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**CONNECTIVE COVERINGS OF BO AND IMMERSIONS OF
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New immersions and nonimmersions of real projective space RP^n in Euclidean space are proved when the number of 1's in the binary expansion of n is 7. The method is obstruction theory, utilizing the connective coverings of BO .

1. Introduction. Let $BO[j]$ (resp. $BO_N[j]$) denote the space obtained from BO (resp. BO_N) by killing π_i for $i < j$. In [4] Mahowald and the author computed the cohomology and stable homotopy groups of $BO[8]/BO_N[8]$ through degree $N + 16$ and used these results to prove some new immersion and nonimmersion results for real projective spaces P^n . In this paper similar computations are performed when $j > 8$ and used to obtain some more new immersion and nonimmersion results.

Let $\alpha(n)$ denote the number of 1's in the binary expansion of n and $\nu(2^a(2b + 1)) = a$.

THEOREM 1.1. *If $\nu(n + 1) \geq \alpha(n) - 4 \geq 3$, then P^n cannot be immersed in $R^{2^{n-2\alpha(n)-3-4}}$.*

THEOREM 1.2. *If $\alpha(n) = 7$, $\nu(n + 1) = 4$ or 5 , then P^n can be immersed in $R^{2^{n-16}}$.*

Theorem 1.2 is within 3 dimensions of best possible (by Theorem 1.1). It provides another counterexample to the previously conjectured nonimmersions ([6], [4]). For $\alpha(n) \geq 8$, Theorem 1.1 is probably not very close to best possible. It gives the densest set of metastable nonimmersion results known to the author. The number of $n < 2^k$ satisfying the condition of 1.1 is $\binom{k-2}{5}$.

Let \mathcal{A} denote the mod 2 Steenrod algebra. For $j \equiv 0, 1, 2$, or $4(8)$ let I_j denote the left ideal generated by

$$\begin{cases} Sq^2 & \text{if } j \equiv 1(8) \\ Sq^3 & \text{if } j \equiv 2(8) \\ Sq^1 \text{ and } Sq^5 & \text{if } j \equiv 0(8) \\ Sq^1 \text{ and } Sq^2 & \text{if } j \equiv 4(8) \end{cases}$$

Let $\mathcal{A}(g_0, g_9)$ be a free \mathcal{A} -module with generators of degree 0 and 9, and I the left ideal generated by $Sq^1g_0, Sq^2g_0, Sq^4g_0, Sq^8g_0, Sq^2g_9,$

and $Sq^{16}g_0 + Sq^7g_9 + Sq^4Sq^2Sq^1g_9$. Let $P_N = RP^\infty/RP^{N-1}$. All cohomology groups have \mathbb{Z}_2 -coefficients. Our other main result is

THEOREM 1.3. (i) *There is an isomorphism of \mathcal{A} -modules through degree $N + 18$*

$$H^*(BO[9], BO_N[9]) \approx \tilde{H}^*(\Sigma P_N) \otimes \mathcal{A}(g_0, g_9)/I$$

(ii) *For $j = 0, 1, 2, 4(8)$ and $j \geq 10$, there is an isomorphism of \mathcal{A} -modules through degree $N + 2j$*

$$H^*(BO[j], BO_N[j]) \approx \tilde{H}^*(\Sigma P_N) \oplus \tilde{H}^*(\Sigma P_N) \otimes \Sigma^j \mathcal{A}/I_j.$$

In Proposition 2.1 we show how 1.3 can be used to compute the Adams spectral sequence (ASS) for $BO[j]/BO_N[j]$ through degree $N + 2j$.

This work owes a heavy debt to Mark Mahowald, who devised this approach to immersions and suggested the validity of 1.3 (ii) and the case $\alpha(n) = 7$ of 1.1.

2. The spaces $BO[j]/BO_N[j]$. In this section we study the \mathbb{Z}_2 -cohomology and stable homotopy groups of the spaces $BO[j]/BO_N[j]$ through degree $N + 2j$.

