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REALIZING PARTIAL ORDERINGS BY CLASSES OF CO-SIMPLE SETS

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We show that we can embed any countable partial ordering into a class of co-r.e. bi-dense subsets of the rationals, each subset of a fixed nonzero r.e. Turing degree, under an order induced by recursive similarity transformations. Also, we show that we can embed any countable partial ordering into the co-simple isols under either the order induced by addition of isols or the order induced by recursive injections.

O. Introduction. Let C denote the continuum, Q denote the rationals, and N denote the natural numbers. We let c denote the cardinality of C and \aleph_0 denote the cardinality of N . Given two linear orderings H and G , we say (i) H is *embeddable* in G , $H < G$, if there is an order preserving map from H into G and (ii) H is *similar* to G if there is an order preserving map from H onto G . H is said to be *bi-dense* in G if $H \subseteq G$ and both H and $G - H$ are dense in G .

Let π be an effective one-one correspondence between Q and the natural numbers. We shall consider π to be an effective Gödel numbering and thus we will identify an element or subset of Q with its image under π . We let \leq or $<$ refer to the usual ordering on N and \subseteq or \subset refer to the usual ordering on Q . Given $\alpha, \beta \subseteq Q$, we say α is *recursively embeddable* in β , $\alpha <_e \beta$, if there is a partial recursive function φ such that $\alpha \subseteq \delta\varphi$, the domain of φ , and the restriction of φ to α , $\varphi \upharpoonright \alpha$, is an order preserving map from α into β .

In [5], Hay, Manaster, and Rosenstein show that complements of recursively enumerable bi-dense subsets of Q of any fixed nonzero r.e. degree under $<_e$ bear a strong resemblance to bi-dense subsets of C of cardinality c under $<$. The main result of this paper answers a question raised by Laver. Based on the results of [5], Laver asked whether or not the following theorem is true.

THEOREM A. *Let β be any recursively enumerable set which is not recursive and let P be any countable partial ordering. Then there is a collection of co-recursively enumerable bi-dense subsets of Q , each Turing equivalent to β , such that, under $<_e$, this collection is order isomorphic to P .*

(A set $A \subseteq N$ is co-recursively enumerable if $N - A$ is recursively enumerable.) In §2 of this paper, we prove Theorem A using methods that Sack's [8] developed to prove that any countable partial ordering

can be embedded in the r.e. Turing degrees under the order induced by Turing reducibility. Theorem A extends Theorems 7 and 8 of [5], where Hay, Manaster, and Rosenstein proved the analogues of Theorem A if the countable partial ordering P in the statement of Theorem A is replaced either by any countable linear ordering or by any finite partial ordering.

The proof of Theorem A will also give a result on the class of co-r.e. isols which have been studied by Hay [3], [4], Ellentuck [2], and others. We will show that one can embed any countable partial ordering P into the class of co-simple isols under either the order induced by addition of isols (due to Ellentuck [2]) or the order induced by recursive injections. (See §1 for the definitions of the co-simple isols and the two orderings.)

1. Preliminaries. Given $B \subseteq N$, we write \bar{B} for the complement of B in N . We write $A \leq_T B$ if A is Turing reducible to B and $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$. Let $\varphi_0, \varphi_1, \dots$ be an effective list of all partial recursive functions where φ_n is the function computed by the n th Turing machine. We write $\varphi_n^s(x) \downarrow$ if the n th Turing machine started on x gives an output in s or less steps. We let I_0, I_1, \dots be an effective list of all intervals of Q of the form $[p, q] = \{x \in Q \mid p \subseteq X \subseteq q\}$ for $p, q \in Q$.

Given a partial ordering P , we say P is an \aleph_0 -universal partial ordering if any countable partial ordering can be embedded in P , that is, if $S < P$ for all countable partial orderings S . The rest of this section will be devoted to defining three partial orderings. The fact that each of the three partial orderings is \aleph_0 -universal will follow easily from the main construction of §2.

