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## **PSEUDO-VALUATION DOMAINS**

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A domain  $R$  is called a pseudo-valuation domain if, whenever a prime ideal  $P$  contains the product  $xy$  of two elements of the quotient field of  $R$  then  $x \in P$  or  $y \in P$ . It is shown that a pseudo-valuation domain which is not a valuation domain is a quasi-local domain  $(R, M)$  such that  $V = M^{-1}$  is a valuation overring with maximal ideal  $M$ . The authors further show that the nonprincipal divisorial ideals of  $R$  coincide with the nonzero ideals of  $V$ . These ideas are then applied to the case of a Noetherian pseudo-valuation domain  $R$ . Such a domain  $R$  is shown to have all its nonzero ideals divisorial if and only if each ideal is two-generated. Examples include valuation rings, certain  $D + M$  constructions, and certain rings of algebraic integers.

**Introduction.** The purpose of this paper is to study *pseudo-valuation domains*, a class of rings closely related to valuation rings. We define a *pseudo-valuation domain* to be a domain  $R$  in which every prime ideal  $P$  has the property that whenever a product of two elements of the quotient field of  $R$  lies in  $P$  then one of the given elements is in  $P$ . One shows easily that valuation rings are pseudo-valuation domains (Prop. 2.1). In the first section of the paper, several characterizations of pseudo-valuation domains are given. For example, a quasi-local domain  $(R, M)$  is a pseudo-valuation domain if and only if  $x^{-1}M \subset M$  whenever  $x$  is an element of the quotient field of  $R$ ,  $x \notin R$  (Th. 1.4).

The name "pseudo-valuation domain" is justified in the second section, first by showing that these rings share many properties with valuation rings. More important is the characterization of a pseudo-valuation domain which is not a valuation domain as a quasi-local domain  $(R, M)$  with the property that  $V = M^{-1}$  is a valuation overring with maximal ideal  $M$ . The second section is concluded with a study of the relationship between the ideals of  $R$  and the ideals of  $V$ ; for example, the set of nonzero ideals of  $V$  and the set of nonprincipal, divisorial ideals of  $R$  are shown to be one and the same (Cor. 2.15).

In the final section, the authors study Noetherian pseudo-valuation domains. Such rings have Krull dimension  $\leq 1$ . Also, a Noetherian pseudo-valuation domain has the 2-generator property if and only if every nonzero ideal is divisorial (Th. 3.5).

Besides valuation rings, two other classes of examples of pseudo-

valuation domains are given. The first (Ex. 2.1) is obtained by taking a valuation ring of the form  $V = K + M$ ,  $K$  a field, and taking  $R$  to be a subring of the form  $F + M$ ,  $F$  a proper subfield of  $K$ . A second class of (Noetherian) pseudo-valuation domains is provided by localizing certain rings of algebraic integers (Ex. 3.6).

## I. Definitions and properties.

**DEFINITION.** Let  $R$  be a domain with quotient field  $K$ . A prime ideal  $P$  of  $R$  is called strongly prime if  $x, y \in K$  and  $xy \in P$  imply that  $x \in P$  or  $y \in P$ .

**DEFINITION.** A domain  $R$  is called a pseudo-valuation domain if every prime ideal of  $R$  is strongly prime.

**PROPOSITION 1.1.** *Every valuation domain is a pseudo-valuation domain.*

*Proof.* Let  $V$  be a valuation domain, and let  $P$  be a prime ideal in  $V$ . Suppose  $xy \in P$  where  $x, y \in K$ , the quotient field of  $V$ . If both  $x$  and  $y$  are in  $V$ , we are done. Suppose that  $x \notin V$ . Since  $V$  is a valuation domain, we have  $x^{-1} \in V$ . Hence  $y = xy \cdot x^{-1} \in P$ , as desired.

As we shall see in the next section, the converse of the above proposition is false. We turn now to some simple properties and characterizations of pseudo-valuation domains.

**PROPOSITION 1.2.** *Let  $P$  be a prime ideal of a domain  $R$  with quotient field  $K$ . Then  $P$  is strongly prime if and only if  $x^{-1}P \subset P$  whenever  $x \in K - R$ .*

*Proof.* Assume that  $P$  is strongly prime. If  $x \in K - R$  and  $p \in P$  then  $p = px^{-1} \cdot x \in P$ , whence  $px^{-1} \in P$  or  $x \in P$ . Since  $x \notin R$  we must have  $px^{-1} \in P$ . Thus  $x^{-1}P \subset P$ .