Proof of Theorem 1.3(ii) Let $k: BO[j] \rightarrow BO$ and $\bar{k}: BO[j]/BO_N[j] \rightarrow BO/BO_N$. Let $i: \Sigma P_N = CP_N/P_N \rightarrow BO[j]/BO_N[j]$ be induced from the $2N$ -equivalence $P_N \rightarrow V_N$ and the fibration $V_N \rightarrow BO_N[j] \rightarrow BO[j]$. The Serre spectral sequence ([9], [10]) of the relative fibration $(CV_N, V_N) \rightarrow (BO[j], BO_N[j]) \rightarrow BO[j]$ is trivial through degree $2N$ because it is mapped onto by that of $(BO, BO_N) \rightarrow BO$. Thus as a vector space $H^*(BO[j], BO_N[j])$ is isomorphic to $\langle \{\bar{k}^*w_m; m > N\} \rangle \otimes H^*(BO[j])$, where $\langle S \rangle$ is the vector space spanned by S . Here we use the external cup product and the fact that $i^*\bar{k}^*w_m = \sigma\alpha^{m-1}$, the nonzero element in $H^m(\Sigma P_N)$.

Stong ([11]) showed $k^* = 0: H^i(BO) \rightarrow H^i(BO[j])$ for $i < 2^{\phi(j)-1}$, where $\phi(j)$ is the number of positive integers $\leq j$ which are $\equiv 0, 1, 2, 4(8)$. Thus for $j \geq 10$ $k^* = 0: H^i(BO) \rightarrow H^i(BO[j])$ for $i \leq 2j$. By the Wu formula

$$S^i q(\bar{k}^*w_m) = \binom{m-1}{i} \bar{k}^*w_{m+1} = \sum_{a=0}^i \binom{m-1-i+a}{a} k^*w_a \cup \bar{k}^*w_{m+i-a},$$

so that for $i \leq 2j$ $S^i q(\bar{k}^*w_m) = \binom{m-1}{i} \bar{k}^*w_{m+i}$. Hence through degree $N + 2j$ $\langle \bar{k}^*w_m; N < m \leq N + 2j \rangle$ is an \mathcal{A} -submodule of $H^*(BO[j], BO_N[j])$ isomorphic to $\tilde{H}^*(\Sigma P_N^{N+2j-1})$. Thus by the Cartan formula

the vector space splitting of the previous paragraph gives an isomorphism of \mathcal{A} -modules through degree $N + 2j$ $H^*(BO[j], BO_N[j]) \approx \tilde{H}^*(\Sigma P_N) \otimes H^*(BO[j])$. Through degree $2j - 1$ $H^*(BO[j]) \approx \mathbb{Z}_2 \oplus \Sigma^j \mathcal{A}/I_j$ as \mathcal{A} -modules ([11]).

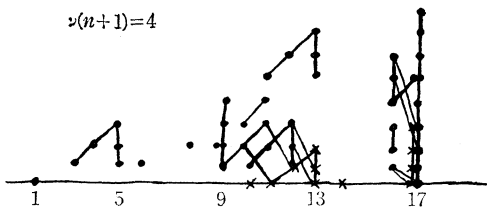
Proof of Theorem 1.3(i) This follows the same outline as the previous proof with a few modifications due to the fact that $2^{\phi(9)-1} < 2 \cdot 9$. This time

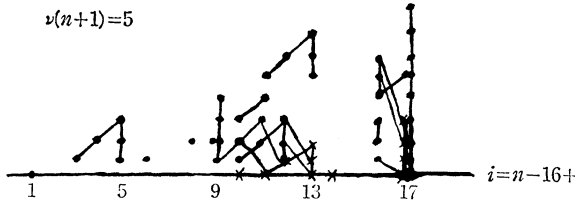
$$Sq^i(\bar{k}^*w_m) = \binom{m-1}{i} \bar{k}^*w_{m+i} + \begin{cases} \bar{k}^*w_{m+i-16} \cup k^*w_{16} & \text{if } i = 16 \text{ or } m \text{ is even and } i = 17 \\ 0 & \text{otherwise.} \end{cases}$$

