SOME THEOREMS ON GENERALIZED DEDEKIND-RADEMACHER SUMS

L. CARLITZ

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Radamacher has defined a generalized Dedekind sum

$$s(h, k; x, y) = \sum_{a \,(\mathrm{mod}\, k)} \left(\left(h\frac{a+y}{k} + x\right)\right) \left(\left(\frac{a+y}{k}\right)\right)$$

**and proved a reciprocity theorem for this sum that generalizes
the well known result for $s(h, k)$. In the present paper we
define**

$$\phi_{r,s}(h, k; x, y) = \sum_{a\,(\mathrm{mod}\, k)} \bar{B}_r\left(h\left(\frac{a+y}{k}\right) + x\right)\bar{B}_s\left(\frac{a+y}{k}\right),$$

$$\psi_{r,s}(h, k; x, y) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}h^{r-j}\phi_{j,r+s-j}(h, k; x, y),$$

where $\bar{B}_n(x)$ is the Bernoulli function, and show that

$$(s+1)k^s\psi_{r+1,s}(h, k; x, y) - (r+1)h^r\psi_{s+1,r}(k, h; y, x)$$
$$= (s+1)k\bar{B}_{r+1}(x)\bar{B}_s(y) - (r+1)h\bar{B}_r(x)\bar{B}_{s+1}(y) \qquad ((h, k) = 1).$$

We also prove the polynomial reciprocity theorem

$$(1-v)\sum_{a=0}^{k-1} u^{h-[(ha+z)/k]}v^a - (1-u)\sum_{b=0}^{h-1} v^{k-[(kb+z)/h]}u^b = u^h - v^k$$
$$((h, k) = 1)$$

as well as some related results.

1. Introduction. For real x put

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \neq \text{integer}) \\ 0 & (x = \text{integer}), \end{cases}$$

where $[x]$ denotes the greatest integer $\leq x$. The Dedekind sum $s(h, k)$
is defined by

$$(1.1) \qquad s(h,k) = \sum_{a \,(\mathrm{mod}\, k)} \left(\!\left(\frac{a}{k}\right)\!\right) \left(\!\left(\frac{ha}{k}\right)\!\right).$$

The most striking property of $s(h,k)$ is the reciprocity theorem

$$(1.2) \quad 12hk\{s(h,k)+s(k,h)\} = h^2 - 3hk + k^2 + 1 \qquad ((h,k)=1).$$

For an excellent introduction and many references to Dedekind sums see [9].

The Bernoulli function $\bar{B}_n(x)$ is defined by

$$\bar{B}_n(x) = B_n(x-[x]),$$

where $B_n(x)$ is the Bernoulli polynomial defined by

$$(1.3) \qquad \frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}.$$

Note that, for $x \neq$ integer, $\bar{B}_1(x) = ((x))$.

Apostol [1], [2] defined the generalized sum

$$(1.4) \qquad s_n(h,k) = \sum_{r \,(\mathrm{mod}\, k)} \bar{B}_1\left(\frac{r}{k}\right) \bar{B}_n\left(\frac{hr}{k}\right)$$

and proved the reciprocity theorem

$$(1.5) \quad (n+1)\{hk^n s_n(h,k) + kh^n s_n(k,h)\} = (Bk + Bh)^{n+1} + nB_{n+1}$$

$$((h,k)=1).$$

This result is indeed valid for all $n \geq 0$.   For a simple proof see [4, §3].

A further generalization of (1.4) is furnished by

$$(1.6) \qquad \phi_{r,s}(h,k) = \sum_{a \,(\mathrm{mod}\, k)} \bar{B}_r\left(\frac{a}{k}\right) \bar{B}_s\left(\frac{ha}{k}\right),$$

where r, s are arbitrary nonnegative integers.   Put

$$(1.7) \qquad \psi_{r,s}(h,k) = \sum_{t=0}^{r} (-1)^t \binom{r}{t} h^t \phi_{r-t,s+t}(h,k).$$

The writer [3], [7] has proved the following reciprocity theorem which includes (1.5) as a special case.

$$(1.8) \quad (s+1)k^s\psi_{r+1,s}(h,k) - (s+1)kB_{r+1}B_s = (r+1)h^r\psi_{s+1,r}(k,h)$$

$$- (r+1)hB_{s+1}B_r \qquad ((h,k)=1).$$

He has also proved the following polynomial reciprocity:

$$(1.9) \qquad (u-1)\sum_{r=1}^{k-1} u^{k-r-1}v^{[hr/k]} - (v-1)\sum_{r=1}^{h-1} v^{h-r-1}u^{[kr/h]}$$

$$= u^{k-1} - v^{h-1} \qquad ((h,k)=1),$$

where u, v are indeterminates.