Conversely, assume  $x^{-1}P \subset P$  whenever  $x \in K - R$ , and let  $ab \in P$ . If  $a, b \in R$  there is nothing to prove. Hence we may assume  $a \notin R$  so that  $a^{-1}P \subset P$  and  $b = a^{-1} \cdot ab \in P$ . This completes the proof.

**COROLLARY 1.3.** *In a pseudo-valuation domain  $R$ , the prime ideals are linearly ordered. In particular  $R$  is quasi-local.*

*Proof.* Let  $P$  and  $Q$  be prime ideals, and suppose  $a \in P -$

*Q.* Then for each  $b \in Q$  we have  $a/b \notin R$ . Hence  $(b/a)P \subset P$  by the proposition. Thus  $b = b/a \cdot a \in P$  and we have  $Q \subset P$ .

**THEOREM 1.4.** *Let  $(R, M)$  be a quasi-local domain. The following statements are equivalent.*

- (1)  *$R$  is a pseudo-valuation domain.*
- (2) *For each pair  $I, J$  of ideals of  $R$ , either  $I \subset J$  or  $MJ \subset MI$ .*
- (3) *For each pair  $I, J$  of ideals of  $R$ , either  $I \subset J$  or  $MJ \subset I$ .*
- (4)  *$M$  is strongly prime.*

*Proof.* (1)  $\Rightarrow$  (2). Assume  $I \not\subset J$  and pick  $a \in I - J$ . For each  $b \in J$  we have  $a/b \notin R$ , so that  $(b/a)M \subset M$  and  $Mb \subset Ma \subset MI$ . It follows that  $MJ \subset MI$ .

(2)  $\Rightarrow$  (3). This requires no comment.

(3)  $\Rightarrow$  (4). Let  $a, b \in R$  with  $a/b \notin R$ . We shall show that  $(b/a)M \subset M$ ; by Proposition 1.2 this will suffice. Since  $a/b \notin R$  we have  $(a) \not\subset (b)$  whence  $Mb \subset (a)$  and  $Mb/a \subset R$ . If  $Mb/a = R$  then  $M = Ra/b$  and  $a/b \in R$ , a contradiction. Hence  $Mb/a \subset M$ , as was to be shown.

(4)  $\Rightarrow$  (1). Let  $x$  be an element of the quotient field of  $R$ ,  $x \notin R$ , and let  $P$  be a prime ideal. Again, by Proposition 1.2, it is enough to show that  $x^{-1}P \subset P$ . Accordingly, let  $p \in P$  and note that since  $P \subset M$ , we have  $x^{-1}p \in M$ . Hence  $x^{-1}p \cdot x^{-1} \in M$ , whence  $(x^{-1}p)^2 = x^{-1}px^{-1} \cdot p \in P$ . Since  $P$  is prime and  $x^{-1}p \in R$ , we therefore have  $x^{-1}p \in P$ .

In the following theorem we characterize pseudo-valuation domains without making the quasi-local assumption.

**THEOREM 1.5.** *Let  $R$  be a domain with quotient field  $K$ . The following statements are equivalent.*

- (1)  *$R$  is a pseudo-valuation domain.*
- (2) *For each  $x \in K - R$  and for each nonunit  $a$  of  $R$ , we have  $(x + a)R = xR$ .*
- (3) *For each  $x \in K - R$  and for each nonunit  $a$  of  $R$ , we have  $x^{-1}a \in R$ .*

*Proof.* (1)  $\Rightarrow$  (2) Let  $x \in K - R$  and let  $a$  be a nonunit of  $R$ . Then  $a \in P$  for some prime ideal  $P$ , so that  $x^{-1}a \in x^{-1}P \subset P \subset R$ . Hence  $(x + a)/x = 1 + a/x \in R$  and  $(x + a)R \subset xR$ . On the other hand,  $x + a \notin R$  so that  $(x + a)^{-1}P \subset P$  and  $a/(x + a) \in R$ . Since  $x/(x + a) = 1 - a/(x + a)$ , we have  $x/(x + a) \in R$  and  $xR \subset (x + a)R$ .

