Then for any $g \in G$, $\text{Ad } g^{-1}(C) \cap H = \text{Ad } g^{-1}(C) \cap H'$. Thus, if $m(X)$ denotes the measure with respect to dg of $X \subseteq G$, we have the relation

$$m_H(\text{Ad } g^{-1}(C) \cap H) = m(C)m(H')m(CgH')^{-1}.$$

Also, the restriction of τ to H' will be an irreducible representation τ' of H' , and we have $I(\delta, \tau, g) = I(\delta, \tau', g)$ for any $g \in G$. Hence we see that in this case (6) may be rewritten

$$(7) \quad I(\delta, \tau) = m(C)^{-1}m(H')^{-1} \int_{G/\Pi} d\dot{g} \left(\int_{H'} \chi(\delta)(ghg^{-1}) \overline{\chi(\tau')(h)} dh \right).$$

Here $d\dot{g}$ denotes the Haar measure on G/Π which makes the natural projection of G onto G/Π measure preserving on sets which are projected one-to-one into G/Π . Note that the projection of G onto G/Π is locally one-to-one since Π is discrete. Note also the integrand of the integral over G/Π is a nonnegative function, so the integral either is convergent or properly divergent to $+\infty$. Compare [9], Chapter V. Since the contribution to the integral from any (C, H') double coset in G/Π is a finite integer, the integral will be convergent if and only if it has compact support.

Let us now specialize further and assume $C = 1 + \mathcal{A}$ and $H' = 1 + N$ for certain small lattices $\mathcal{A}, N \subseteq \mathfrak{G}$. Let us take δ and τ' to be shallow characters of C and H respectively. With these choices, we see the inner integral in (7) has the value

$$m(\text{Ad } g^{-1}(C) \cap H') I(\delta, \tau', g),$$

and $I(\delta, \tau', g)$ is one or zero according as δ and τ coincide on $C \cap H$ or not. We see we can choose a Haar measure dz on \mathfrak{G} such that, if $\mathcal{M}(X)$ denotes the measure of $X \subseteq \mathfrak{G}$ with respect to dz , then $\mathcal{M}(X) = m(1 + X^*)$ whenever X is a small lattice. Since if $X_1 \subseteq X_2$ are two lattices in \mathfrak{G} we have $*(X_2/X_1) = *(X_1^*/X_2^*)$, there is a constant ζ such that $m(X)m(X^*) = \zeta$ for any lattice $X \subseteq \mathfrak{G}$. (Since we have set things up so that $L_0^* = L_0$, we may calculate ζ by the formula $\zeta = \mathcal{M}(L_0)^2$.) We also note the relation

$$\mathcal{M}(X_1)\mathcal{M}(X_2) = \mathcal{M}(X_1 + X_2)\mathcal{M}(X_1 \cap X_2).$$

From these formulas we compute $\mathcal{M}(X_1^* \cap X_2^*) = \zeta \mathcal{M}((X_1^* \cap X_2^*)^*)^{-1} = \zeta \mathcal{M}(X_1 + X_2)^{-1} = \zeta \mathcal{M}(X_1)^{-1} \mathcal{M}(X_2)^{-1} \mathcal{M}(X_1 \cap X_2)$. Denote by $\beta(\delta)$ the set of elements in \mathfrak{G} which represent δ in the sense of §2, and let $\beta(\tau')$ be the set of representatives for τ' . Then $\beta(\delta)$ is a certain coset of A^* in \mathfrak{G} , and $\beta(\tau')$ is a coset of N^* . Thus $\text{Ad } g^{-1}(\beta(\delta)) \cap \beta(\tau')$, if it is nonempty, is a coset of $\text{Ad } g^{-1}(A^*) \cap N^*$. On the other hand, Lemma 2.3 says $I(\delta, \tau', g)$ is one or zero according as $\text{Ad } g^{-1}(\beta(\delta)) \cap \beta(\tau')$ is nonempty or not. Putting these facts together and plugging them in (7), we see that for this special situation (7) becomes

$$(8) \quad I(\delta, \tau) = \zeta^{-1} \int_{G/H} \mathcal{M}(\text{Ad } g^{-1}(\beta(\delta)) \cap \beta(\tau')) d\dot{g}.$$

REMARK. As a simple but suggestive preliminary observation from (8), let us remark that, in order for $\text{ind}_H^G \tau$ to be admissible, it is necessary and sufficient that $\beta(\tau')$ be contained in the set of regular elliptic elements. If this happens, then $\text{ind}_H^G \tau$ decomposes into a finite number of supercuspidal representations. By a regular elliptic element of \mathfrak{G} we mean a regular element which belongs to a minimally split Cartan subalgebra. For our $\mathfrak{G} = M_n(F)$, a minimally split Cartan subalgebra is just a subfield of $M_n(F)$ of degree n over F . The observation follows because the isotropy group under $\text{Ad } G$ of $m \in \mathfrak{G}$ is compact modulo the center of G if and only if m is regular elliptic.

We will now use (8) to compute $\mathcal{I}(\delta, \tau)$ when τ' is one of the representations of §3. Specifically, let \mathfrak{A} be a minimally split Cartan subalgebra, and let $\bar{a} \in \mathfrak{A}$ be sufficiently regular in the sense of §3. (Note that the geometric condition on \mathfrak{A} required in §3 is vacuous for minimally split Cartans.) Let $\text{ord}(\bar{a}) = -\mu$, and put $\eta = [(\mu + 1)/2]$, and let ψ be the shallow character of K_η represented by \bar{a} . Let J' and \mathcal{J} be the groups constructed for ψ in §3. Briefly, if A is the Cartan subgroup of \mathfrak{G} corresponding to \mathfrak{A} , and if $\nu = [(\mu + 2)/3]$, then $J' = (A \cap K_\nu) \cdot X$, where $X = \bigcap_{b \in A} \text{Ad } b(K_\nu \cap (1 + \mathfrak{A}^*))$ and $\mathcal{J} = A \cdot J'$. Let $\tilde{\varphi}$ be any character of A which agrees with ψ on $A \cap K_\eta$, and let $\tau'(\tilde{\varphi})$ be the corresponding irreducible representation of \mathcal{J} lying above ψ on K_η . Recall the bilinear form B_ψ defined on J' by the formula $B_\psi(x, y) = \psi(xy x^{-1} y^{-1})$, and let \mathcal{A} be the radical of this form. (See the discussion following Lemma 3.5.) We have seen $\mathcal{A} = (J' \cap A) \cdot (A \cap X)$ and $\mathcal{A} \cap X \subseteq K_\eta$. Put $\mathcal{A}' = \mathcal{A} \cap K_\eta$. Then $\mathcal{A}' = 1 + Y$ for a small lattice Y , and $Y = (Y \cap \mathfrak{A}) \oplus (Y \cap \mathfrak{A}^*)$. Let ψ' now denote the restriction of ψ to \mathcal{A}' . Then ψ' is a shallow character of \mathcal{A}' and \bar{a} represents ψ' , and $\beta(\psi') = \bar{a} + Y^*$. Moreover, the restriction of $\tau'(\tilde{\varphi})$ to \mathcal{A}' , for any $\tilde{\varphi}$, is a multiple of ψ' . The precise