Through degree 17, $H^*(BO[9]) \approx \mathbb{Z}_2 \oplus \Sigma^9 \mathcal{A}/\mathcal{A}(Sq^2)$. Let u_9 denote the nonzero element of $H^9(BO[9])$. Then $k^*w_{16} = (Sq^7 + Sq^4Sq^2Sq^1)u_9$ because $(Sq^7 + Sq^4Sq^2Sq^1)u_9$ is the only nonzero element of $H^{16}(BO[9])$ annihilated by Sq^1 and Sq^2 , which is true of k^*w_{16} . The homomorphism $\psi: \tilde{H}^*(\Sigma P_N) \otimes \mathcal{A}(g_0, g_9)/I \rightarrow \tilde{H}^*(BO[9]/BO_N[9])$ defined by $\psi(s\alpha^m \otimes g_0) = \bar{k}^*w_{m+1}$, $\psi(s\alpha^m \otimes Sq^1g_9) = \bar{k}^*w_{m+1} \cup Sq^1u_9$ is easily seen to be an \mathcal{A} -module isomorphism in the desired range. For example, $\psi(Sq^{16}(s\alpha^m \otimes g_0)) = \psi\left(\binom{m}{16}s\alpha^{m+16} \otimes g_0 + s\alpha^m \otimes (Sq^7 + Sq^4Sq^2Sq^1)g_9\right) = \binom{m}{16}\bar{k}^*w_{m+17} + \bar{k}^*w_{m+1} \cup (Sq^7 + Sq^4Sq^2Sq^1)u_9 = Sq^{16}(\bar{k}^*w_{m+1}) = Sq^{16}(\psi(s\alpha^m \otimes g_0))$.

Theorem 1.3 enables one to compute $\text{Ext}_{\mathcal{A}}(\tilde{H}^*(BO[j]/BO_N[j]), \mathbb{Z}_2)$, the E_2 -term of the ASS converging to the stable homotopy groups of $BO[j]/BO_N[j]$. We exemplify with the case which will be used in proving Theorem 1.2. Other cases are treated similarly. Ext groups are graphed as in [2]–[8], with vertical lines indicating multiplication by $h_0 \in \text{Ext}_{\mathcal{A}}^{1,1}(\mathbb{Z}_2, \mathbb{Z}_2)$ which corresponds (up to elements of higher filtration) to multiplication by 2 in homotopy groups, and diagonal (/) lines indicating multiplication by $h_1 \in \text{Ext}_{\mathcal{A}}^{1,2}(\mathbb{Z}_2, \mathbb{Z}_2)$, which corresponds to $\eta \in \pi_{n+1}(S^n)$. Differentials in the ASS are indicated by diagonal (\) lines.

PROPOSITION 2.1. *Suppose $n \equiv 7(8)$. The ASS chart for $BO[9]/BO_{n-16}[9]$ is given by*





with some differentials omitted in the top degree.

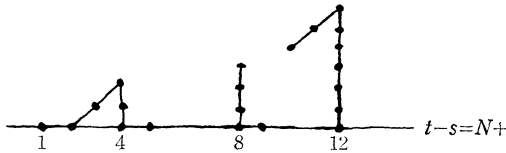
Proof. Let $N = n - 16$. By 1.3 (i) there is a short exact sequence of \mathcal{A} -modules

$$0 \rightarrow \tilde{H}^*(\Sigma P_N) \otimes \Sigma^9 \mathcal{A} / \mathcal{A}(Sq^2) \rightarrow H^*(BO[9], BO_N[9]) \rightarrow \tilde{H}^*(\Sigma P_N) \rightarrow 0 ,$$

inducing a long exact sequence in $\text{Ext}_{\mathcal{A}}(, \mathbf{Z}_2)$. $\text{Ext}_{\mathcal{A}}(\tilde{H}^*(\Sigma P_N), \mathbf{Z}_2)$ is given in [7; 8.16]. Let \mathcal{A}_1 denote the subalgebra of \mathcal{A} generated by Sq^1 and Sq^2 . By the method of [1; § 6] $\mathcal{A} / \mathcal{A}(Sq^2) \approx \mathcal{A} // \mathcal{A}_1 \otimes M$, where M is the \mathcal{A}_1 -module with nonzero element $Sq^0, Sq^1, Sq^2 Sq^1$. By the change-of-rings theorem ([2; 3.1])

$$\text{Ext}_{\mathcal{A}}(\tilde{H}^*(\Sigma P_N) \otimes \mathcal{A} // \mathcal{A}_1 \otimes M, \mathbf{Z}_2) \approx \text{Ext}_{\mathcal{A}_1}(\tilde{H}^*(\Sigma P_N) \otimes M, \mathbf{Z}_2) .$$