Rademacher [10] has generalized $s(h,k)$ in the following way:

$$(1.10) \qquad s(h,k;x,y) = \sum_{a \,(\mathrm{mod}\, k)} \left(\left(h\frac{a+y}{k} + x\right)\right)\left(\left(\frac{a+y}{k}\right)\right),$$

where x, y are arbitrary real numbers. He proved that

$$s(h,k;x,y) + s(k,h;y,x)$$

$$= -\tfrac{1}{4}\delta(x)\delta(y) + ((x))((y))$$

$$(1.11)$$

$$+ \frac{1}{2}\left\{\frac{h}{k}\bar{B}_2(y) + \frac{1}{hk}\bar{B}_2(hy+kx) + \frac{k}{h}\bar{B}_2(x)\right\},$$

where $(h,k)=1$ and

$$\delta(x) = \begin{cases} 1 & (x = \text{integer}) \\ 0 & (x \neq \text{integer}) . \end{cases}$$

For a simplified version of the proof see [5].

In the present paper we define

$$(1.12) \qquad \phi_{r,s}(h,k;x,y) = \sum_{a \,(\mathrm{mod}\, k)} \bar{B}_r\left(h\frac{a+y}{k} + x\right)\bar{B}_s\left(\frac{a+y}{k}\right)$$

and

$$(1.13) \qquad \psi_{r,s}(h,k;x,y) = \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j}h^{r-j}\phi_{j,r+s-j}(h,k;x,y),$$

corresponding to (1.6) and (1.7), respectively. We prove the reciprocity theorem

$$(s+1)k^s\psi_{r+1,s}(h,k;x,y) - (r+1)h'\psi_{s+1,r}(k,h;y,x) = (s+1)k\bar{B}_{r+1}(x)\bar{B}_s(y)$$

(1.14) $$-(r+1)h\bar{B}_r(x)\bar{B}_{s+1}(y) \qquad ((h,k)=1).$$

It should be observed that there is no loss in generality in assuming that

(1.15) $$0 \le x < 1, \qquad 0 \le y < 1.$$

We show also, assuming (1.15), that

(1.16) $$(1-v)\sum_{a=0}^{k-1} u^{h-[(ha+z)/k]}v^a - (1-u)\sum_{b=0}^{h-1} v^{k-[(kb+z)/h]}u^b = u^h - v^k$$

$$((h,k)=1),$$

where $z = kx + hy$. For $x = y = 0$, (1.16) reduces to (1.9) after a little manipulation. Clearly (1.16) holds for all z such that $0 \le z < h + k$.

For some additional results see §4 below, in particular (4.1), (4.2), (4.3), (4.4), (4.5), (4.6).

2.   Proof of (1.14).   We recall that [8, Ch. 2]

(2.1) $$\bar{B}_n(hx) = h^{n-1}\sum_{b \,(\mathrm{mod}\, h)} \bar{B}_n\left(x + \frac{b}{h}\right).$$

Thus (1.12) becomes

$$\phi_{r,s}(h,k;x,y) = h^{r-1}\sum_{\substack{a\,(\mathrm{mod}\,k)\\b\,(\mathrm{mod}\,h)}} \bar{B}_r\left(\frac{a+y}{k} + \frac{b+x}{h}\right)\bar{B}_s\left(\frac{a+y}{h}\right).$$

We shall write this in the abbreviated form

(2.2) $$\phi_{r,s}(h,k;x,y) = h^{r-1}\sum_{a,b}\bar{B}_r(\alpha+\beta)\bar{B}_s(\alpha),$$

where

(2.3) $$\alpha = \frac{a+y}{k}, \qquad \beta = \frac{b+x}{h}$$

and the summation on the right of (2.2) is over complete residue systems (mod k) and (mod h), respectively.