multiple is, of course, $\dim \tau'(\tilde{\varphi})$, which by the standard theory of two-step nilpotent groups ([3]), is equal to $^*(J'/\Delta)^{1/2}$. It follows easily from Lemma 3.7 and the corollary to Proposition 3.3 that ψ' on Δ' occurs with uniform multiplicity $^*(J'/\Delta)^{1/2}$ in each representation $\pi(\tilde{\varphi}) = \text{ind}_{\mathcal{J}}^G J'(\tilde{\varphi})$, and that these are the only unitary representations in which ψ' can occur. Thus $\mathcal{J}(\delta, \psi')$ will indeed give us fairly sharp information on the occurrence of δ in the $\pi(\tilde{\varphi})$.

Notice that Δ' is normal in \mathcal{J} , and ψ' on Δ' is $\text{Ad}^* \mathcal{J}$ -invariant. It follows that $\text{Ad}^* x(\beta(\psi')) = \beta(\psi')$ for any $x \in \mathcal{J}$. On the other hand, if $g \in G$, and $\text{Ad}^* g(\beta(\psi'))$ intersects $\beta(\psi')$, then g intertwines ψ' with itself, and it follows almost directly from Lemma 3.6 that $g \in \mathcal{J}$. Thus we see $\text{Ad } G(\beta(\psi')) = \bigcup_{g \in G/\mathcal{J}} \text{Ad } g(\beta(\psi'))$, the union being disjoint.

Now in (8) we let $H = \Pi \cdot \Delta'$ and let τ be any extension of ψ' from Δ' to H , so that $\tau' = \psi'$. Then, observing that the right hand side of (8) is actually symmetric in δ and τ' , and using the decomposition of $\text{Ad } G(\beta(\psi'))$ given above, we see

$$I(\delta, \tau) = \zeta^{-1} m(\mathcal{J}/\Pi) \mathcal{M}(\beta(\delta) \cap \text{Ad } G(\beta(\psi'))),$$

where $m(\mathcal{J}/\Pi)$ is the measure of $\mathcal{J}/\Pi \subseteq G/\Pi$ with respect to $d\hat{g}$. That is, $m(\mathcal{J}/\Pi) = ^*(\mathcal{J}/J' \cdot \Pi) m(J')$ where $m(J')$ is as before the measure of $J' \subseteq G$ with respect to dg . Now according to Proposition 3.2 $\text{ind}_H^G \tau$ consists of $^*(\mathcal{J}/J' \cdot \Pi) ^*(\Delta/\Delta')$ different irreducible representations, each occurring with multiplicity $^*(J'/\Delta)^{1/2}$. Moreover, for two different extensions τ_1 and τ_2 of ψ' to H , the components of $\text{ind}_H^G \tau_1$ are all distinct from the components of $\text{ind}_H^G \tau_2$. Proposition 3.3 says that as τ runs over all possible unitary extensions of ψ' to H , the components of $\text{ind}_H^G \tau$ run over $\hat{G}(\psi')$, the set of unitary representations of G containing the Δ' -type ψ' . Proposition 3.4 says the Plancherel measure of $\hat{G}(\psi')$ is $^*(\Delta/\Delta') m(J')^{-1} (J'/\Delta)^{1/2}$. Putting these facts together, we conclude that

$$\begin{aligned} \mathcal{J}(\delta, \psi') &= ^*(\Delta/\Delta') m(J')^{-1} (J'/\Delta)^{1/2} (^*(\mathcal{J}/J' \cdot \Pi) ^*(\Delta/\Delta'))^{-1} \zeta^{-1} \\ &\quad \times ^*(\mathcal{J}/J' \cdot \Pi) \mathcal{M} m(J') (\beta(\delta) \cap \text{Ad } G(\beta(\psi'))). \end{aligned}$$

Cancelling terms gives

$$(9) \quad \mathcal{J}(\delta, \psi') = \zeta^{-1} (J'/\Delta)^{1/2} \mathcal{M}(\beta(\delta) \cap \text{Ad } G(\beta(\psi'))).$$

Of course to compute the integral of $\omega(\rho, \delta)$ over $\hat{G}(\psi')$, we must divide (9) by $(J'/\Delta)^{1/2}$, since $\omega(\rho, \psi')$ is identically equal to $(J'/\Delta)^{1/2}$ on $\hat{G}(\psi')$. We write the result explicitly.

$$(10) \quad \int_{\hat{G}(\psi')} \omega(\rho, \delta) d\rho = \zeta^{-1} \mathcal{M}(\beta(\delta) \cap \text{Ad } G(\beta(\psi'))).$$

We will now consider the general case. Our goal is to develop formulas analogous to (9) and (10) for the case when \mathfrak{A} is not minimally split. Considerations of length force us to treat some parts of the argument sketchily. We begin by re-establishing our notation. We will have frequent recourse to the results of §3, and our notation is borrowed more or less wholesale from that section.

Let \mathfrak{A} be a Cartan subalgebra of \mathfrak{G} . Suppose \mathfrak{A} satisfies the geometric condition of §3. A is the Cartan subgroup of G corresponding to \mathfrak{A} . A_s is the maximal split torus in A ; M is the centralizer of A_s in G ; P is a parabolic subgroup of G such that M is a Levi factor for P ; U_P is the unipotent radical of P ; $U_{\bar{P}}$ is the unipotent group opposite to U_P ; \mathcal{P} , \mathcal{M} , \mathcal{U} , and $\overline{\mathcal{U}}$ are the Lie algebras of P , M , U_P , and $U_{\bar{P}}$ respectively. Let $\bar{a} \in \mathfrak{A}$ be a sufficiently regular element. Put $\mu = -\text{ord}(a)$, and $\eta = [(\mu + 1)/2]$. Let ψ be the shallow character of K_η represented by \bar{a} . Let J' and \mathcal{J} be the subgroups constructed for ψ in §3. Recall J' is open and compact, and $U_P \subseteq \mathcal{J} \subseteq P \cdot J' \cap P \subseteq \mathcal{J}$. Let $\tilde{A}(\psi)$ be the set of quasicharacters of A which agree with ψ on $A \cap K_\eta$. For $\tilde{\varphi} \in \tilde{A}(\psi)$, let $\tau'(\tilde{\varphi})$ denote the representation of \mathcal{J} lying over ψ on $K_\eta \cap P$, and corresponding to $\tilde{\varphi}$, as constructed in §3. Let $\pi(\tilde{\varphi}) = \text{ind}_{\mathcal{J}}^G \tau'(\tilde{\varphi})$ be the corresponding irreducible representation of G .