This is computed as in [2; Ch. 3] or [8, Ch. 4] to begin



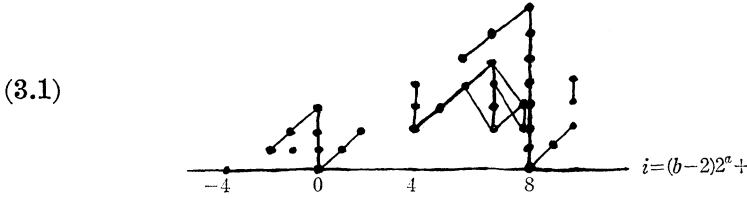
There is a nonzero boundary homomorphism in the Ext-sequence, which we picture as a d_1 -differential in the ASS. The d_2 -differentials are deduced by applying $\pi_*()$ to the diagram

$$\begin{array}{ccc} \Sigma^{-1}(BO[9]/BO_{n-16}[9]/\Sigma P_{n-16}) & & \\ \downarrow & \searrow & \\ \Sigma^{-1}(BO[8]/BO_{n-16}[8]/\Sigma P_{n-16}) & \nearrow & \Sigma P_{n-16} \end{array}$$

using the results of [4; Ch. 3]. The higher differentials in the top degree which are present in ΣP_{n-16} are deduced by considering the map of ASS induced by $P_{n-16} \rightarrow P_{n-16} \wedge bJ$ (see [5]). We have also used $d_2(h_0(\iota_{16}1)) = 0$ in ASS (P_{n-16}) , which is proved by going back to P_{n-24} . When $\nu(n+1) = 6$, the element in $s = 5, t - s = n$ may not be killed, which is the reason no immersion result is stated in this case.

3. Proof of nonimmersions (Theorem 1.1). This proof is very similar to that in [4; Ch.5]. Let $f: P^{n-2\alpha(n)-3+11} \rightarrow BO[8]$ classify the restriction of the stable normal bundle $(2^L - n - 1)\xi$ of P^n . We will prove the composite $P^{n-2\alpha(n)-3+11} \xrightarrow{f} BO[8] \xrightarrow{k} BO[8]/BO_{n-2\alpha(n)-3-4}[8]$ is essential.

Let $a = \alpha(n) - 4$ and $b = (n + 1)/2^a$. Then $\alpha(b - 1) = 4$ and $a \geq 3$. From [4; 3.2] the ASS chart for $\pi_i(BO[8]/BO_{(b-2)2^a-5}[8])$ is



In order to use the main result of [3], we “factor” f through QP by going into $BSpin$ as in [4; 4.1]. Thus we have

$$(3.2) \quad \begin{array}{ccc} BO[8] & \longrightarrow & BSpin \longleftarrow BSp \\ f \uparrow & & g \uparrow \\ P^{(b-2)2^a+10} & \longrightarrow & QP^{(b-2)2^{a-2}+2} \end{array}$$

where g classifies $(2^{L-2} - b2^{a-2})H$.

LEMMA 3.3. *If $\alpha(b - 1) = 4$ and $a \geq 3$, then $\nu\left(\frac{2^{L-2} - b2^{a-2}}{(b-2)2^{a-2} + \Delta}\right) = 3$ if $\Delta = 0$ and is greater than 3 if $\Delta = -1, 1$, or 2.*

Proof. By [3; 4.1]

$$\begin{aligned} \nu\left(\frac{2^{L-2} - b2^{a-2}}{(b-2)2^{a-2}}\right) &= \alpha((b-2)2^{a-2}) + \alpha(2^{L-2} - 2^{a-2}(2b-2)) - \alpha(2^{L-2} - b2^{a-2}) \\ &= \alpha(b-2) + L - a - \alpha(2b-3) - (L - a - \alpha(b-1)) \\ &= \alpha(b-2) - (\alpha(2b-4) + 1) + \alpha(b-1) = -1 + 4 = 3. \end{aligned}$$

The case $\Delta \neq 0$ is handled by similar techniques.