Substituting from (2.2) in (1.13), we get

$$(2.4) \qquad \psi_{r,s}(h,k;x,y) = h^{r-1} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \sum_{a,b} \bar{B}_j(\alpha+\beta)\bar{B}_{r+s-j}(\alpha).$$

Now consider

$$\Phi(h,k;x,y;u,v)$$

$$(2.5) \qquad = \sum_{r,s=0}^{\infty} sk^{s-1}\psi_{r,s-1}(h,k;x,y)\frac{u^r v^s}{r!s!}$$

$$= \sum_{r,s} \frac{h^{r-1}k^{s-1}u^r v^s}{r!(s-1)!} \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} \sum_{a=0}^{k-1}\sum_{b=0}^{h-1} \bar{B}_j(\alpha+\beta)\bar{B}_{r+s-j-1}(\alpha).$$

We assume in what follows that

$$(2.6) \qquad\qquad 0 \leqq x < 1, \qquad 0 \leqq y < 1,$$

which implies

$$(2.7) \qquad\qquad 0 \leqq \alpha < 1, \qquad 0 \leqq \beta < 1.$$

Thus

$$\bar{B}_{r+s-j-1}(\alpha) = B_{r+s-j-1}(\alpha).$$

Taking $m = r + s - j - 1$, (2.5) becomes

$$\Phi(h,k;x,y;u,v)$$

$$= h^{-1}v \sum_{a=0}^{k-1}\sum_{b=0}^{h-1}\sum_{j=0}^{\infty} \frac{(hu)^j}{j!} \bar{B}_j(\alpha+\beta) \sum_{m=0}^{\infty} \frac{1}{m!} B_m(\alpha)$$

$$\sum_{r=j}^{m+j} (-1)^{r-j} \frac{m!}{(r-j)!(m-r+j)!} (hu)^{r-j}(kv)^{m-r+j}$$

$$(2.8) \qquad = h^{-1}v \sum_{a=0}^{k-1}\sum_{b=0}^{h-1}\sum_{j=0}^{\infty} \frac{(hu)^j}{j!} \bar{B}_j(\alpha+\beta) \sum_{m=0}^{\infty} \frac{(-hu+kv)^m}{m!} B_m(\alpha)$$

$$= h^{-1}v \sum_{j,m=0}^{\infty} \frac{(hu)^j(-hu+kv)^m}{j!m!} \sum_{a,b} \bar{B}_j(\alpha+\beta)B_m(\alpha).$$

Since

$$B_j(x+1) - B_j(x) = jx^{j-1},$$

the inner sum

$$\sum_{a=0}^{k-1}\sum_{b=0}^{h-1}\bar{B}_j(\alpha+\beta)B_m(\alpha)$$

$$= \sum_{a,b} B_j(\alpha+\beta-[\alpha+\beta])B_m(\alpha)$$

$$= \sum_{a,b} B_j(\alpha+\beta)B_m(\alpha) - \sum_{\substack{a,b \\ \alpha+\beta\geq 1}} (B_j(\alpha+\beta)-B_j(\alpha+\beta-1))B_m(\alpha)$$

$$= \sum_{a,b} B_j(\alpha+\beta)B_m(\alpha) - j\sum_{\substack{a,b \\ \alpha+\beta\geq 1}} (\alpha+\beta-1)^{j-1}B_m(\alpha).$$

Thus (2.8) becomes

(2.9) $\Phi(h,k;x,y;u,v) = \Phi_1(h,k;x,y;u,v) - \Phi_2(h,k;x,y;u,v),$

where

$$\Phi_1(h,k;x,y;u,v) = h^{-1}v\sum_{j,m=0}^{\infty}\frac{(hu)^j(-hu+kv)^m}{j!m!}$$

$$\cdot\sum_{a,b}B_j(\alpha+\beta)B_m(\alpha),$$

$$\Phi_2(h,k;x,y;u,v) = h^{-1}v\sum_{j,m=0}^{\infty}\frac{(hu)^j(-hu+kv)^m}{j!m!}$$

$$\cdot\sum_{\substack{a,b \\ \alpha+\beta\geq 1}}(\alpha+\beta-1)^{j-1}B_m(\alpha).$$

Clearly, by (1.3) and (2.3),

$$\Phi_1(h,k;x,y;u,v) = h^{-1}v\frac{hu}{e^{hu}-1}\frac{-hu+kv}{e^{-hu+kv}-1}$$

(2.10)

$$\cdot\sum_{a,b}e^{hu(\alpha+\beta)}e^{(-hu+kv)\alpha}$$

$$= \frac{uv}{e^{hu}-1}\frac{-hu+kv}{e^{-hu+kv}-1}\cdot e^{xu+yv}\frac{e^{hu}-1}{e^u-1}\frac{e^{kv}-1}{e^v-1}$$

$$= \frac{uv}{e^u-1}\frac{e^{kv}-1}{e^v-1}\frac{hu-kv}{e^{hu}-e^{kv}}e^{hu+xu+yv},$$

$$
\Phi_2(h, k\,;\,x, y\,;\,u, v) = uv\,\frac{-hu + kv}{e^{-hu+kv} - 1} \cdot \sum_{\substack{a,b \\ \alpha+\beta\geqq 1}} e^{hu(\alpha+\beta-1)}\, e^{(-hu+kv)\alpha}
$$