The first step in our computation is an integral formula for $\theta_{\pi(\tilde{\varphi})}$. This formula is more or less standard (see [10], [22]) and is essentially the Frobenius formula for induced characters in this situation. Let dg be the Haar measure on G normalized so that the total measure of K_0 with respect to dg is equal to one. Let dk be the restriction of dg on K_0 . Recall we have $G = K_0 P$. Let $d_{\tau,p}$ denote the right Haar measure on P such that $dg = dk d_{\tau,p}$ in the sense that for $f \in C_c^\infty(G)$

$$(11) \quad \int_G f(g) dg = \int_{K \times P} f(kp) dk dp.$$

Putting f equal to the characteristic function of K_0 shows that the measure of $K_0 \cap P$ with respect to $d_{\tau,p}$ is equal to one. Let dm and du denote the Haar measures of M and U_P respectively, normalized so that the measures of $K_0 \cap M$ and $K_0 \cap U_P$ are both equal to one. Since $K_0 \cap P = (K_0 \cap M) \cdot (K_0 \cap U)$ we see that $d_{\tau,p} = du dm$ in the sense that for $f \in C_c^\infty(P)$

$$(12) \quad \int_P f(p) d_{\tau,p} = \int_{U \times M} f(um) du dm.$$

Using (11), (12) we find the familiar formula:

$$(13) \quad \int_G f(g)dg = \int_{K \times U_P \times M} f(kum)dkdudm, \quad \text{for } f \in C_c^\infty(G).$$

Put $\sigma(\tilde{\varphi}) = \text{ind}_P^G \tau'(\tilde{\varphi})$, so that $\pi(\tilde{\varphi}) = \text{ind}_P^G \sigma(\tilde{\varphi})$. Let $\theta_{\sigma(\tilde{\varphi})}$ and $\theta_{\pi(\tilde{\varphi})}$ denote the characters of $\sigma(\tilde{\varphi})$ and $\pi(\tilde{\varphi})$ respectively. Since P is not unimodular, $\theta_{\sigma(\tilde{\varphi})}$ is not invariant under inner automorphisms of P . However, we may express $\theta_{\sigma(\tilde{\varphi})}$ in terms of an invariant distribution on M as follows. For convenience, we will suppress $\tilde{\varphi}$ in this discussion. Thus $\sigma = \sigma(\tilde{\varphi})$, $\theta_\sigma = \theta_{\sigma(\tilde{\varphi})}$, and so forth. Let σ^0 be the restriction of σ to M , and let θ_σ^0 be the character of σ^0 , so θ_σ^0 is an invariant distribution on M . For $f \in C_c^\infty(P)$, define $f^0 \in C_c^\infty(M)$ by the formula

$$(14) \quad f^0(m) = \int_{U_P} f(mu)du.$$

Since σ is trivial on U_P , one may verify directly from the definitions that

$$(15) \quad \sigma(f) = \sigma^0(f^0) \quad \text{for } f \in C_c^\infty(P).$$

Thus we may immediately assert

$$(16) \quad \theta_\sigma(f) = \theta_\sigma^0(f^0),$$

the desired equation.

We may also express θ_π in terms of θ_σ . In fact, for $f \in C_c^\infty(G)$, define $\alpha(f) \in C_c^\infty(P)$ by the formula

$$(17) \quad \alpha(f)(p) = \int_{K_0} f(kpk^{-1})dk.$$

Then the equation

$$(18) \quad \theta_\pi(f) = \theta_\sigma(\alpha(f)) \quad \text{holds for } f \in C_c^\infty(G).$$

We will not prove (18). Proofs are available in [10] and [22]. We remark that in case f is supported in K_0 , equation (18) is immediate from the standard Frobenius formula for induced characters of compact groups, and we shall only be interested in the formula for such f . For general f , some checking is necessary to verify (18).

Now write $\mathcal{F} = \mathcal{F}^0 \cdot U_P$, where $\mathcal{F}^0 = \mathcal{F} \cap M$. Let τ^0 be the restriction of τ' to \mathcal{F} . Then τ^0 is irreducible since τ' is trivial on U_P , and it is clear that $\sigma^0 = \text{ind}_{\mathcal{F}^0}^M \tau^0$. Since σ^0 is an irreducible supercuspidal representation, and in particular admissible, and since \mathcal{F}^0 is open in M , it is easy to check that the analogue of the usual Frobenius formula holds here. Thus, let dj^0 be the restriction of dm to \mathcal{F}^0 , and let $d\tilde{m}$ be counting measure on M/\mathcal{F}^0 , which is

discrete. Let $\chi(\tau^0)$ be the character of τ^0 . Then for $f \in C_c^\infty(M)$ we have the formula

$$(19) \quad \theta_\sigma^0(f) = \int_{M/\mathcal{F}^0} d\tilde{m} \left(\int_{\mathcal{F}^0} f(mjm^{-1}) \chi(\tau^0)(j^{-1}) dj^0 \right).$$

Since σ^0 is admissible, the outer integral will in fact reduce to a finite sum, so there is no problem of convergence. Combining equations (14) through (19) we are led to the following formula for θ_π .

$$(20) \quad \begin{aligned} \theta_\pi(f) \\ = \int_{M/\mathcal{F}^0} d\tilde{m} \left(\int_{\mathcal{F}^0} dj^0 \left(\int_{U_P} du \left(\int_{K_0} dk (f(kmj m^{-1} u k^{-1}) \chi(\tau^0)(j^{-1})) \right) \right) \right), \end{aligned}$$

for $f \in C_c^\infty(G)$.