Given $\alpha, \beta \subseteq Q$, we define $\alpha \sim_c \beta$ iff $\alpha <_c \beta$ and $\beta <_c \alpha$. It is clear that \sim_c is an equivalence relation. Let \mathbf{a} be any nonzero r.e. Turing degree. We let $B(\mathbf{a}, Q) = \{\alpha : \alpha \text{ is a co-r.e. bi-dense subset of } Q \text{ of degree } \mathbf{a}\}$ and $\bar{B}(\mathbf{a}, Q) = B(\mathbf{a}, Q) / \sim_c$. Given equivalence classes, $[\alpha], [\beta] \in \bar{B}(\mathbf{a}, Q)$, we define $[\alpha] \leq_c [\beta]$ iff there exists $\alpha \in [\alpha]$ and $\beta \in [\beta]$ such that $\alpha <_c \beta$. It is easy to check that \leq_c is a well defined partial order on $\bar{B}(\mathbf{a}, Q)$. Thus, Theorem A is equivalent to saying that $\langle \bar{B}(\mathbf{a}, Q), \leq_c \rangle$ is an \aleph_0 -universal partial ordering for any nonzero r.e. degree \mathbf{a} .

Given $\alpha, \beta \subseteq N$, we say α is *recursively equivalent* to β if there is a 1-1 partial recursive function p such that $\alpha \subseteq \delta p$ and $p \upharpoonright \alpha$ maps α onto β . The recursive equivalence type or RET of α , denoted by $\langle \alpha \rangle$, is the class of all β recursively equivalent to α . A set $\alpha \subseteq N$ is *immune* if α is infinite and α has no infinite r.e. subset. A r.e. set $\beta \subseteq N$ is *simple* if $\bar{\beta}$ is immune. A set $\alpha \subseteq N$ is *isolated* if α is either finite or immune. The RETs of isolated sets are called

isols and their collection is denoted by \mathcal{A} . The elements of \mathcal{A} can be considered as an “effective” analogue of the Dedekind finite cardinals and have been extensively studied by Dekker, Manaster, Myhill, Nerode, and others. Isols $\langle \alpha \rangle$ of sets α such that α is co-r.e. are called *co-simple isols* and their collection is denoted by \mathcal{A}_z . We shall define two distinct partial orders on \mathcal{A}_z . Addition of RETs is defined by $\langle \alpha \rangle + \langle \beta \rangle = \langle \{2x \mid x \in \alpha\} \cup \{2x + 1 \mid x \in \beta\} \rangle$. The partial ordering \leq_i is defined on the RETs by $A \leq_i B$ iff $\exists C(A + C = B)$. Given sets $\alpha, \beta \subseteq N$, we define $\alpha <_i \beta$ iff $\alpha \subseteq \beta$ and there are disjoint r.e. sets W_1 and W_2 such that $W_1 \cap \beta = \alpha$ and $W_2 \cap \beta = \beta - \alpha$. It is proved in [1], that for RETs $\langle \alpha \rangle$ and $\langle \beta \rangle$, $\langle \alpha \rangle \leq_i \langle \beta \rangle$ iff there exists $\alpha' \in \langle \alpha \rangle$ and $\beta' \in \langle \beta \rangle$ such that $\alpha' <_i \beta'$. Given sets $\alpha, \beta \subseteq N$, we define $\alpha <_e \beta$ iff there is a partial recursive function p such that $\alpha \subseteq \delta p$ and $p \upharpoonright \alpha$ is a $1 - 1$ map from α into β . Given RETs $\langle \alpha \rangle$ and $\langle \beta \rangle$, we define $\langle \alpha \rangle \leq_e \langle \beta \rangle$ iff there exists $\alpha' \in \langle \alpha \rangle$ and $\beta' \in \langle \beta \rangle$ such that $\alpha' <_e \beta'$. It is easy to check that \leq_e is a well defined partial order on the class of RETs.

In §2, we shall prove that $\langle \bar{B}(\alpha, Q), \leq_e \rangle$, $\langle \mathcal{A}_z, \leq_i \rangle$, and $\langle \mathcal{A}_z, \leq_e \rangle$ are all \aleph_0 -universal partial orderings. We shall discuss the differences between $<_e$, $<_i$, and $<_e$ on the class of co-r.e. sets and the differences between \leq_i and \leq_e on \mathcal{A}_z in §3.