(2)  $\Rightarrow$  (3). By (2)  $(x + a)/x = 1 + a/x \in R$ , whence  $x^{-1}a \in R$  also.

(3)  $\Rightarrow$  (1). Let  $P$  be prime and take  $ab \in P$  with  $a, b \in K$ . We

may assume  $b \notin R$ . By hypothesis since  $ab$  is a nonunit of  $R$ ,  $a = b^{-1} \cdot ab \in R$ . We claim that  $a$  is a nonunit; otherwise  $b = a^{-1} \cdot ab \in P$ , a contradiction. We apply the hypothesis again to get  $b^{-1}a \in R$ . Thus  $a^2 = b^{-1}a \cdot ab \in P$  and  $a \in P$ , as desired.

We close this section with a brief study of overrings of pseudo-valuation domains. (By an overring of a domain  $R$ , we mean a domain between  $R$  and its quotient field.)

**LEMMA 1.6.** *Let  $R$  be a pseudo-valuation domain and let  $T$  be an overring. If  $Q$  is prime in  $T$ , then every prime ideal of  $R$  contained in  $Q \cap R$  is also a prime ideal of  $T$ .*

*Proof.* Let  $P$  be prime in  $R$  with  $P \subset Q \cap R$ . To show that  $P$  is an ideal of  $T$ , it suffices to show  $tp \in P$  for all  $t \in T$ ,  $p \in P$ . Now  $p = tp \cdot t^{-1} \in P \Rightarrow tp \in P$  or  $t^{-1} \in P$ . However, if  $t^{-1} \in P \subset Q \cap R$ , we have that  $t^{-1} \in Q$ . This implies that  $t^{-1}$  is a nonunit of  $T$ , contradicting that  $t \in T$ . Thus  $tp \in P$  and  $P$  is indeed an ideal of  $T$ . That  $P$  is a prime ideal of  $T$  follows easily from the fact that  $P$  is strongly prime in  $R$ .

**THEOREM 1.7.** *Let  $R$  be a pseudo-valuation domain with overring  $T$ . If the pair  $R \subset T$  satisfies incomparability, then  $T$  is also a pseudo-valuation domain, and every prime ideal of  $T$  is a prime of  $R$ .*

*Proof.* Let  $Q$  be a prime ideal of  $T$ . We claim that  $Q$  is also prime in  $R$ . Clearly  $Q \cap R$  is prime in  $R$ , whence  $Q \cap R$  is prime in  $T$  by the lemma. Thus  $Q \cap R \subset Q$  are primes of  $T$  lying over  $Q \cap R$  in  $R$ . Since incomparability holds, we must have  $Q = Q \cap R$ , so that  $Q$  is a prime of  $R$ . Since  $R$  and  $T$  have the same quotient field and  $Q$  is strongly prime in  $R$ , it follows easily that  $Q$  is strongly prime in  $T$ . Thus  $T$  is a pseudo-valuation domain.

**II. Valuation overrings.** We begin this section with an example which anticipates most of the results in the section.

**EXAMPLE 2.1.** Let  $V$  be a valuation domain of the form  $K + M$ , where  $K$  is a field and  $M$  is the maximal ideal of  $V$ . If  $F$  is a proper subfield of  $K$ , then  $R = F + M$  is a pseudo-valuation domain which is not a valuation domain. To see this, note that by [3, Theorem A, p. 560]  $R$  and  $V$  have the same quotient field  $L$  and that  $M$  is the maximal ideal of  $R$ . Therefore, since valuation domains are pseudo-valuation domains, we see that  $M$  is strongly prime in  $V$ . It follows from the fact that  $R$  and  $V$  have the same quotient field that  $M$  is strongly prime in  $R$ . Thus by

Theorem 1.4  $R$  is a pseudo-valuation domain. Note that  $R$  is not a valuation ring, again by [3, Theorem A, p. 560].

**PROPOSITION 2.2.** *If a GCD domain  $R$  is also a pseudo-valuation domain, then  $R$  is a valuation domain.*

*Proof.* By Theorem 1.3 the primes of  $R$  are linearly ordered. Thus the result follows from [7, Theorem 1].

**REMARK 2.3.** It is not enough in the above proposition to take  $R$  to be an integrally closed pseudo-valuation domain, for if in Example 2.1 we take  $F$  to be algebraically closed in  $K$ , then we have by [3, Theorem A, p. 560] that  $R$  is integrally closed.