Now return to J' . Recall the T -valued bilinear form B_ψ on J' , defined by $B_\psi(x, y) = \psi(xy x^{-1} y^{-1})$ for $x, y \in J'$. Let $\mathcal{A} \subseteq J'$ be the radical of B_ψ . That is $\mathcal{A} = \{x \in J' : B_\psi(x, y) = 1 \text{ for all } y \in J'\}$. Let $Z \subseteq \mathfrak{G}$ be the small lattice such that $1 + Z = J'$. As we saw in §3, if $z_1, z_2 \in Z$, then $B_\psi(1 + z_1, 1 + z_2) = \psi(1 + [z_1, z_2]) = \Omega(\bar{a})([z_1, z_2]) = \Omega_0(\langle \bar{a}, [z_1, z_2] \rangle) = \Omega_0(\langle \bar{a}, z_1 \rangle, z_2)$. Thus $\mathcal{A} = \{1 + z : z \in Z \text{ and } \langle \bar{a}, z \rangle \in Z^*\}$. Suppose $\mathcal{A} = 1 + Z_1$. We know by the construction of J' that $Z = (Z \cap \mathfrak{U}) \oplus (Z \cap \mathfrak{U}^*)$. Hence also $Z_1 = (Z_1 \cap \mathfrak{U}) \oplus (Z_1 \cap \mathfrak{U}^*)$. Moreover $Z_1 \cap \mathfrak{U} = Z \cap \mathfrak{U}$. Put $Y = (Z_1 \cap \mathfrak{U}^*) + \mathfrak{U}_\eta$, and put $\mathcal{A}' = 1 + Y$. It is easy to see \mathcal{A}' is normal in J' , and since $\mathcal{A}' \subseteq \mathcal{A}$, we see $\text{Ad } x(\bar{a} + Y^*) = \bar{a} + Y^*$ for any $x \in J'$. I claim that $\text{Ad } J'((\bar{a} + Y^*) \cap \mathfrak{U}) = \bar{a} + Y^*$. This fact is implicit in the construction of J' . We summarize the pertinent details. First we had $\text{Ad } (J' \cap M)(\bar{a} + \mathfrak{U}_{-\gamma-c_1}) \supseteq \bar{a} + \mathfrak{M}_{-\gamma}$, where c_1 was the constant used in the definition of sufficient regularity. Next, we note $\mathfrak{M}_{-\gamma} = \mathfrak{M}_\eta^* \cap \mathfrak{M}$, and $1 + \mathfrak{M}_\eta \subseteq J' \cap M$. By the theory of two-step nilpotent groups, the characters $\text{Ad } (J' \cap M)(\psi)$, restricted to $1 + \mathfrak{M}_\eta$, constitute all characters of $1 + \mathfrak{M}_\eta$ agreeing with ψ on $\mathcal{A}' \cap M \subseteq 1 + \mathfrak{M}_\eta$. Thus we may conclude that $\text{Ad } (J' \cap M)(\bar{a} + \mathfrak{U}_{-\gamma-c_1}) \supseteq \bar{a} + (Y^* \cap \mathcal{M})$. Next, we recall that by construction $J' \cap U_P \subseteq \mathcal{A} \supseteq J' \cap U_{\bar{P}}$. Hence $Y \cap \mathcal{U} = Z \cap \mathcal{U}$ and $Y \cap \bar{\mathcal{U}} = Z \cap \bar{\mathcal{U}}$. Moreover, again by construction, we had $[\bar{a}, Z \cap \mathcal{U}] = (Z \cap \bar{\mathcal{U}})^* \cap \mathcal{U}$, and the same holds with \mathcal{U} and $\bar{\mathcal{U}}$ received. Combining these facts, using an expansion in series as in Lemma 3.1, the claim follows. The same reasoning also shows that $[\bar{a}', Z] = Y^* \cap \mathfrak{U}^*$ for any $\bar{a}' \in (\bar{a} + Y^*) \cap \mathfrak{U}$, or in other words, $\text{Ad } G(\bar{a}') \cap (\bar{a} + Y^*) = \text{Ad } J'(\bar{a}')$.

Put $\mathfrak{U}(\psi) = \bar{a} + (Y^* \cap \mathfrak{U})$. By what we have just seen

$$\text{Ad } G(\bar{a} + Y^*) = \text{Ad } G(\mathfrak{U}(\psi)).$$

Since every element of $\mathfrak{A}(\psi)$ is a regular element of \mathfrak{A} , it is seen that $\text{Ad } G(\mathfrak{A}(\psi))$ is an open and closed set in \mathfrak{G} . The second step in our computation is to give a formula analogous to (20) for the integral of a function $f \in C_c^\infty(\mathfrak{G})$ over $\text{Ad } G(\mathfrak{A}(\psi))$.

The isotropy group under $\text{Ad } G$ of any point of $\mathfrak{A}(\psi)$ is A . Thus we have a bijection $\zeta: G/A \times \mathfrak{A}(\psi) \rightarrow \text{Ad } G(\mathfrak{A}(\psi))$ given by $\zeta(gA, \bar{a}') = \text{Ad } g(\bar{a}')$ for $g \in G$, $\bar{a}' \in \mathfrak{A}(\psi)$. Let dz be a Haar measure on \mathfrak{G} , and let $d\tilde{z}$ be the pullback by ζ of dz to $G/A \times \mathfrak{A}(\psi)$. The fact that dz is invariant by $\text{Ad } G$ translates into the decomposition $d\tilde{z} = dg_1 da$ where dg_1 is a G -invariant measure on G/A and da is a measure on $\mathfrak{A}(\psi)$. See [9]. Thus, using (13), we see that if dm_1 is a suitably normalized invariant measure on M/A , we have for $f \in C_c^\infty(\mathfrak{G})$

$$(21) \quad \begin{aligned} \int_{\text{Ad } G(\mathfrak{A}(\psi))} f(z) dz &= \int_{G/A \times \mathfrak{A}(\psi)} f \cdot \zeta(\tilde{z}) d\tilde{z} = \int_{G/A \times \mathfrak{A}(\psi)} f \cdot \zeta(g, \bar{a}') dg_1 da \\ &= \int_{K \times U_P \times M/A \times \mathfrak{A}(\psi)} f \circ \zeta(kum, \bar{a}') dk dudm_1 da. \end{aligned}$$

