

# Pacific Journal of Mathematics

**A NOTE ON RADON-NIKODÝM THEOREM FOR FINITELY  
ADDITIVE MEASURES**

**SURJIT SINGH KHURANA**

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**The Radon-Nikodyn theorem for finitely additive measures  
 is deduced from the corresponding result for countably addi-  
 tive measures.**

In ([4], Theorem 1, p. 35) a Radon-Nikodyn type result is proved for finitely additive measures. In this note we prove that this result is a simple consequence of the corresponding result for the countably additive case.

Let  $\mathfrak{A}_0$  be an algebra of subsets of a set  $X$ ; without loss of generality we assume that  $\mathfrak{A}_0$  is reduced, i.e., separates points of  $X$  ([5], p. 68). We denote by  $\rho$  the isomorphism between  $\mathfrak{A}_0$  and  $\mathfrak{A}$  the algebra of all clopen subsets of  $\hat{X}$ , the compact Hausdorff, totally disconnected space which is the Boolean space for  $\mathfrak{A}_0$  ([5], p. 70).

**THEOREM** ([4], Theorem 1, p. 35). *Let  $\lambda$  and  $\mu$  be two complex-valued finite-additive measures on  $\mathfrak{A}_0$  such that  $\mu$  is bounded and  $\lambda$  is absolutely continuous relative to  $\mu$  ( $\varepsilon - \delta$  meaning of absolute continuity). Then there exists a sequence  $\{f_n\}$  of  $\mathfrak{A}_0$ -simple functions on  $X$  such that*

$$(1) \quad \lim \int_A f_n d\mu = \lambda(A), \quad \text{unif. for } A \in \mathfrak{A}_0$$

and

$$(2) \quad \lim_{m, n \rightarrow \infty} \int |f_n - f_m| d|\mu| = 0, \quad |\mu| \text{ being the total variation of } \mu \text{ ([2]).}$$

*Proof.* For any disjoint sequence  $\{A_n\} \subset \mathfrak{A}_0$ ,  $|\mu|(A_n) \rightarrow 0$  (note  $\mu$  is bounded) and so  $\lambda(A_n) \rightarrow 0$ . This means  $\lambda$  is exhaustive ( $\equiv$  strongly bounded) and so  $\lambda$  is bounded ([1]).  $\lambda$  and  $\mu$  naturally give rise to countably additive measures  $\lambda'$  and  $\mu'$  on  $\mathfrak{A}$  and as such can be uniquely extended to the  $\sigma$ -algebra  $\mathfrak{B}_\infty$  generated by  $\mathfrak{A}$ ;  $\mathfrak{B}_\infty$  is also the class of all Baire subsets of  $\hat{X}$  ([5], p. 70). We claim  $|\lambda'|$  is absolutely continuous with respect to  $|\mu'|$ : suppose  $|\mu'| (B) = 0$  but  $|\lambda'| (B) > 0$  for some  $B \in \mathfrak{B}_\infty$ . This means there exists a  $C \subset B$ ,  $C \in \mathfrak{B}_\infty$  such that  $|\lambda'| (C) > \varepsilon$  for some  $\varepsilon > 0$ . Fix  $\delta > 0$  such that  $P \in \mathfrak{A}_0$ ,  $|\mu|(P) < \delta$  implies  $|\lambda|(P) < \varepsilon$ . Since Baire measures are regular, there exists an open subset  $V$  of  $\hat{X}$  such that  $V \supset C$ ,  $|\mu'| (V) < \delta$ , and  $|\lambda'| (V) > \varepsilon$ . Again by regularity and total disconnectedness of  $\hat{X}$  there is a clopen subset  $U \subset V$  such that  $|\mu'| (U) < \delta$  and  $|\lambda'| (U) > \varepsilon$ . Taking  $P = \rho^{-1}(U)$  we get  $|\mu|(P) < \delta$  and  $|\lambda|(P) > \varepsilon$ , a contradiction.

By ([2], Theorem 7, p. 181) there exists an  $f \in \mathcal{L}_1(X, \mathcal{B}_\infty, |\mu'|)$  such that  $\lambda' = f\mu'$ . Since  $\mathcal{A}$ -simple functions are dense in  $\mathcal{L}_1(X, \mathcal{B}_\infty, |\mu'|)$  there exists a sequence  $\{f_n\}$  of  $\mathcal{A}$ -simple functions such that  $\lim \int |f_n - f|d|\mu'| = 0$ . From this it follows that  $\int_E f_n d|\mu'| \rightarrow \int_E f d|\mu'|$  uniformly for  $E \in \mathcal{A}$ . Note on  $\mathcal{A}$  the variation  $|\mu'|_E$  of  $\mu'$  is the same whether this variation is calculated relative to  $\mathcal{A}$  or  $\mathcal{B}_\infty$  ([2], Theorem 3, p. 76). The results (1) and (2) of the theorem are obvious now.

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