Let $8l = (b-2)2^a$ and $Q = BO[8]/BO_{8l-5}[8]$. Let $Q\langle 3 \rangle$ be the space formed from Q by killing Ext^s for $s < 3$. ([4; 2.1]). $Q\langle 3 \rangle$ has cohomology generators $k_{8l}, k_{8l+4}, k_{8l+5}, k_{8l+7}, k_{8l+8}, k'_{8l+8}$, and k_{8l+10} corresponding to the elements of (3.1) of filtration 3.

LEMMA 3.4. *kf lifts to a map $P^{8l+10} \rightarrow Q\langle 3 \rangle$ which sends only k_{8l} nontrivially.*

Proof. By the method of [4; Ch. 2 and 4.2], which is based upon [3; 1.8], and using Lemma 3.3 and the fact that $\pi_{4i}^s(\Sigma P_{8l-5}) \rightarrow \pi_{4i}^s(\Sigma P_{8l-5} \wedge bo)$ is injective for $i \leq 2l + 2$ (by [7; 8.4 and 8.12]), there exists a lifting of g to E_3 in the modified Postnikov tower (MPT) of the fibration $\widetilde{BSp}_{8l-5} \rightarrow BSp$ which sends only the $8l$ -dimensional k -invariant nontrivially. Thus, since (3.2) effectively gives a factorization through QP , as in the proof of [4; 5.1] f lifts to a map $\tilde{f}: P^{8l+10} \rightarrow E_3''$, where E_3'' is the third stage of the MPT of $BO_{8l-5}[8] \rightarrow BO[8]$, sending only the $8l$ -dimensional k -invariant \bar{k}_{8l} nontrivially. There is a map $E_3'' \xrightarrow{j} Q\langle 3 \rangle$ and its behaviour on k -invariants is computed by computing the induced morphism of minimal resolutions. In particular, for the k -invariant $k'_{8l+8} \in H^{8l+8}(Q\langle 3 \rangle)$ corresponding to the element in coker $(\text{Ext}_{\mathcal{A}}^{3,8l+11}(\tilde{H}^*(\Sigma P_{8l-5}), \mathbb{Z}_2) \rightarrow \text{Ext}_{\mathcal{A}}^{3,8l+11}(\tilde{H}^*(Q), \mathbb{Z}_2))$, if l is odd, $j^*(k'_{8l+8}) = (Sq^8 + w_8)\bar{k}_{8l}$ so that $\tilde{f}^*j^*k'_{8l+8} = Sq^8\alpha^{8l} + w_8((2^L - 8l - 2^{a+1})\varepsilon) \cup \alpha^{8l} = \alpha^{8l+8} + \alpha^{8l+8} = 0$, while if l is even, $\mathcal{A}(BO[8])$ annihilates \bar{k}_{8l} in the range under consideration.

Theorem 1.1 follows from Lemma 3.4 together with

PROPOSITION 3.5. *If $P^{8l+10} \rightarrow Q\langle 3 \rangle$ sends only k_{8l} nontrivially, then $P^{8l+10} \rightarrow Q\langle 3 \rangle \rightarrow Q$ is essential.*

Proof. We show that such a map represents a nontrivial class of $\text{Ext}_{\mathcal{A}}^{3,3}(\tilde{H}^*(Q), \tilde{H}^*(P^{8l+10}))$ which is not hit by a differential in the ASS converging to $[P^{8l+10}, Q]$.

A minimal \mathcal{A} -resolution of $\tilde{H}^*(Q)$ corresponds to a minimal \mathcal{A}_2 -resolution of $\tilde{H}^*(\Sigma P_{8l-5})$. (See [4; 3.1].) This is listed in the Appendix. $\text{Ext}_{\mathcal{A}}^{3,3}(\tilde{H}^*(Q), \tilde{H}^*(P^{8l+10})) \approx \ker d_3^*/\text{im } d_2^*$ in

$$\text{Hom}(\Sigma^{-3}C_2, \tilde{H}^*P^{8l+10}) \xrightarrow{d_2^*} \text{Hom}(\Sigma^{-3}C_3, \tilde{H}^*P^{8l+10}) \xrightarrow{d_3^*} \text{Hom}(\Sigma^{-3}C_4, \tilde{H}^*P^{8l+10}).$$