We observe that da must be the restriction to $\mathfrak{A}(\psi)$ of a Haar measure on \mathfrak{A} . This follows because $\text{Ad } G(\bar{a}') \cap \bar{a} + Y^* = \text{Ad } J'(\bar{a}')$ for any $\bar{a}' \in \mathfrak{A}(\psi)$. Now the restriction of ζ to $M/A \times \mathfrak{A}(\psi)$ gives a bijection of $M/A \times \mathfrak{A}(\psi)$ onto $\text{Ad } M(\mathfrak{A}(\psi))$, which is an open set in \mathfrak{M} . The same argument used to identify $d\tilde{z}$ shows that the direct image of $dm_1 da$ under ζ is the restriction to $\text{Ad } M(\mathfrak{A}(\psi))$ of a Haar measure dx on \mathfrak{M} . Furthermore, for any $m \in M$, $\bar{a}' \in \mathfrak{A}(\psi)$, the restriction of ζ to $(U_P m, \bar{a}')$ gives a bijection of this set onto $\text{Ad } m(\bar{a}') + \mathcal{U}$. The direct image of du under this mapping is, again by the invariance argument, seen to be a suitably normalized Haar measure dn on \mathcal{U} . Although the normalization of dn might depend on m and \bar{a}' , in this case it does not. This is because the normalization is well-known to depend only on $\|\det(\text{ad}(\text{Ad } m(\bar{a}')))\|$, and this quantity is always independent of m , and is independent of $\bar{a}' \in \mathfrak{A}(\psi)$ by construction (see the remark accompanying Lemma 3.1). Using these remarks we may retranslate the last expression in (21) to obtain, for $f \in C_c^\infty(\mathfrak{G})$,

$$(22) \quad \begin{aligned} \int_{\text{Ad } G(\mathfrak{A}(\psi))} f(z) dz &= \int_{\text{Ad } M(\mathfrak{A}(\psi))} dx \left(\int_{\mathcal{U}} dn \left(\int_{K_0} dk (f(\text{Ad } k(x + n))) \right) \right) \\ &= \int_{M/A \times \mathfrak{A}(\psi)} dm_1 da \left(\int_{\mathcal{U}} dn \left(\int_{K_0} dk (f(\text{Ad } k(\text{Ad } m(\bar{a}') + n))) \right) \right). \end{aligned}$$

Next, observe $A \subseteq \mathcal{F}^0$, and $\text{Ad } \mathcal{F}^0(\mathfrak{A}(\psi)) = (\bar{a} + Y^*) \cap \mathfrak{M} = S$ is an open subset of \mathfrak{M} . Thus we may write

$$\text{Ad } M(\mathfrak{A}(\psi)) = \bigcup_{m \in M | \mathcal{F}^0} \text{Ad } m(S),$$

the union being disjoint. Thus we may convert (22) into

$$(23) \quad \int_{\text{Ad } G(\mathfrak{H}(\psi))} f(z) dz \int_{M| \mathcal{L}^0} d\tilde{m} \left(\int_S dx \left(\int_{\mathcal{H}} dn \left(\int_{K_0} dk (f(\text{Ad } k(\text{Ad } m(x) + n))) \right) \right) \right).$$

Now consider a small lattice $A \subseteq \mathfrak{G}$, and let $C = 1 + A \subseteq G$, and let δ be a shallow character of C , with set of representatives $\beta(\delta) \subseteq \mathfrak{G}$. We will apply formula (20) to $\chi(\delta)$, the function on G which equals (the character of) δ on C and vanishes off C , and we will apply formula (23) to the characteristic function of $\beta(\delta)$. Then we will compare the results.

In (20), recall that $\tau^0 = \tau^0(\tilde{\varphi})$ depends on the quasicharacter $\tilde{\varphi} \in \tilde{A}(\psi)$. Fix a discrete, torsion free subgroup $\Pi \subseteq A_s$ such that A_s/Π is compact. Put $H = \Pi \cdot (A' \cap M)$. Let ψ_1 be the restriction of ψ to $A' \cap M$, and let τ_1 be any extension of ψ_1 to H . Specifying τ_1 essentially amounts to specifying a quasicharacter of Π . We see that $\text{ind}_{H^0}^{\tau_1}$ consists of $^*(J' \cap M/A \cap M)^{1/2} = ^*(J'/A)^{1/2}$ copies of each of those $\tau^0(\tilde{\varphi})$ such that $\tilde{\varphi}$ agrees with τ_1 on $A \cap H$. There are $^*(A/A \cap H) = ^*(\mathcal{L}^0/\Pi \cdot (J' \cap M)) \cdot ^*(A/A')$ such $\tilde{\varphi}$. Extend τ_1 to $H \cdot U_P$ by letting it be trivial on U_P . Put $\pi_1 = \text{ind}_{H \cdot U_P}^{\tau_1}$. Then performing the indicated sum in (20) gives

$$(24) \quad \theta_{\pi_1}(f) = ^*(\mathcal{L}^0/H) \int_{M| \mathcal{L}^0} d\tilde{m} \times \left(\int_H dh \left(\int_{U_P} du \left(\int_{K_0} dk (f(kmh m^{-1} u k^{-1}) \chi(\tau_1(h^{-}))) \right) \right) \right).$$

Here dh is the restriction of dj^0 to H .

In (24), take $f = \chi(\delta)$ and consider the integrals over H and U_P first. This is legal since all three inner integrals in (24) converge absolutely and so may be interchanged at will by Fubini's theorem. By inspection the integral over U_P is zero unless $\text{Ad}^* k^{-1}(\delta)$ is trivial on $\text{Ad } k^{-1}(C) \cap U_P$, in which case it is equal to the measure with respect to du of $\text{Ad } k^{-1}(C) \cap U_P$ independently of m and h . Suppose $\text{Ad}^* k^{-1}(\delta)$ is trivial on $\text{Ad } k^{-1}(C) \cap U_P$. Let $C_P(k)$ be the projection of $\text{Ad } k^{-1}(C) \cap P$ onto M under the identification $M \simeq P/U_P$. Let $\delta_P(k)$ be the character of $C_P(k)$ gotten from $\text{Ad}^* k^{-1}(\delta)$ via this identification. Then $C_P(k) = 1 + A_P(k)$ for a small lattice $A_P(k) \subseteq \mathfrak{M}$ and $\delta_P(k)$ is a shallow character of $C_P(k)$. By construction of H , $C_P(k) \cap H = C_P(k) \cap (A' \cap M)$. The integral over H is zero unless $\text{Ad}^* m^{-1}(\delta_P(k))$ agrees with ψ on $\text{Ad } m^{-1}(C_P(k)) \cap H$. If $\text{Ad}^* m^{-1}(\delta_P(k))$ and ψ do agree, then the integral over H is just the volume with respect to dm (e.g., dh) of $\text{Ad } m^{-1}(C_P(k)) \cap H$. It is clear these conditions are equivalent to, the integral over H is zero unless $\delta_P(k)$ and $\text{Ad } m(\psi)$ agree on $C_P(k) \cap \text{Ad } m(A' \cap M)$, in which case it is equal

to the measure of $C_P(k) \cap \text{Ad } m(\mathcal{A}' \cap M)$. Put $\mathcal{A}'' = (\mathcal{A}' \cap M) \cdot U_P$. The restriction of $dmdu$ to \mathcal{A}'' is a two-sided invariant Haar measure on \mathcal{A}'' . The same is true of the restriction of $dmdu$ to $\text{Ad } m(\mathcal{A}'') = \text{Ad } m(\mathcal{A}' \cap M) \cdot U_P$. (Note, however, that the transform of $dmdu$ by $\text{Ad } m$ is not necessarily $dmdu$, but some multiple of it.) If we put the integrations over U_P and H together, we see that the result is zero unless $\text{Ad}^* k^{-1}(\delta)$ agrees with $\text{Ad}^* m(\psi)$ on $\text{Ad}^* k^{-1}(C) \cap \text{Ad } m(\mathcal{A}'')$, and if they do agree, the result is the measure of $\text{Ad}^* k^{-1}(C) \cap \text{Ad } m(\mathcal{A}'')$, with respect to $dmdu$.

