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**The Radon-Nikodyn theorem for finitely additive measures is deduced from the corresponding result for countably additive measures.**

In ([4], Theorem 1, p. 35) a Radon-Nikodyn type result is proved for finitely additive measures. In this note we prove that this result is a simple consequence of the corresponding result for the countably additive case.

Let $\mathfrak{A}_0$ be an algebra of subsets of a set $X$; without loss of generality we assume that $\mathfrak{A}_0$ is reduced, i.e., separates points of $X$ ([5], p. 68). We denote by $\rho$ the isomorphism between $\mathfrak{A}_0$ and $\mathfrak{A}$ the algebra of all clopen subsets of $\hat{X}$, the compact Hausdorff, totally disconnected space which is the Boolean space for $\mathfrak{A}_0$ ([5], p. 70).

THEOREM ([4], Theorem 1, p. 35). *Let $\lambda$ and $\mu$ be two complex-valued finite-additive measures on $\mathfrak{A}_0$ such that $\mu$ is bounded and $\lambda$ is absolutely continuous relative to $\mu$ ($\varepsilon - \delta$ meaning of absolute continuity). Then there exists a sequence $\{f_n\}$ of $\mathfrak{A}_0$-simple functions on $X$ such that*

(1) $\lim \int_A f_n d\mu = \lambda(A)$ , *unif. for* $A \in \mathfrak{A}_0$

*and*

(2) $\lim_{m,n\to\infty} \int |f_n - f_m| d|\mu| = 0$, $|\mu|$ *being the total variation of* $\mu$ ([2]).

*Proof.* For any disjoint sequence $\{A_n\} \subset \mathfrak{A}_0$, $|\mu|(A_n) \to 0$ (note $\mu$ is bounded) and so $\lambda(A_n) \to 0$. This means $\lambda$ is exhaustive ($\equiv$strongly bounded) and so $\lambda$ is bounded ([1]). $\lambda$ and $\mu$ naturally give rise to countably additive measures $\lambda'$ and $\mu'$ on $\mathfrak{A}$ and as such can be uniquely extended to the $\sigma$-algebra $\mathscr{B}_\infty$ generated by $\mathfrak{A}$; $\mathscr{B}_\infty$ is also the class of all Baire subsets of $\hat{X}$ ([5], p. 70). We claim $|\lambda'|$ is absolutely continuous with respct to $|\mu'|$: suppose $|\mu'|(B) = 0$ but $|\lambda'|(B) > 0$ for some $B \in \mathscr{B}_\infty$. This means there exists a $C \subset B$, $C \in \mathscr{B}_\infty$ such that $|\lambda'(C)| > \varepsilon$ for some $\varepsilon > 0$. Fix $\delta > 0$ such that $P \in \mathfrak{A}_0$, $|\mu|(P) < \delta$ implies $|\lambda(P)| < \varepsilon$. Since Baire measures are regular, there exists an open subset $V$ of $\hat{X}$ such that $V \supset C$, $|\mu'|(V) < \delta$, and $|\lambda'(V)| > \varepsilon$. Again by regularity and total disconnectedness of $\hat{X}$ there is a clopen subset $U \subset V$ such that $|\mu'|(U) < \delta$ and $|\lambda'(U)| > \varepsilon$. Taking $P = \rho^{-1}(U)$ we get $|\mu|(P) < \delta$ and $|\lambda(P)| > \varepsilon$, a contradiction.

By ([2], Theorem 7, p. 181) theae exists an $f \in \mathscr{L}_1(X, \mathscr{B}_\infty, |\mu'|)$ such that $\lambda' = f\mu'$. Since $\mathfrak{A}$-simple functions are dense in $\mathscr{L}_1(X, \mathscr{B}_\infty, |\mu'|)$ there exists a sequence $\{f_n\}$ of $\mathfrak{A}$-simple functions such that $\lim \int |f_n - f| d|\mu'| = 0$. From this it follows that $\int_E f_n d|\mu'| \to \int_E f d|\mu'|$ uniformly for $E \in \mathfrak{A}$. Note on $\mathfrak{A}$ the variation $|\mu'|$ of $\mu'$ is the same whether this variation is calculated relative to $\mathfrak{A}$ or $\mathscr{B}_\infty$ ([2]), Theorem 3, p. 76). The results (1) and (2) of the theorem are obvious now.

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