2. The main construction. In [5], Hay, Manaster, and Rosenstein constructed a set $\alpha \subseteq Q$ with the following property.

(\mathcal{P}) If φ is a partial recursive function such that $\alpha \subseteq \delta \varphi$ and $\varphi \upharpoonright \alpha$ is a $1 - 1$ map from α into α , then $\{a \in \alpha \mid \varphi(a) \neq a\}$ is finite. If α has property \mathcal{P} then α is isolated. For if α contains an infinite r.e. set, then α contains an infinite recursive set $R = \{a_0 < a_1 < a_2 < \dots\}$. Let φ be the recursive function defined by

$$\varphi(x) = \begin{cases} a_{i+1} & \text{if } x = a_i \text{ and } i \text{ is even} \\ a_{i-1} & \text{if } x = a_i \text{ and } i \text{ is odd} \\ x & \text{otherwise.} \end{cases}$$

$\varphi \upharpoonright \alpha$ thus would be a $1 - 1$ map from α into α such that $R = \{a \in \alpha \mid a \neq \varphi(a)\}$ contradicting property \mathcal{P} . If α is isolated, then α has the property that for no proper subset β of α is $\alpha <_e \beta$. For if $\beta \subset \alpha$ and $\alpha <_e \beta$, then let φ be the partial recursive function such that $\alpha \subseteq \delta \varphi$ and $\varphi \upharpoonright \alpha$ is an order isomorphism from α into β . Let $x \in \alpha - \beta$. Thus either $x \otimes \varphi(x)$ or $\varphi(x) \otimes x$. If $x \otimes \varphi(x)$, then $\{x \otimes \varphi(x) \otimes \varphi(\varphi(x)) \otimes \varphi(\varphi(\varphi(x))) \otimes \dots\}$ is an infinite r.e. subset of α and if $\varphi(x) \otimes x$, then $\{x \otimes \varphi(x) \otimes \varphi(\varphi(x)) \otimes \dots\}$ is an infinite r.e. subset of α contradicting the fact that α is isolated. All sets α we construct in this section will have property \mathcal{P} so that we will always have $\langle \alpha \rangle \in \mathcal{A}$.

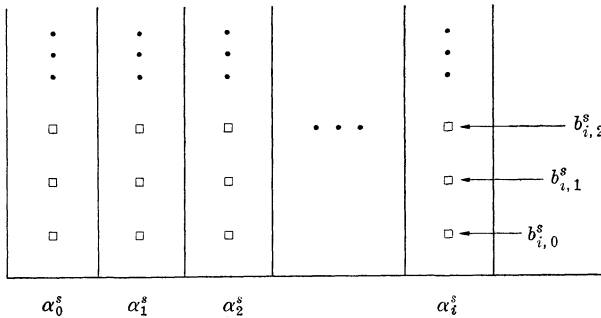
The proof that $\langle \bar{B}(a, Q), \leq_c \rangle$, $\langle A_z, \leq_i \rangle$, and $\langle A_z, \leq_e \rangle$ are \aleph_0 -universal ordering will proceed in two steps in the same manner as Sack's proof [8] of the fact that the r.e. degrees under Turing reducibility is an \aleph_0 -universal partial ordering. The first step is to construct an infinite sequence of 'incomparable' elements.

THEOREM 1. *Let β be a nonrecursive r.e. set. There is a recursive sequence of co-r.e. subsets of Q , $\alpha_0, \alpha_1, \dots$, such that*

- (a) *For each i , α_i is bi-dense in Q ,*
- (b) *For each recursive set $R \subseteq N$, $\bigcup_{i \in R} \alpha_i$ has property \mathcal{P} ,*
- (c) *For each i , $\alpha_i \cap \bigcup_{i \neq j} \alpha_j = \emptyset$ and moreover $\alpha_i \not\leq_i \bigcup_{i \neq j} \alpha_j$, and*
- (d) *For each i , $\alpha_i \equiv_T \beta$.*