If we denote by \hat{k}_i the \mathcal{A}_2 -homomorphism $\Sigma^{-3}C_3 \rightarrow \tilde{H}^*(P^{8l+10})$ sending only $\Sigma^{-3}k_i$ nontrivially, then $d_3^*(\hat{k}_0) = 0$ because, for example $d_3^*\hat{k}_0(\Sigma^{-3}l_4) = \hat{k}_0 d_3(\Sigma^{-3}l_4) = \hat{k}_0(\Sigma^{-3}(Sq^1k_4 + Sq^2Sq^3k_0)) = Sq^2Sq^3\alpha^{8l} = 0$. The image of d_2^* is generated by $\hat{k}_8 = d_2^*(\hat{h}_8)$, $\hat{k}'_8 + \hat{k}_{10} = d_2^*(\hat{h}_7)$, $\hat{k}_0 + \hat{k}'_8 = d_2^*(\hat{h}_{-1})$ and $\hat{k}_4 + \hat{k}_5 = d_2^*(\hat{h}_4)$. Thus \hat{k}_0 gives a nonzero element of Ext .

Similarly $\text{Ext}_{\mathcal{A}}^{j,j+1}(\tilde{H}^*Q, \tilde{H}^*P^{8l+10}) = \mathbb{Z}_2$ for $j = 0$ and 1 . The non-zero elements in these groups survive to give the nontrivial elements $2^j[f_0]$, where f_0 is the map defined after (4.6). Thus there are no elements which could support a differential hitting \hat{k}_0 .

4. Proof of immersions (Theorem 1.2). The proof is very similar to that of [4; 1.1]. We let $g: P^n \rightarrow BO[9]$ classify the stable normal

bundle and let \mathcal{E} denote the fibre of $k_0: BO[9] \rightarrow C$, where $C = BO[9]/BO_{n-16}[9]$. We consider the diagram

$$\begin{array}{ccccc} & & BO_{n-16}[9] & & \\ & & \downarrow & & \\ \Omega C \times \mathcal{E} & \xrightarrow{\mu} & \mathcal{E} & \xrightarrow{k_1} & \mathcal{E}/BO_{n-16}[9] \\ & & \downarrow & & \\ P^n & \xrightarrow{g} & BO[9] & \xrightarrow{k_0} & C \end{array}$$

As in [4; 1.4(c)] the fibre of k_1 has the same n -type as $BO_{n-16}[9]$. (This is the main reason for using $BO[9]$ instead of $BO[8]$.) It suffices to prove

(4.1) $k_0 g$ is null-homotopic, so that there is a lifting l of g , and

(4.2) there is a map $P^n \rightarrow \Omega C$ such that $k_1 \mu(f \times l)$ is null-homotopic.

Proof of 4.1. We use the charts of $\pi_*(C)$ given in 2.1.

Similarly to [4; 4.1] one shows that $k_0 g$ has filtration ≥ 5 . This is accomplished by noting that if $n = 16l + 15$, then $\nu \binom{2^l - 16l - 16}{4l + \varepsilon} = \begin{cases} 5 & \varepsilon = 1 \text{ or } 3 \\ 4 & \varepsilon = 2 \end{cases}$, so that QP^{4l+3} lifts to E_4 sending only k_{16l+8} nontrivially. (QP gets past the irregular element in $s = 2, t - s = 16l + 8$ as in [4; 4.2]. It gets by the $x'd$ tower in $t - s = 16l + 12$ because they are not present in the MPT for $\widetilde{BSp}_{n-16} \rightarrow BSp$, where the liftings are first performed. (See [4; 4.1, 4.2].) Thus P^n lifts to E_6 since primary indeterminacy enables one to vary k_{16l+8} without varying the other k^4 -invariants.