As the following results show, pseudo-valuation domains enjoy many of the same properties that valuation domains do.

**PROPOSITION 2.4.** *If  $I$  is an ideal in a pseudo-valuation domain, then  $P = \cap \{I^k : k = 1, 2, \dots\}$  is a prime ideal.*

*Proof.* Let  $xy \in P$  with  $x \notin P$ . Since  $x \notin P$  we have that  $x \notin I^n$  for some  $n > 0$ . Thus by Theorem 1.4  $I^{2n} \subset (x)$ . Hence for each positive integer  $k$ , we have  $(xy) \subset P \subset I^{2n+k} = I^{2n} \cdot I^k \subset xI^k$ , whence  $y \in I^k$ . Therefore  $y \in P$  and  $P$  is prime.

**COROLLARY 2.5.** *Let  $I, J$  be ideals in a pseudo-valuation domain  $R$ . If  $I \not\subseteq \sqrt{J}$  then  $J$  contains some power of  $I$ .*

*Proof.* Suppose  $I^k \not\subset J$  for all  $k > 0$ . Then by Theorem 1.4 we have  $J^2 \subset I^k$  for all  $k$  so that  $J^2 \subset \cap \{I^k : k = 1, 2, \dots\} = P$ , a prime ideal. Hence  $J \subset P \subset I$  and  $\sqrt{J} \subset P \subset I$ , a contradiction.

**PROPOSITION 2.6.** *Let  $R$  be a pseudo-valuation domain with maximal ideal  $M$ . If  $P$  is a nonmaximal prime ideal of  $R$ , then  $R_P$  is a valuation domain.*

*Proof.* Let  $K$  denote the quotient field of  $R$ , and let  $x \in K$ . If  $x \in R$  then  $x \in R_P$ . If  $x \notin R$  then since  $R$  is a pseudo-valuation domain  $x^{-1}M \subset M$ . Choose  $m \in M - P$ . Then  $x^{-1} = x^{-1}m/m \in R_P$ .

We now characterize pseudo-valuation domains in terms of valuation overrings.

**THEOREM 2.7.** *The following statements are equivalent for a quasi-local domain  $(R, M)$ .*

- (1)  $R$  is a pseudo-valuation domain.
- (2)  $R$  has a (unique) valuation overring  $V$  with maximal ideal  $M$ .
- (3) There exists a valuation overring  $V$  in which every prime ideal of  $R$  is also a prime ideal of  $V$ .

*Proof.* (1)  $\Rightarrow$  (2) By [5, Theorem 56] there is a valuation overring  $(W, N)$  with  $N \cap R = M$ . By Lemma 1.6  $M$  is a prime ideal of  $W$ . Put  $V = W_M$ , then  $V$  is a valuation overring with maximal ideal  $M_M$ . Since  $M$  is strongly prime, it follows easily that  $M = M_M$ . The uniqueness of  $V$  follows from the fact that valuation overrings of  $R$  are determined by their maximal ideals [3, Theorem 14.6].

(2)  $\Rightarrow$  (3). Let  $P$  be prime in  $R$ ,  $p \in P$ , and  $v \in V$ . Then  $p \in M$  so that  $vp \in M$ . Thus  $v^2p \in M$ , whence  $(vp)^2 \in P$ . Hence  $vp \in P$  and  $P$  is an ideal of  $V$ . Now let  $xy \in P$  with  $x, y \in V$ . If both  $x$  and  $y$  are in  $R$  then  $x \in P$  or  $y \in P$ . Thus assume  $x \notin R$  so that  $x \notin M$  and  $x^{-1} \in V$ . Thus, since  $P$  is an ideal of  $V$ , we have  $y = x^{-1} \cdot xy \in P$ . Hence  $P$  is a prime ideal of  $V$ .

(3)  $\Rightarrow$  (1). Let  $V$  be the given valuation overring. Then since every prime ideal  $P$  of  $R$  is also prime in  $V$ , and since  $V$  is a pseudo-valuation domain,  $P$  is strongly prime. Thus  $R$  is a pseudo-valuation domain.