Proof. Let f be a 1-1 recursive function whose range is β and let $\beta^s = \{y | \exists x(x \leq s \text{ \& } f(x) = y)\}$. Let $k: N \times N \rightarrow N$ be a 1-1, onto, recursive function. Let r and c be recursive functions such that $k(i, j) = n$ iff $c(n) = i$ and $r(n) = j$. Moreover, we assume k is chosen so that for each i , $N_i = \{y | \exists x(k(i, x) = y)\}$ is a bi-dense recursive subset of Q . We shall give a procedure to enumerate a r.e. set A in stages such that if $\alpha_i = \bar{A} \cap N_i$, then $\alpha_0, \alpha_1, \dots$ is the recursive sequence of sets required by the theorem. Each α_i is co-r.e. since $\bar{\alpha}_i = (A \cap N_i) \cup \bigcup_{i \neq j} N_j$ and clearly the sets $\alpha_0, \alpha_1, \dots$ are pairwise disjoint.

A convenient picture for the construction of A will be to imagine an infinite sequence of infinite columns of windows



At the end of stage s , the windows in the i th column will be occupied consecutively from the bottom up by $b_{i,0}^s < b_{i,1}^s < \dots$ where

$$\{b_{i,0}^s, b_{i,1}^s, \dots\} = N_i \cap \bar{A}^s = \alpha_i^s$$

and A^s is the set of elements enumerated into A by the end of stage s . Thus the windows give us a picture of the complement of A^s at the end of stage s . Then during stage $s+1$, certain elements from

the columns will be put into A^{s+1} and the elements left in each column will drop down to fill in any vacant windows. We shall ensure that for each stage $s > 0$, $A^s \cap N_i$ will be finite so that α_i^s will be infinite and every window will be occupied. For $s > 0$, A^s will always be an infinite recursive set.

We will meet three sets of requirements in the course of the construction. To ensure that each α_i is bi-dense, we must meet the following, requirements.

$$D(i, n): \alpha_i \cap I_n \neq \emptyset.$$

We will employ a set of markers $\Delta(i, n)$. At stage s , $\Delta(i, n)$ will rest on an $x \in \alpha_i^s \cap I_n$. Then for the sake of requirement $D(i, n)$ we will try to keep the element marked by $\Delta(i, n)$ out of A . If we are successful for all i and n , then each α_i will be dense in Q and hence each α_i will be bi-dense in Q since $\bar{\alpha}_i \supseteq \bigcup_{i \neq j} N_j$.

To ensure that condition (b) is satisfied by the α_i 's, we will meet the following set of requirements.

$Q(n)$: $\varphi_n \upharpoonright \bar{A}$ is a 1-1 map from \bar{A} into \bar{A} only if $\{a \in \bar{A} \mid a \neq \varphi(a)\}$ is finite. Suppose there is a recursive set $R \subseteq N$ and a partial recursive function φ_e such that $\varphi_e \upharpoonright \bigcup_{i \in R} \alpha_i$ is a 1-1 map from $\bigcup_{i \in R} \alpha_i$ into $\bigcup_{i \in R} \alpha_i$ and $\{a \in \bigcup_{i \in R} \alpha_i \mid a \neq \varphi_e(a)\}$ is infinite. Let φ_n be the recursive function defined by

$$\varphi_n(x) = \begin{cases} \varphi_e(x) & \text{if } x \in \bigcup_{i \in R} N_i \text{ and } x \in \delta\varphi_e \\ x & \text{if } x \in \overline{\bigcup_{i \in R} N_i} = \bigcup_{i \in \bar{R}} N_i \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then φ_n would violate requirement $Q(n)$. Thus if we meet all the requirements $Q(n)$, condition (b) will automatically follow.