Finally we show that any filtration 5 map $P^n \rightarrow C$ is null-homotopic. The only possible map not trivial by Ext-considerations (or by the differentials in the top degree) is an extension over P^n of the map $P^{n-3} \xrightarrow{k} S^{n-3} \xrightarrow{f} C$, where k is the collapse and $[f]$ the filtration 5 generator. But $[f]$ is divisible by 2 by [4; 3.5] since $BO[9]/BO_{n-16}[9] \rightarrow BO[8]/BO_{n-16}[8]$ sends the filtration 2 class in π_{n-3} to the filtration 3 class. Thus $P^{n-3} \rightarrow S^{n-3} \xrightarrow{f} C$ is trivial and hence so is fk . But there is a unique filtration 5 extension of fk over P^n since for $i = 0, 1, 2$, $\pi_{n-i}(C)$ has no elements of filtration ≥ 5 . Hence the extension over P^n is trivial.

Proof of 4.2. This is very similar to [4; Ch. 4 beginning with 4.3]. $[k_1 \mu(f \times l)]$ is considered as the homotopy sum of three stable maps

$$(4.4) \quad P^n \xrightarrow{l} \mathcal{E} \longrightarrow \mathcal{E}/BO_{n-16}[9]$$

$$(4.5) \quad P^n \xrightarrow{f} \Omega C \longrightarrow \mathcal{E}/BO_{n-16}[9]$$

$$(4.6) \quad P^n \xrightarrow{f \wedge l} \Omega C \wedge \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/BO_{n-16}[9].$$

Let f_0 denote the composite $P^n \rightarrow V_{n-16} \rightarrow \Omega \Sigma V_{n-16} \rightarrow \Omega C$ and f_1 the composite $P^n \rightarrow S^n \xrightarrow{u} \Omega C$ where $[u]$ has smallest possible filtration (≤ 2). As in [4] we have the following results.

PROPOSITION 4.7. (4.5) with $f = f_0$ and (4.6) with $f = f_1$ are null-homotopic.

PROPOSITION 4.8. $[P^n, \mathcal{E}/BO_{n-16}[9]] \approx \mathbf{Z}_2 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_{16}$.

Proof. As in [4; 4.3] $H^*(\mathcal{E}, BO_{n-16}[9]) \approx \tilde{H}^*(\Sigma^9 P_{n-16}) \otimes \mathcal{A}/I(Sq^2)$, so that its homotopy groups are as in (2.2, $j \equiv 1$) reindexed. $[P^n, \mathcal{E}/BO_{n-16}[9]]$ is computed as in [4; 4.9] or by computing $\text{Ext}_{\mathcal{A}}^{*,*}(H^*(\mathcal{E}, BO_{n-16}[9]), P^n)$ as in 3.5. The \mathbf{Z}_2 's are due to the \mathbf{Z}_2 homotopy groups in degrees $n-3$ and $n-7$.

Let G_1, G_2, G_3 , and G_4 denote generators under a splitting of Proposition 4.8.

PROPOSITION 4.9. Some multiple of (4.5) with $f = f_1$ equals $4G_2$.

PROPOSITION 4.10. If $\nu(n+1) = 4$, then (4.6) with $f = f_0$ has odd coefficient of G_1 . If $\nu(n+1) = 5$, then (4.6) with $f = f_0$ is $2aG_1 + 2bG_2$ where a is an odd integer and b is an integer.

Proof. This can be seen by using the map $\mathcal{E}/BO_{n-16}[9] \rightarrow \overline{\mathcal{E}}/BO_{n-16}[8]$, where $\overline{\mathcal{E}} = \text{fibre}(BO[8] \rightarrow BO[8]/BO_{n-16}[8])$, and [4; 4.11, 4.12].

To deduce 1.2, we use the fact ([4; 1.2]) that P^n immerses in R^{2n-14} and argue as in the last paragraphs of [4; Ch. 4]. We consider the diagram

$$\begin{array}{ccccc} \Omega(BO[9]/BO_{n-16}[9]) \times \mathcal{E} & \xrightarrow{\mu} & \mathcal{E} & \xrightarrow{k_1} & \mathcal{E}/BO_{n-16}[9] \\ \downarrow j_1 \times j_2 & & \downarrow j_2 & & \downarrow j_3 \\ \Omega(BO[9]/BO_{n-14}[9]) \times \mathcal{E}' & \xrightarrow{\mu'} & \mathcal{E}' & \xrightarrow{k'_1} & \mathcal{E}'/BO_{n-14}[9] \end{array}$$

$[P^n, j_3]$ is onto with kernel $\langle 4G_2, 8G_4 \rangle$. $[P^n, j_1]$ is onto all elements except a filtration 1 map f_2 trivial on P^{n-2} . The analogue of (4.6)

with $f = f_2$ is null-homotopic and the analogue of (4.5) with $f = f_2$ is 0 or $4G'_2$.