In Theorem 2.10 we shall give more information about the valuation overring in the above theorem. We have need of the following:

**PROPOSITION 2.8.** *Let  $(R, M)$  be a pseudo-valuation ring which is not a valuation ring, and let  $(V, M)$  be the valuation overring (of Theorem 2.7). If  $I$  is a nonzero principal ideal of  $R$ , then  $I$  is not an ideal of  $V$ .*

*Proof.* Suppose  $I = Ra$  is a nonzero ideal of  $V$ . Then  $I = VI = VRa = Va$ . Choose  $v \in V - R$ . Then  $va \in I$  so that  $va = ra$  with  $r \in R$  and  $v = r \in R$ , a contradiction.

**COROLLARY 2.9.** *If a pseudo-valuation domain  $R$  has a nonzero principal prime ideal, then  $R$  is a valuation domain.*

*Proof.* Assume that  $R$  is not a valuation domain. Let  $V \neq R$  be a valuation overring with the same maximal ideal. If  $P$  is a nonzero principal prime ideal of  $R$  then  $P$  is not an ideal of  $V$  by Proposition 2.8. This contradicts Lemma 1.6.

We now show that the valuation overring of Theorem 2.7 (2) is simply  $M^{-1}$ .

**THEOREM 2.10.** *Let  $(R, M)$  be a quasi-local domain which is not a*

*valuation domain. Then  $R$  is a pseudo-valuation domain if and only if  $V = M^{-1}$  is a valuation overring with maximal ideal  $M$ .*

*Proof.* Assume that  $R$  is a pseudo-valuation domain. Let  $x \in V = M^{-1}$ . We claim that  $xM \subset M$ . Otherwise  $xM = R$ , whence  $M = Rx^{-1}$  is principal and  $R$  is a valuation domain by Corollary 2.9. Since  $R$  was assumed not valuation, our claim is verified. To show that  $V$  is an overring, it suffices to show that  $xy \in V$  whenever  $x, y \in V$ . This follows from our claim since  $x, y \in V$  implies  $xyM \subset xM \subset M \subset R$  so that  $xy \in M^{-1} = V$ . To see that  $V$  is a valuation domain, let  $z$  be an element of the quotient field. If  $z \in R$  then  $z \in V$ . Otherwise,  $z^{-1}M \subset M$ , whence  $z^{-1} \in M^{-1} = V$ . That  $M$  is an ideal of  $V$  also follows from  $xM \subset M$  whenever  $x \in V$ . To see that  $M$  is the maximal ideal of  $V$ , let  $x$  be a nonunit of  $V$ . If  $x \notin M$  then  $x \notin R$ , whence  $x^{-1}M \subset M$  and  $x^{-1} \in V$ , a contradiction. Thus  $M$  is the maximal ideal of  $V$ .

Conversely, assume that  $V = M^{-1}$  is a valuation ring with maximal ideal  $M$ . Then  $R$  is a pseudo-valuation domain by Theorem 2.7.

Throughout the rest of this section,  $(R, M)$  will denote a pseudo-valuation domain which is not a valuation ring, and  $V = M^{-1}$  will denote the valuation overring with the same maximal ideal. As we have seen (Theorem 2.7), every prime ideal of  $R$  is also a prime ideal of  $V$ . Conversely, since every ideal of  $V$  is contained in  $M$ , it is clear that every ideal of  $V$  is an ideal of  $R$ . Thus  $R$  and  $V$  have the same set of prime ideals. As Proposition 2.8 shows, however, if  $A$  is a nonzero ideal of  $V$  then  $A$  is not a principal ideal of  $R$ ; hence there are ideals of  $R$  which are not ideals of  $V$ . We shall now study further the relationship between ideals of  $R$  and ideals of  $V$ . This study is motivated by Bastida and Gilmer's investigation of divisorial ideals in rings of the form  $D + M$  [1, §4]. In particular, compare [1, Theorem 4.1] with Lemma 2.12 and [1, Theorem 4.3 (1)] with Theorem 2.13.

**PROPOSITION 2.11.** *If  $A$  is an ideal of  $R$ , then either  $A$  is an ideal of  $V$  or  $AV$  is a principal ideal of  $V$ .*

*Proof.* Assume that  $A$  is not an ideal of  $V$ , and choose  $x \in AV - A$ . We shall show that  $AV = xV$ . Suppose, on the contrary, that  $y \in AV - xV$ . Then  $y/x \notin V$ , so that  $x/y \in M$  and  $x = x/y \cdot y \in M(AV) = MA \subset A$ , a contradiction. Thus  $AV = xV$  is a principal ideal of  $V$ .