The strategy to meet requirement $Q(n)$ at stage $s+1$ will be to try to find an $x \in \bar{A}^s$ such that $\varphi_n^s(x) \downarrow$ and $\varphi_n(x) \neq x$ and then put $\varphi_n(x)$ into A^{s+1} , put a marker $\lambda(n)$ on x , and then try to keep x out of A . If $x \in \bar{A}$, then x will witness that $\varphi_n(\bar{A}) \not\subseteq \bar{A}$. However, there may be two reasons why we cannot put $\varphi_n(x)$ into A^{s+1} . The first reason is that $\varphi_n(x)$ may already have another marker on it which means we want to keep $\varphi_n(x)$ out of A for the sake of some other requirement. Thus, we must put a priority ranking on our list of requirements. We shall ensure that requirements with higher priority than $Q(n)$ restrict only finitely many elements from being put into A so that if $\varphi_n \upharpoonright \bar{A}$ is 1-1 and $\{a \in \bar{A} \mid a \neq \varphi_n(a)\}$ is really infinite, we will be able to find a pair $(x, \varphi_n(x))$ for which $\varphi_n(x)$ is never restricted by higher priority requirements. Then we will be able to put $\varphi_n(x)$ into A and keep x out of A . The second reason is that

to ensure each $\alpha_i \leq_T \beta$, we use a Yates permitting argument which puts some restrictions on which $b_{i,n}^s$ can be put into A^{s+1} . Thus it is also possible that $\varphi_n(x)$ is not 'permitted' to be put into A^{s+1} . In such a case, we shall place a $\lambda(n)$ marker on x and try to keep x out of A in the hope that sometime later we will be permitted to put $\varphi_n(x)$ into A . We say requirement $Q(n)$ is *satisfied* at stage s if there is an $x \in \bar{A}^s$ with a $\lambda(n)$ marker on it such that $\varphi_n^s(x) \downarrow$ and $\varphi_n^s(x) \in A^s$.

To ensure that each α_i has property (c), we must meet the following set of requirements.

$R(i, n)$: If $\alpha_i \subseteq \delta\varphi_n$ and $\varphi_n \upharpoonright \alpha_i$ is 1-1, then $\varphi_n(\alpha_i) \not\subseteq \bigcup_{i \neq j} \alpha_j$.

The requirements $R(i, n)$ have basically the same character as the requirements $Q(n)$. The strategy to meet requirement $R(i, n)$ at stage $s+1$ is to try to find an $x \in \alpha_i^s$ such that $\varphi_n^s(x) \downarrow$ and $x \neq \varphi_n^s(x)$ and either we can put $\varphi_n(x)$ into A^{s+1} or $\varphi_n(x) \in N_i$. Then we put $\varphi_n(x)$ into A^{s+1} , if possible, and place a $\Gamma(i, n)$ marker on x and try to keep x out of A . If $x \in \bar{A}$, then $x \in \alpha_i$ and x will witness that $\varphi_n(\alpha_i) \not\subseteq \bigcup_{i \neq j} \alpha_j$. Again the same type of restrictions as described above can restrict us from placing $\varphi_n(x)$ into A^{s+1} . We say that requirement $R(i, n)$ is *satisfied* at stage s if there is an $x \in \bar{A}^s$ with a $\Gamma(i, n)$ marker on it such that $\varphi_n^s(x) \downarrow$ and $\varphi_n(x) \in A^s \cup N_i$.

It is clear that $\alpha_i \leq_T A$ for each i . Thus to ensure that each $\alpha_i \leq_T \beta$, we shall ensure that $A \leq_T \beta$, using a Yates permitting argument where $b_{i,n}^s$ is allowed to be put into A^s only if $\max(i, n) \geq f(s)$. Finally to force $\alpha_i \geq_T \beta$, we shall use a coding argument where at each stage s either $b_{i,f(s)}^s$ or $b_{i,f(s)+1}^s$ will be put into A^{s+1} for each i . Thus at each stage $s > 0$, A^s will be an infinite but recursive set.

We make the following priority ranking of requirements:

$D(c(0), r(0)), Q(0), R(c(0), r(0)), D(c(1), r(1)), Q(1), R(c(1), r(1)), \dots$

(That is, $D(c(0), r(0))$ has highest priority, $Q(0)$ has the second highest priority, and so on.)