Let $l: P^n \rightarrow \mathcal{E}$ be some lifting of g . There exists $f': P^n \rightarrow \Omega(BO[9]/BO_{n-14}[9])$ such that $k'_1\mu'(f' \times j_2l) = 0$. Either f' or $f' - f_2$ factors as $P^n \xrightarrow{f} \Omega C \xrightarrow{j_1} \Omega(BO[9]/BO_{n-14}[9])$. Then $k'_1\mu'(j_1 \times j_2)(f \times l) = 0$ or $4G'_2$. Hence $k_1\mu(f \times l)$ is $4aG_1 + 4bG_2$ for some $a \in \mathbb{Z}_2, b \in \mathbb{Z}_4$. If $a=0$, by 4.7 and 4.9 there is some multiple df_1 such that $k_1\mu((f - df_1) \times l) = 0$. If $a=1$, there exists d such that $k_1\mu((f + 2^{8-\nu(n+1)}f_0 - df_1) \times l) = 0$.

5. Appendix. A minimal \mathcal{A}_2 -resolution through degree $8l + 10$ of $\tilde{H}^*(SP_{8l-5})$ is given by $C_0 \xleftarrow{d_0} C_1 \xleftarrow{d_1} C_2 \xleftarrow{d_2} C_3 \xleftarrow{d_3} C_4 \leftarrow$, where C_s is a free \mathcal{A}_2 -module generated by elements x_i, g_i, h_i, k_i , or l_i for $s=0, 1, 2, 3$, or 4 with subscripts indicating $t - s - 8l$, where t is the degree of the generator. (See [4; Ch. 6].) We omit Sq for Steenrod squares; thus, $62g$ denotes Sq^6Sq^2g . This resolution corresponds to (3.1).

C_0 has generators x_{-4}, x_0 , and x_8

$$g_{-2}: 21x_{-4} \quad (\text{This means } d_0(g_{-2}) = Sq^2Sq^4x_{-4})$$

$$g_{-1}: 4x_{-4}$$

$$g_0: 1x_0 + 41x_{-4}$$

$$g_1: 2x_0 + 42x_{-4}$$

$$g_8: 1x_8 + 27x_0$$

$$g_9: 2x_8 + 424x_0$$

$$h_{-1}: 2g_{-2}$$

$$h_0: 1g_0 + 21g_{-2}$$

$$h_2: 2g_1 + 3g_0 + 4g_{-1}$$

$$h_4: 51g_{-1} + (7 + 421)g_{-2}$$

$$h_7: 621g_{-1} + (91 + 46)g_{-2}$$

$$h_8: 1g_8 + 521g_1 + 54g_0 + 46g_{-1}$$

$$h'_8: (46 + 73 + 631)g_{-1} + 461g_{-2}$$

$$h_{10}: 2g_9 + 3g_8 + (46 + 91)g_1 + 47g_0$$

$$k_0: 1h_0 + 2h_{-1}$$

$$k_4: 1h_4 + 41h_0$$

$$k_8: 2h_4 + (7 + 421)h_{-1}$$

$$k_7: 1h_7 + 4h_4 + 51h_2 + 72h_{-1}$$

$$k_8: 1h_8 + 43h_2 + (27 + 72)h_0 + 631h_{-1}$$

$$k'_8: 1h'_8 + 2h_7 + 43h_2 + (27 + 72)h_0 + 46h_{-1}$$

$$k_{10}: 3h'_8 + 4h_7 + 423h_2 + (66 + 75)h_{-1}$$

$$l_4: 1k_4 + 23k_0$$

$$l_6: 2k_6 + 3k_4 + (7 + 421)k_0$$

$$l_7: 1k_7 + 21k_5 + 4k_4 + 62k_0$$

$$l_8: 1k_8 + 72k_0 + 41k_4$$

$$l_{10}: 1k_{10} + 21k'_8 + 4k_7 + (6 + 51)k_5 + 7k_4$$

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