To complete our discussion of ideals we have need of the  $v$ -operation, a discussion of which may be found in [1, p. 87]. To simplify our notation, we shall use " $v$ " for the  $v$ -operation on  $R$  and " $w$ " for the  $v$ -operation on  $V$ . Recall that an ideal  $A$  is called divisorial  $\Leftrightarrow A$  is a



$v$ -ideal  $\Leftrightarrow A = A_v = (A^{-1})^{-1}$  = the intersection of principal fractional ideals containing  $A$ .

LEMMA 2.12.  *$M$  is divisorial.*

*Proof.* Otherwise  $M^{-1} = R$ , contradicting that  $M^{-1}$  is a valuation overring.

THEOREM 2.13. *If  $A$  is a nonzero ideal of  $V$ , then  $A$  is a divisorial ideal of  $R$ .*

*Proof.* We have already noted that  $A$  is an ideal of  $R$ . Assume that  $A$  is not divisorial in  $R$ , and pick  $x \in A_v - A$ . We assert that  $Mx = MA$ . Since  $Rx \not\subset A$  we have  $MA \subset Mx$  by Theorem 1.4. Furthermore, if  $Mx \not\subset MA$  then  $A \subset Rx$ , also by Theorem 1.4. Hence if  $a \in A$  then  $a = rx$ , whence  $r \in M$  since  $x \notin A$ . Thus  $a \in Mx$  and  $A \subset Mx$ . This implies that  $Rx \subset A_v \subset (Mx)_v = M_v x = Mx$ , the last equality following from the lemma. We have arrived at the absurdity that  $Rx \subset Mx$ ; therefore,  $Mx = MA$  as asserted.

Now in  $V$  either  $M_w = V$  or  $M$  is principal [1, Lemma 4.2]. In either case  $M_w$  is principal. Thus  $M_w x = (Mx)_w = (MA)_w = (M_w A_w)_w = M_w A_w$ , the last equality following from the fact that  $M_w$  is principal. Again, since  $M_w$  is principal, we cancel  $M_w$  from the equation  $M_w x = M_w A_w$ , yielding  $Vx = A_w$ . If  $A_w = A$  then  $x \in A_w = A$ , a contradiction. Thus  $A$  is not divisorial in  $V$ , whence by [1, Lemma 4.2],  $A = bM$  for some  $b \in K$ , the quotient field of  $V$ . But then  $A_v = (bM)_v = bM_v = bM = A$ , and the theorem is established.

PROPOSITION 2.14. *If  $A$  is an ideal of  $R$ , then either  $A$  is principal in  $R$  or  $A_v = AV$ .*

*Proof.* Suppose  $A$  is not principal. Since  $AV$  is an ideal of  $V$ ,  $AV$  is a divisorial ideal of  $R$  by the preceding theorem. Thus since  $A \subset AV$  we have  $A_v \subset (AV)_v = AV$ . We must prove that  $AV \subset A_v$ ; thus if  $x \in A^{-1}$  we must show  $AVx \subset R$ . But  $x \in A^{-1}$  implies that  $xA \subset R$  whence  $xA \subset M$  since  $A$  is not principal. Hence  $VxA \subset VM = M \subset R$ , as desired.

COROLLARY 2.15.  *$A$  is a divisorial ideal of  $R$  if and only if  $A$  is a nonzero principal ideal of  $R$  or  $A$  is a nonzero ideal of  $V$ .*

*Proof.* If  $A$  is a nonzero principal ideal of  $R$ , then  $A$  is clearly divisorial. If  $A$  is a nonzero ideal of  $V$ , then  $A$  is divisorial in  $R$  by Theorem 2.13.

Conversely, assume that  $A$  is a divisorial ideal of  $R$ . If  $A$  is not principal, then  $A_v = AV$  by the preceding result. Hence  $A = A_v = AV$  is an ideal of  $V$ .