(2.11)

$$
= uv\,\frac{hu - kv}{e^{hu} - e^{kv}}\, e^{xu+yv} \sum_{\substack{a,b \\ \alpha+\beta\geqq 1}} e^{av+bu}.
$$

It follows from (2.10) that

$$
\Phi_1(h, k\,;\,x, y\,;\,u, v) - \Phi_1(k, h\,;\,y, x\,;\,v, u)
$$

(2.12)
$$
= \frac{u}{e^u - 1}\,\frac{v}{e^v - 1}\,\frac{hu - kv}{e^{hu} - e^{kv}}\, e^{xu+yv}\{e^{hu}(e^{kv} - 1) - e^{kv}(e^{hu} - 1)\}.
$$

$$
= (-hu + kv)\,\frac{ue^{xu}}{e^u - 1}\,\frac{ve^{yv}}{e^v - 1}\,,
$$

while

(2.13) $\Phi_2(h, k\,;\,x, y\,;\,u, v) - \Phi_2(k, h\,;\,y, x\,;\,v, u) = 0.$

Therefore, by (2.9), (2.12) and (2.13),

$$
\Phi(h, k\,;\,x, y\,;\,u, v) - \Phi(k, h\,;\,y, x\,;\,v, u) = (-hu + kv)\,\frac{ue^{xu}}{e^u - 1}\,\frac{ve^{yv}}{e^v - 1}\,.
$$

(2.14)

By (2.5), the left hand side of (2.14) is equal to

$$
\sum_{r,s=0}^{\infty} \{sk^{s-1}\psi_{r,s-1}(h, k\,;\,x, y) - rh^{r-1}\psi_{s,r-1}(k, h\,;\,y, x)\}\,\frac{u^r v^s}{r!s!}.
$$

By (1.3), the right hand side of (2.14) is equal to

$$
(-hu + kv) \sum_{r,s=0}^{\infty} B_r(x)B_s(y)\,\frac{u^r v^s}{r!s!}
$$

$$
= \sum_{r,s=0}^{\infty} \{sk B_r(x)B_{s-1}(y) - rh B_{r-1}(x)B_s(y)\}\,\frac{u^r v^s}{r!s!}.
$$

Hence, equating coefficients of $u^r v^s/r!s!$, we get

$$
sk^{s-1}\psi_{r,s-1}(h, k\,;\,x, y) - rh^{r-1}\psi_{s,r-1}(k, h\,;\,y, x) = sk B_r(x)B_{s-1}(y)
$$

$$
- rh B_{r-1}(x)B_s(y).
$$

Finally, dropping the restriction (2.6), we have

$$(2.15) \quad sk^{s-1}\psi_{r,s-1}(h,k;x,y) - rh^{r-1}\psi_{s,r-1}(k,h;y,x) = sk\bar{B}_r(x)\bar{B}_{s-1}(y)$$
$$- rh\bar{B}_{r-1}(x)\bar{B}_s(y),$$

for all nonnegative r, s and all real x, y.

3. *Proof of* (1.16). We again assume that

$$(3.1) \qquad\qquad 0 \leqq x < 1, \quad 0 \leqq y < 1.$$

By (1.12) and (1.13) we have

$$\psi_{r,s}(h,k;x,y) = \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j}h^{r-j}\sum_{a=0}^{k-1}\bar{B}_j\left(h\frac{a+y}{k}+x\right)B_{r+s-j}\left(\frac{a+y}{k}\right).$$