In (23) put f equal to the characteristic function of $\beta(\delta)$, and do the integrations over S and \mathcal{U} . An analysis precisely parallel to that of the preceding paragraph shows that the result is the measure, with respect to $dxdm$, of $\text{Ad } k^{-1}(\beta(\delta)) \cap (\text{Ad } m(S) + \mathcal{U})$. Now S is a coset in \mathfrak{M} of $Y^* \cap \mathfrak{M}$, and $\beta(\delta)$ is a coset of \mathcal{A}^* . Hence if this intersection is nonempty, then its measure is equal to the measure of $\text{Ad } k^{-1}(\mathcal{A}^*) \cap (\text{Ad } m(Y^* \cap \mathfrak{M}) \oplus \mathcal{U})$.

Now $\mathcal{A}'' = 1 + ((Y \cap \mathfrak{M}) \oplus \mathcal{U})$, and since

$$Y = (Y \cap \mathfrak{M}) \oplus (Y \cap \mathcal{U}) \oplus (Y \cap \mathcal{U}),$$

we see that $((Y \cap \mathfrak{M}) \oplus \mathcal{U})^* = (Y^* \cap \mathfrak{M}) \oplus \mathcal{U}$. Therefore, we can assert that $\text{Ad}^* k^{-1}(\delta)$ and $\text{Ad}^* m(\psi)$ will agree on $\text{Ad } k^{-1}(C) \cap \text{Ad } m(\mathcal{A}'')$ if and only if intersection of $\text{Ad } k^{-1}(\mathcal{A}^*)$ and $\text{Ad } m(Y^* \cap \mathfrak{M}) \oplus \mathcal{U}$ is nonempty. Moreover, there is a relation between the measures of the intersections. Let $m(X)$ denote the measure of a set $X \subseteq \mathfrak{G}$ with respect to dz . Similarly, let $\mathcal{M}_P(X)$ indicate the measure of a set $X \subseteq \mathcal{P}$ with respect to $dxdn$, and $\mathcal{M}_m(X)$ the measure of $X \subseteq \mathfrak{M}$ with respect to dx . Let us assume we have so normalized dz so that if $X \subseteq \mathfrak{G}$ is a small lattice, then $\mathcal{M}(X)$ equals the measure of $1 + X \subseteq G$ with respect to dg . Once dz is fixed $dxdn$ is determined by (22) or (23). There will be a positive number α_0 such that if X is a small lattice in \mathcal{P} , then $\mathcal{M}_P(X)$ is equal to α_0 times the measure of $1 + X$ with respect to $dmdu$. In particular, the measure of $\text{Ad } k^{-1}(C) \cap \text{Ad } m(\mathcal{A}'')$ equals

$$\alpha_0^{-1} \mathcal{M}_P(\text{Ad } k^{-1}(\mathcal{A}) \cap (\text{Ad } m(Y \cap \mathfrak{M}) \oplus \mathcal{U})).$$

As we have seen before, there is a constant ζ such that

$$\mathcal{M}(X) \mathcal{M}(X^*) = \mathcal{M}(L_0)^2 = \zeta.$$

We may again deduce for lattices $X, Z \subseteq \mathfrak{G}$, the relation

$$\mathcal{M}(X^* \cap Z^*) = \zeta \mathcal{M}(X)^{-1} \mathcal{M}(Z)^{-1} \mathcal{M}(X \cap Z).$$

Suppose $Z = \overline{\mathcal{U}}_+ \oplus \mathcal{U}_- \oplus (Z \cap \mathfrak{M})$. Then $Z^* = \overline{\mathcal{U}}_- \oplus \mathcal{U}_+ \oplus (Z^* \cap \mathfrak{M})$. Put $\zeta_M = \mathcal{M}_M(\mathfrak{M}_0)_2$ and $\zeta_P = \mathcal{M}_P(\mathfrak{M}_0 \oplus \mathcal{U}_0)^2$. Then we have the

formula $\mathcal{M}(Z) = \zeta^{1/2} \zeta_M^{-1/2} \mathcal{M}_M(Z \cap \mathfrak{M})$. For ν sufficiently large, we have $X \cap Z = \overline{\mathcal{U}}_\nu \oplus (X \cap Z \cap \mathcal{P})$ and $X^* \cap Z^* = \overline{\mathcal{U}}_\nu \oplus (X^* \cap Z^* \cap \mathcal{P})$. When this is so, we see that $\mathcal{M}(X \cap Z) = \gamma \zeta^{1/2} \zeta_P^{-1/2} \mathcal{M}_P(X \cap Z \cap \mathcal{P})$, and $\mathcal{M}(X^* \cap Z^*) = \gamma \zeta^{1/2} \zeta_P^{-1/2} \mathcal{M}_P(X^* \cap Z^* \cap \mathcal{P})$ hold, where $\gamma = q^{-\dim \overline{\mathcal{U}}}$. Plugging these facts in the relation between $\mathcal{M}(X^* \cap Z^*)$ and $\mathcal{M}(X \cap Z)$, and taking the limit as ν goes to ∞ , we conclude that for a lattice $Z_1 \subseteq \mathfrak{M}$, we have

$$(25) \quad \begin{aligned} & \mathcal{M}_P(X^* \cap ((Z_1^* \cap \mathfrak{M}) \oplus \mathcal{U})) \\ &= \zeta^{1/2} \zeta_M^{1/2} \mathcal{M}(X)^{-1} \mathcal{M}_M(Z_1)^{-1} \mathcal{M}_P(X \cap (X_1 \oplus \mathcal{U})). \end{aligned}$$

If we put $X = \text{Ad } k^{-1}(A)$ and $Z_1 = \text{Ad } m(Y \cap \mathfrak{M})$, then (25) reads

$$(26) \quad \begin{aligned} & \mathcal{M}_P(\text{Ad } k^{-1}(A^*) \cap (\text{Ad } m(Y^* \cap \mathfrak{M}) \oplus \mathcal{U})) \\ &= \zeta^{1/2} \zeta_M^{1/2} \mathcal{M}(A)^{-1} \mathcal{M}_M(Y \cap \mathcal{M})^{-1} \\ & \quad \times \mathcal{M}_P(\text{Ad } k^{-1}(A) \cap (\text{Ad } m(Y \cap \mathcal{M}) \oplus \mathcal{U})). \end{aligned}$$