REMARK. A summary of the results in 2.7–2.15 is in order. Let  $(R, M)$  be a pseudo-valuation domain which is not a valuation ring. Then  $V(=M^{-1})$  is a valuation overring whose prime ideals coincide with those of  $R$  (Theorem 2.7 and 2.10). Recall that each nonzero ideal of  $V(=M^{-1})$  is a nonprincipal ideal of  $R$  (Proposition 2.8). On the other hand, a nonprincipal ideal  $I$  of  $R$  is an ideal of  $V \Leftrightarrow I$  is divisorial in  $R$  (Corollary 2.15). Thus the nonprincipal divisorial ideals of  $R$  coincide with the nonzero ideals of  $V$ .

### III. Noetherian pseudo-valuation domains.

THEOREM 3.1. *Let  $R$  be a Noetherian domain with quotient field  $K$  and integral closure  $R'$ . Then  $R$  is a pseudo-valuation domain if and only if  $x^{-1} \in R'$  whenever  $x \in K - R$ .*

*Proof.* Assume that  $R$  is a pseudo-valuation domain with maximal ideal  $M$ . If  $x \in K - R$  then  $x^{-1}M \subset M$ . Since  $M$  is finitely generated, we have  $x^{-1} \in R'$  by [5, Theorem 12].

Conversely, assume  $x \in K - R$  and let  $P$  be prime in  $R$ . We must show  $x^{-1}P \subset P$ .

Let  $P'$  be a prime ideal of  $R'$  such that  $P' \cap R = P$  [5, Theorem 44]. Since  $x^{-1} \in R'$ ,  $x^{-1}P \subset x^{-1}P' \subset P'$ . We claim  $x^{-1}P \subset R$ , in which case  $x^{-1}P \subset P' \cap R = P$ , and we are done. To prove the claim, suppose there exists  $p \in P$  with  $x^{-1}p \notin R$ . Then  $xp^{-1} \in R'$  by hypothesis, whence  $1 = xp^{-1} \cdot x^{-1}p \in P'$ , a contradiction.

PROPOSITION 3.2. *If  $R$  is a Noetherian pseudo-valuation domain, then  $R$  has Krull dimension  $\leq 1$ .*

*Proof.* This follows from [5, Theorem 144] and the fact that the primes of  $R$  are linearly ordered (Corollary 1.3).

COROLLARY 3.3. *If  $R$  is a Noetherian pseudo-valuation domain, then every overring of  $R$  is a pseudo-valuation domain.*

*Proof.* By the Krull-Akizuki Theorem [5, Theorem 93], every overring  $T$  has Krull dimension  $\leq 1$  (and is Noetherian). Hence the pair  $R \subset T$  satisfies incomparability, and  $T$  is a pseudo-valuation domain by Theorem 1.7.

**COROLLARY 3.4.** *If  $R$  is a Noetherian pseudo-valuation domain, then the integral closure  $R'$  of  $R$  is a discrete rank one valuation ring.*

*Proof.* We noted in the proof of Corollary 3.3 that  $R'$  is a pseudo-valuation ring, hence  $R'$  is local of Krull dimension one and integrally closed. Thus  $R'$  is a discrete rank one valuation ring.

**REMARK.** A Noetherian pseudo-valuation domain which is a *GCD* domain is a discrete rank one valuation ring by Proposition 2.2.

In Theorem 3.5 we prove that each nonzero ideal of a Noetherian pseudo-valuation domain is divisorial if and only if every ideal of  $R$  requires at most two generators. The result is a consequence of Matlis [6, Theorems 40 and 57]. We include our direct proof due to the considerable simplification of the Matlis results in the case where  $R$  is a pseudo-valuation domain. It should be noted that the conditions on  $R$  in Theorem 3.5 do not imply that  $R$  is a pseudo-valuation domain, as one can show using the example in [2, Exercise 1, p. 81].

**THEOREM 3.5.** *Let  $(R, M)$  be a Noetherian pseudo-valuation domain with  $V = M^{-1} (\neq R)$  its valuation overring. Then the following statements are equivalent.*

- (1) *Each nonzero ideal of  $R$  is divisorial.*
- (2) *Each ideal of  $R$  may be generated by two elements.*
- (3)  *$M$  may be generated by two elements.*
- (4)  *$V$  is a two-generated  $R$ -module.*
- (5) *Each nonprincipal ideal of  $R$  is an ideal of  $V$ .*

*Proof.* (1)  $\Leftrightarrow$  (5) This is a restatement of Corollary 2.15.