Then, as in the previous proof,

$$\Phi(h,k;x,y;u,v)$$
$$= \sum_{r,s=0}^{\infty} sk^{s-1}\psi_{r,s-1}(h,k;x,y)\frac{u^r v^s}{r!s!}$$
$$= v\sum_{r,s=0}^{\infty}\frac{u^r(kv)^s}{r!s!}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}h^{r-j}\sum_{a=0}^{k-1}\bar{B}_j\left(h\frac{a+y}{k}+x\right)B_{r+s-j}\left(\frac{a+y}{k}\right)$$
$$= v\sum_{j,m=0}^{\infty}\frac{u^j(-hu+kv)^m}{j!m!}\sum_{a=0}^{k-1}B_j\left(\frac{ha+z}{k}-\left[\frac{ha+z}{k}\right]\right)B_m\left(\frac{a+y}{k}\right)$$
$$= v\frac{u}{e^u-1}\frac{-hu+kv}{e^{-hu+kv}-1}\sum_{a=0}^{k-1}\exp\left\{\left(\frac{ha+z}{k}-\left[\frac{ha+z}{k}\right]\right)u\right.$$
$$\left.+\frac{a+y}{k}(-hu+kv)\right\}$$
$$= \frac{uv}{e^u-1}\frac{hu-kv}{e^{hu}-e^{kv}}e^{xu+yv}\sum_{a=0}^{k-1}\exp\left\{\left(h-\left[\frac{ha+z}{k}\right]\right)u+av\right\},$$

where

$$(3.2) \qquad\qquad z = kx + hy.$$

It follows that

$$\Phi(h, k; x, y; u, v) - \Phi(k, h; y, x; v, u)$$

$$(3.3) \quad = \frac{uv}{e^u - 1} \frac{hu - kv}{e^{hu} - e^{kv}} e^{xu + yv} \sum_{a=0}^{k-1} \exp\left\{\left(h - \left[\frac{ha + z}{k}\right]\right)u + av\right\}$$

$$- \frac{uv}{e^v - 1} \frac{hu - kv}{e^{hu} - e^{kv}} e^{xu + yv} \sum_{b=0}^{h-1} \exp\left\{\left(k - \left[\frac{kb + z}{h}\right]\right)v + bu\right\}.$$

Comparing (3.3) with (2.14) and simplifying, we get

$$(3.4) \quad (e^v - 1) \sum_{a=0}^{k-1} \exp\left\{\left(h - \left[\frac{ha + z}{k}\right]\right)u + av\right\}$$

$$- (e^u - 1) \sum_{b=0}^{h-1} \exp\left\{\left(k - \left[\frac{kb + z}{h}\right]\right)v + bu\right\} = -e^{hu} + e^{kv}.$$

Replacing e^u, e^v by u, v, respectively, this becomes

$$(3.5) \quad (1 - v) \sum_{a=0}^{k-1} u^{h-[(ha+z)/k]} v^a - (1 - u) \sum_{b=0}^{h-1} v^{k-[(kb+z)/h]} u^b = u^h - v^k$$

$$((h, k) = 1).$$

Clearly (3.5) is a polynomial identity in the indeterminates u, v. It is not evident how the restriction (3.1) can be removed.

To show that (3.5) includes (1.9), take $x = y = z = 0$ and replace a by $k - a$, b by $k - b$. Thus the left hand side of (3.5) becomes

$$(1 - v)u^h - (1 - u)v^k$$

$$+ (1 - v) \sum_{a=1}^{k-1} u^{h-[h-(ha/k)]} v^{k-a} - (1 - u) \sum_{b=1}^{h-1} v^{k-[k-(kb/h)]} u^{h-b}$$

$$= (u^h - v^k) - uv(u^{h-1} - v^{k-1})$$

$$+ (1 - v) \sum_{a=1}^{k-1} u^{[ha/k]+1} v^{k-a} - (1 - u) \sum_{b=1}^{h-1} v^{[kb/h]+1} u^{h-b},$$

since

$$[m - x] = m - 1 - [x] \quad (m = \text{integer}, x \neq \text{integer}).$$

Thus we get

$$(1 - v) \sum_{a=1}^{k-1} u^{[ha/k]} v^{k-a-1} - (1 - u) \sum_{b=1}^{h-1} v^{[kb/h]} u^{h-b-1} = u^{h-1} - v^{k-1},$$

which is (1.9) in a slightly different notation.

4. Additional results.   We have

$$(e^v - 1) \sum_{a=0}^{k-1} \exp \left\{ \left(h - \left[\frac{ha + z}{k} \right] \right) u + av \right\}$$

$$= \sum_{a=0}^{k-1} \sum_{r=0}^{\infty} \left(h - \left[\frac{ha + z}{k} \right] \right)^r \frac{u^r}{r!} \sum_{s=0}^{\infty} ((a + 1)^s - a^s) \frac{v^s}{s!}.$$

Thus the left hand side of (3.4) is equal to

$$\sum_{r,s=0}^{\infty} \frac{u^r v^s}{r! s!} \left\{ \sum_{a=0}^{k-1} \left(h - \left[\frac{ha + z}{k} \right] \right)^r ((a + 1)^s - a^s) \right.$$

$$\left. - \sum_{b=0}^{h-1} \left(k - \left[\frac{kb + z}{h} \right] \right)^s ((b + 1)^r - b^r) \right\}.$$