Now taking (26) and using it to compare (23) and (24) for our choices of functions, we find

$$(27) \quad \begin{aligned} & \mathcal{M}(\beta(\delta) \cap \text{Ad } G(\mathfrak{A}(\psi))) \\ &= *(\mathcal{I}^0/H)^{-1} \mathcal{M}(A)^{-1} \mathcal{M}_M(Y \cap \mathfrak{M})^{-1} \alpha_0 \zeta^{1/2} \zeta_M^{1/2} \theta_{\pi_1}(\chi(\delta)). \end{aligned}$$

Recall $\pi_1 = \text{ind}_{H \cdot V_P}^G \tau_1$, where τ_1 was an arbitrary extension of ψ_1 on $A' \cap M$ to H . Now let τ_1 vary in such a way that π_1 varies over all unitary representations in $\hat{G}(\psi)$. Taking into account the discussion preceding (24) concerning the structure of π_1 , using the Plancherel measure of $\hat{G}(\psi)$ as given by Proposition 3.4, and remembering that $\chi(\delta)^* \chi(\delta) = \mathcal{M}(A) \chi(\delta)$, we get

$$(28) \quad \begin{aligned} & m(J')^{-1} * (A/A') \mathcal{M}(\beta(\delta) \cap \text{Ad } G(\mathfrak{A}(\psi))) \\ &= \alpha_0 \zeta^{1/2} \zeta_M^{1/2} \mathcal{M}_M(Y \cap \mathfrak{M})^{-1} \int_{\hat{G}(\psi)} \omega(\rho, \delta) d\rho. \end{aligned}$$

Here in (J') is the measure of $J' \subseteq G$ with respect to dm .

It remains only to consolidate the constants appearing in (28). To do this, the simplest thing is to take $\delta = \psi'$, the restriction of ψ to A' . This is not strictly legal, but is easily justified. Then

$$\beta(\psi') = \bar{a} + Y^* \subseteq \text{Ad } G(\mathfrak{A}(\psi')) = \text{Ad } G(\beta(\psi')).$$

Hence $\mathcal{M}(\beta(\psi') \cap \text{Ad } G(\mathfrak{A}(\psi))) = \mathcal{M}(Y^*) = \zeta \mathcal{M}(Y)^{-1}$. On the other hand, the total Plancherel measure of $\hat{G}(\psi)$ is equal to

$$m(J')^{-1} * (J'/A)^{1/2} * (A/A'),$$

and ψ' occurs in representations of $\hat{G}(\psi)$ with uniform multiplicity

$(J'/\Delta)^{1/2}$. Thus

$$\int_{\hat{G}(\psi)} \omega(\rho, \psi') d\rho = m(J')^{-1} (J'/\Delta)^*(\Delta/\Delta') = m(\Delta')^{-1} = \mathcal{M}(Y)^{-1}.$$

Plugging these into (28) gives

$$(29) \quad m(J')^{-1} (\Delta/\Delta') \zeta = \alpha_0 \zeta^{1/2} \zeta_M^{1/2} \mathcal{M}_M(Y \cap \mathfrak{M})^{-1}$$

and also gives the final formula, which we formally enshrine.

THEOREM 4.1. *Let \mathfrak{A} be a Cartan subalgebra of \mathfrak{G} satisfying the geometric condition of §3. Let $\bar{a} \in \mathfrak{A}$ be sufficiently regular. Put $\mu = -\text{ord}(\bar{a})$ and $\eta = [(\mu + 1)/2]$. Let ψ be the character of K_η represented by ψ . Then for any shallow character δ of some compact group $C = 1 + A \subseteq G$, with A a small lattice of \mathfrak{G} , we have the formula*

$$(30) \quad \int_{\hat{G}(\psi)} \omega(\rho, \delta) d\rho = \zeta^{-1} \mathcal{M}(\beta(\delta) \cap \text{Ad } G(\beta(\psi))) .$$

Here, for $X \subseteq \mathfrak{G}$, $\mathcal{M}(X)$ denotes the measure of X with respect to the Haar measure dz , which satisfies the relation $\mathcal{M}(X)\mathcal{M}(X^*) = \zeta$ and $\mathcal{M}(X) = m(1 + X)$, where $m(T)$, for $T \subseteq G$ is the measure of T with respect to the Haar measure dg of G . The measure $d\rho$ is Plancherel measure on $\hat{G}(\psi)$, which is that subset of the unitary dual \hat{G} of G consisting of representations whose restrictions to K_η contain ψ .

Now we shall consider many ψ 's simultaneously in order to place our results in a larger perspective. We seek here to give a general view of the state of affairs, and will give no proofs. Let us fix representatives $\{\mathfrak{A}_i\}$ for the conjugacy classes of Cartan subalgebras of G . If F is of characteristic zero, these will be finite in number, but for F of positive characteristic there may be infinitely many \mathfrak{A}_i 's. (There will be if $n \geq p$.) Let A_i be the Cartan subgroup of G attached to \mathfrak{A}_i . It is not hard to see (see [12]) that we may arrange that for each i , $(A_0)_i$, the maximal compact subgroup of A_i , is contained in K_0 , and we assume we have arranged this. Then the Weyl group of each A_i will have representatives in K_0 , and so will act by isometries on \mathfrak{A}_i . (It is also not hard to see (see [12] again) that one may arrange for the constant c_0 occurring in the definition of sufficient regularity to be less than or equal to 2 for any \mathfrak{A}_i . The constant c_1 , however, is not controllable, it depends in an essential way on ramification-theoretic properties of \mathfrak{A}_i .) In any case, given these normalizations of the \mathfrak{A}_i , it is clear

that $\mathcal{R}(i, \mu)$ is the set of sufficiently regular $x \in \mathfrak{A}_i$ with $\text{ord}(a) = -\mu$, $\mu > 0$, then $\text{Ad } G(\mathcal{R}(i, \mu))$ and $\text{Ad } G(\mathcal{R}(j, \nu))$ are disjoint unless $i = j$ and $\mu = \nu$. As always, if $\bar{a} \in \mathfrak{A}_i(\mu)$, and $\eta = [(\mu + 1)/2]$, \bar{a} represents a shallow character of K_η . Since by Lemma 3.2,

$$\bar{a} + L_{-\eta} = \beta(\psi) \subseteq \text{Ad } G(\mathcal{R}(i, \mu)),$$

we see that as \mathfrak{A}_i and μ vary, the sets $\hat{G}(i, \mu)$, where $\hat{G}(i, \mu)$ is the union of $\hat{G}(\psi)$ for shallow characters ψ of K_η having representatives in $\mathcal{R}(i, \mu)$, form a collection of disjoint compact open and closed subsets (with no limit point in \hat{G}), each of finite Plancherel measure.