(1)  $\Rightarrow$  (2) By [4, Lemma 2.2],  $V = R + Rx$  with  $x \in V - R$ . Let  $I$  be a nonprincipal ideal of  $R$ . By (5)  $I = IV = kV$  for some  $k \in I$  since  $V$  is a discrete rank one valuation ring. Hence  $I = kV = k(R + Rx) = Rk + Rkx$ , and  $I$  is two-generated.

(2)  $\Rightarrow$  (3). This is trivial.

(3)  $\Rightarrow$  (4). Let  $M = (a, b)$ . Then in  $V$ ,  $M$  is generated by one of  $a$  and  $b$ , say  $M = aV$ . Then  $V = 1/aM = 1/a(Ra + Rb) = R + Rb/a$ , and  $V$  is two-generated.

(4)  $\Rightarrow$  (5). Write  $V = Rx + Ry$ . We first reduce to the case  $y = 1$ . To this end pick  $r, s \in R$  with  $1 = rx + sy$ . Then either  $r$  or  $s$ , say  $s$ , is a unit, and  $y = s^{-1} - s^{-1}rx \in R + Rx$ . Thus  $V = R + Rx$ . Now let  $I$  be a nonprincipal ideal of  $R$ . Then  $IV = kV$  for some  $k \in I$ , and, since  $I$  is not principal in  $R$ , we may pick  $i \in I - kR$ . Now  $i = kv = k(a + bx)$  for some  $a, b \in R$ ,  $v \in V$ . If  $b \in M$ , then  $bx \in M$  whence  $a + bx \in R$  and  $i \in kR$ , a contradiction. Hence  $b$  is a unit of  $R$ , and we have

$kx = b^{-1}i - b^{-1}ka \in I$ . Thus  $IV = kV = kR + kxR \subset I$ , proving (5).

We close this section with an example of a Noetherian pseudo-valuation domain which is not a valuation ring. The example given is easily seen to satisfy the equivalent conditions of Theorem 3.5.

EXAMPLE 3.6. Let  $m$  denote a square-free positive integer,  $m \equiv 5 \pmod{8}$ . Let  $Z$  denote the ring of integers and set  $D = Z[\sqrt{m}]$ . Since  $m \equiv 1 \pmod{4}$ ,  $D$  does not contain the algebraic integers of the form  $(a + b\sqrt{m})/2$ , where  $a$  and  $b$  are odd integers. Thus,  $D$  is not integrally closed [8, Theorem 6.6]. It is routine to check that  $(2, 1 + \sqrt{m}) = N$  is a maximal ideal of  $D$ . The desired example is  $R = D_N$ , which has  $K = Q[\sqrt{m}]$  as its quotient field.  $R$  is not a valuation ring since neither  $(1 + \sqrt{m})/2$  nor its inverse lies in  $R$ .

To show that  $R$  is a pseudo-valuation ring we apply Theorem 3.1 to the integral closure  $R'$  of  $R$ . Since  $R' = (D_N)' = (D')_S$ , where  $S = D - N$  and  $(\cdot)'$  denotes integral closure, we must show  $x \in K - R$  implies  $1/x \in (D')_S$ . Now  $x = (a + b\sqrt{m})/c$  where  $a, b, c \in Z$  and  $\gcd(a, b, c) = 1$ . Since  $x \notin R$ ,  $c \in N \cap Z = 2Z$  so 2 divides  $c$ . But then  $a$  or  $b$  must be odd since  $\gcd(a, b, c) = 1$ . Now  $x^{-1} = c(a - b\sqrt{m}) \cdot (a^2 - b^2m)^{-1}$ . If  $a^2 - b^2m \notin S$  then  $a^2 - b^2m \in N \cap Z = 2Z$ , but  $m \equiv 1 \pmod{4}$ ; so  $a$  and  $b$  are both odd integers. It follows that  $a^2 - b^2m \equiv 0 \pmod{4}$ , but  $a^2 - b^2m \equiv 1 - m \equiv 4 \pmod{8}$ . Thus  $a^2 - b^2m = 4t$  with  $t$  an odd integer, and so  $x^{-1} = (c/2((a - b\sqrt{m})/2))/t \in D'_S = R$  because with  $a, b$  odd integers we have  $(a - b\sqrt{m})/2$  an algebraic integer, hence an element of  $D'$ .

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