Since the right hand side of (3.4) is equal to

$$- \sum_{r=0}^{\infty} \frac{h^r u^r}{r!} + \sum_{s=0}^{\infty} \frac{k^s v^s}{s!},$$

we get

$$\sum_{a=0}^{k-1} \left(h - \left[\frac{ha + z}{k} \right] \right)^r ((a + 1)^s - a^s) - \sum_{b=0}^{h-1} \left(k - \left[\frac{kb + z}{h} \right] \right)^s ((b + 1)^r - b^r)$$

(4.1) $$= - h^r \delta_{s,0} + k^s \delta_{r,0}, \qquad ((h, k) = 1)$$

for all nonnegative r, s and all z such that

$$0 \leqq z < h + k.$$

Hence, in particular,

$$\sum_{a=0}^{k-1} \left(h - \left[\frac{ha + z}{k} \right] \right)^r ((a + 1)^s - a^s) = \sum_{b=0}^{h-1} \left(k - \left[\frac{kb + z}{h} \right] \right)^s ((b + 1)^r - b^r)$$

(4.2) $$(r > 0, s > 0; \quad 0 \leqq z < h + k).$$

For example, for $r = s = 2$,

$$\sum_{a=0}^{k-1} (2a + 1)\left(h - \left[\frac{ha + z}{k}\right]\right)^2 = \sum_{b=0}^{h-1} (2b + 1)\left(k - \left[\frac{kb + z}{h}\right]\right)^2$$

$$(0 \leq z < h + k).$$

For $s = 1$ we get

(4.3)
$$\sum_{a=0}^{k-1} \left(h - \left[\frac{ha + z}{k}\right]\right)^r = \sum_{b=0}^{h-1} \left(k - \left[\frac{kb + z}{h}\right]\right)((b + 1)^r - b^r)$$

$$(r > 0, \quad 0 \leq z < h + k).$$

Recall [8, Ch. 2] that

$$nx^{n-1} = B_n(x + 1) - B_n(x)$$

$$= \sum_{j=0}^{n} \binom{n}{j} B_{n-j}((x + 1)^j - x^j),$$

where $B_n = B_n(0)$ is the nth Bernoulli number. Thus if in (4.1) we replace r, s by i, j, respectively, multiply both sides by

$$\binom{r}{i}\binom{s}{j} B_{r-i} B_{s-j}$$

and sum over i, j, we get

(4.4)
$$s\sum_{a=0}^{k-1} a^{s-1} B_r\left(h - \left[\frac{ha + z}{k}\right]\right) - r\sum_{b=0}^{h-1} b^{r-1} B_s\left(k - \left[\frac{kb + z}{h}\right]\right)$$

$$= B_s(k)B_r - B_r(h)B_s \qquad (0 \leq z < h + k).$$

A more general result is

(4.5)
$$s\sum_{a=0}^{k-1} (a + \eta)^{s-1} B_r\left(h + \xi - \left[\frac{ha + z}{k}\right]\right)$$

$$- r\sum_{b=0}^{h-1} (b + \xi)^{r-1} B_s\left(k + \eta - \left[\frac{kb + z}{h}\right]\right)$$

$$= B_r(\xi)B_s(k + \eta) - B_r(h + \xi)B_s(\eta) \qquad (0 \leq z < h + k),$$

where ξ and η are arbitrary. In particular, for $\xi = 1 - h$, $\eta = 1 - k$, (4.5)

reduces to

$$s\sum_{a=0}^{k-1}(a+1-k)^{s-1}B_r\left(1-\left[\frac{ha+z}{k}\right]\right)-r\sum_{b=0}^{h-1}(b+1-h)^{r-1}B_s\left(1-\left[\frac{kb+z}{h}\right]\right)$$

$$=B_r(1-h)B_s(1)-B_r(1)B_s(1-k).$$

Since

$$B_n(1-x)=(-1)^nB_n(x),$$

we get

$$(4.6)\quad s\sum_{a=0}^{k-1}(k-a-1)^{s-1}B_r\left(\left[\frac{ha+z}{k}\right]\right)-r\sum_{b=0}^{h-1}(h-b-1)^{r-1}B_s\left(\left[\frac{kb+z}{h}\right]\right)$$

$$=-B_r(h)B_s+B_rB_s(k)\qquad(0\leqq z<h+k).$$

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