Now consider K_ν and an arbitrary shallow character φ of K_ν . According to Theorem 4.1, we have

$$\int \bigcup_{i, \mu} \hat{G}(i, \mu) \omega(\rho, \varphi) d\rho = \zeta^{-1} \mathcal{M}(\beta(\varphi) \cap (\bigcup_{i, \mu} \text{Ad } G(\mathcal{R}(i, \mu)))) .$$

If in particular $\beta(\varphi) \subseteq \bigcup_{i, \mu} \text{Ad } G(\mathcal{R}(i, \mu))$, then according to the discussion at the beginning of this section, all the harmonic analysis of $\mathcal{H}(\varphi)$, as far as $L^2(G)$ is concerned, is wrapped up in the $G(i, \mu)$. It seems most likely that one could show without too much difficulty that in fact $\hat{G}(\varphi) \subseteq \bigcup_{i, \mu} \hat{G}(i, \mu)$. In any case, since it is clear that $\mathcal{H}(\varphi)$ is mildly non-abelian in the sense of §2, since it is clearly faithfully represented in $L^2(G)$.

Consider again $\bar{a} \in \mathcal{R}(i, \mu)$, representing ψ on K_η . The totality of representations of G in which ψ occurs corresponds to the set $\tilde{A}_i(\psi)$ quasicharacters of A_i which agree with ψ on $A_i \cap K_\eta$. Let $\tilde{A}(i, \mu)$ be the set of quasicharacters of A which are in $\tilde{A}_i(\psi)$, where ψ is a shallow character of K_η with representative in $\mathcal{R}(i, \mu)$.

The following statements seem fairly clear from the definition of sufficient regularity.

(1) As $\mu \rightarrow \infty$, most of the quasicharacters of A_i are in $\bigcup_\mu \tilde{A}(i, \mu)$, in the following sense. The ratio of the number of characters of A_0 vanishing on $A_0 \cap K_\mu$ to the number of such characters which are the restriction of elements of $\bigcup_{\mu'=1}^\infty \tilde{A}(i, \mu')$ decreases to one as μ goes to ∞ .

(2) As $\mu \rightarrow \infty$, the ratio of the number of shallow characters of K_η (with $\eta = [(\mu + 1)/2]$) of conductor K_μ , to the number of such characters whose dual blobs are contained in $\bigcup_{i, \nu} \text{Ad } G(\mathcal{R}(i, \nu))$ decreases to one. Even more quickly, the ratios of the number of essential shallow characters of K_η of conductor K_μ to the number of such essential shallow characters with dual blob in $\bigcup_{i, \nu} \text{Ad } G(\mathcal{R}(i, \nu))$ decreases to one,

(3) As $\mu \rightarrow \infty$, the ratio of $\mathcal{M}(L_{-\eta})$ to

$$\mathcal{M}(L_{-\eta} \cap (\bigcup_{i, \nu} \text{Ad } G(\mathcal{R}(i, \nu))))$$

decreases to one.

These statements together can be taken to mean that, in an asymptotic sense, most of the quasicharacters of the tori of G have been associated to representations of G , and these representations account for most of the representations and most of the Plancherel measure of G .

One the other hand, the representations we have constructed are so uniformly well-behaved, they are almost dull. This is, of course, the reason we could construct them. More interesting series of representations, with nonconstant Plancherel measure, and reducibility in the analytic continuation, will be encountered in trying to extend the above analysis to more singular shallow characters. It will be in these series that the non-supercuspidal discrete series, complementary series, and so forth will be found. Not only will these representations be interesting at the local level, but globally they are unavoidable, since K_ϕ -spherical representations, in same sense the most singular representations of G , will occur as factors at almost all places of an adèle group over a global field. It is clear that the analysis of these series will not proceed along the straightforward lines followed here. One will need the techniques of Harish-Chandra [10] and MacDonald [17], and probably other methods, as yet undeveloped.

In closing, I would like to pose a question suggested by Theorem 4.1. Is it possible to extend the set collection $\{\hat{G}(i, \mu)\}$ so as to write $\hat{G} = \cup \hat{G}_\alpha$, where each \hat{G}_α is a compact, open and closed subset in \hat{G} of finite Plancherel measure, and characterized by the K_ϕ -types occuring in $\rho \in \hat{G}_\alpha$, and to associate to each \hat{G}_α a set $S_\alpha \subseteq \mathfrak{G}$, with S_α open, closed, Ad G -invariant and equal to Ad $G(X_\alpha)$ with X_α compact, in such a way that for any shallow character δ of $C \subseteq G$, with $C = 1 + A$ for A a small lattice in \mathfrak{G} the formula

$$\int_{\hat{G}_\alpha} \omega(\rho, \delta) d\rho = \zeta^{-1} \mathcal{M}(\beta(\delta) \cap S_\alpha)$$

holds? If so, this would make a very pleasant version of the full Plancherel formula. Moreover, if the characters turn out to be as incomputable as they now seem, it might also be the most practical version.

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Received May 21, 1977. Supported in part by National Science Foundation grant GP-7952X3.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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Roger Evans Howe, <i>On representations of discrete, finitely generated, torsion-free, nilpotent groups</i>	281
Roger Evans Howe, <i>The Fourier transform for nilpotent locally compact groups. I</i>	307
Roger Evans Howe, <i>On a connection between nilpotent groups and oscillatory integrals associated to singularities</i>	329
Roger Evans Howe, <i>Kirillov theory for compact p-adic groups</i>	365
Roger Evans Howe, <i>Topics in harmonic analysis on solvable algebraic groups</i>	383
Roger Evans Howe, <i>Tamely ramified supercuspidal representations of GL_n</i>	437
Lawrence Jay Corwin and Roger Evans Howe, <i>Computing characters of tamely ramified p-adic division algebras</i>	461
Roger Evans Howe, <i>Some qualitative results on the representation theory of GL_n over a p-adic field</i>	479
Herbert Stanley Bear, Jr., <i>Corrections to: "Ordered Gleason parts"</i>	539
Andreas Blass, <i>Corrections to: "Exact functors and measurable cardinals"</i>	540
Robert M. DeVos, <i>Corrections to: "Subsequences and rearrangements of sequences in FK spaces"</i>	540