Only finitely many markers will be placed on elements at any given stage s . We assume we have infinitely many $\mathcal{A}(i, n)$, $\lambda(n)$, and $\Gamma(i, n)$ markers at our disposal and if at stage $s+1$ we place a marker Φ on an $x \in \bar{A}^s$ such that at stage s , x was unmarked or had a marker different from Φ on it, then Φ has never been used at any previous stage. If an $x \in \bar{A}^s$ drops to a lower window at stage $s+1$, the marker on x , if any, will stay with x unless specifically stated otherwise. If an $x \in \bar{A}^s$ is put into A^{s+1} , then we automatically remove any marker on x . We say a marker Φ is *active* at stage s if it rests

on an $x \in \bar{A}^s$ and Φ is *inactive* otherwise. For simplicity, each x will have at most one marker on it at any stage s . It will be possible for several markers of the same type to be active at a stage s . We say a marker Φ_1 has *higher priority* than marker Φ_2 if Φ_1 corresponds to a higher priority requirement than Φ_2 does. Finally, we define $\mathcal{H}(\Delta(i, n), s) = \{x \mid x \text{ has a marker } \Phi \text{ on it at stage } s \text{ and } \Phi \text{ has higher priority than } \Delta(i, n)\}$. $\mathcal{H}(\lambda(n), s)$ and $\mathcal{H}(\Gamma(i, n), s)$ are defined similarly.

Construction.

Stage 0. Let $A^0 = \emptyset$. Put a marker $\Delta(c(0), r(0))$ on the least x in $N_{e(0)} \cap I_{r(0)}$.

Stage $s + 1$. Assume that A^s is recursive and that at stage s

(a) $A^s \cap N_i$ is finite for each i ,

(b) only finitely many markers are active and no $x \in \bar{A}^s$ has more than one marker on it,

(c) for all $j \leq s$, exactly one $\Delta(c(j), r(j))$ marker is active and it rests on an $x \in N_{e(j)} \cap I_{r(j)}$,

(d) a $\lambda(n)$ marker rests on x only if $\varphi_n^s(x) \downarrow$ and $x \neq \varphi_n(x)$ and a $\Gamma(i, n)$ marker rests on x only if $\varphi_n^s(x) \downarrow$, $x \neq \varphi_n(x)$, and $x \in \alpha_i^s$,

(e) if requirement $Q(j)(R(j, n))$ is satisfied, then exactly one $\lambda(j)(\Gamma(j, n))$ marker is active.

Look for a $j \leq s + 1$ such that at stage s either

(1) $Q(j)$ is not satisfied and there is an $x \leq s + 1$ such that $x \in \bar{A}^s - \mathcal{H}(\lambda(j), s)$, $\varphi_j^{s+1}(x) \downarrow$, $x \neq \varphi_j(x)$, and either $x \notin \{b_{i, f(s)}^s, b_{i, f(s)+1}^s\}$ for any i or if $x \in \{b_{i, f(s)}^s, b_{i, f(s)+1}^s\}$, then $y \in \{b_{i, f(s)}^s, b_{i, f(s)+1}^s\} - \{x\}$ implies $y \notin \mathcal{H}(\lambda(j), s)$, and moreover either

(1A) $\varphi_j(x) \notin \{b_{i, n}^s \mid \max(i, n) < f(s)\} \cup \mathcal{H}(\lambda(j), s)$ or

(1B) $\varphi_j(x) \in \{b_{i, n}^s \mid \max(i, n) < f(s)\} - \mathcal{H}(\lambda(j), s)$ and if $b_{i, n}^s = \varphi_j(x)$, then for all $b_{e, k}^s = \varphi_j(y)$, where y has a $\lambda(n)$ marker on it, $\max(i, n) > \max(e, k) + 1$,

(2) Condition (1) fails and $R(c(j), r(j))$ is not satisfied and there is an $x \leq s + 1$ such that $x \in \bar{A}^s - \mathcal{H}(\Gamma(c(j), r(j)), s)$, $\varphi_{r(j)}^{s+1}(x) \downarrow$, $x \neq \varphi_{r(j)}(x)$, and either $x \notin \{b_{i, f(s)}^s, b_{i, f(s)+1}^s\}$ for any i or if $x \in \{b_{i, f(s)}^s, b_{i, f(s)+1}^s\}$, then $y \in \{b_{i, f(s)}^s, b_{i, f(s)+1}^s\} - \{x\}$ implies $y \notin \mathcal{H}(\Gamma(c(j), r(j)), s)$, and moreover either

(2A) $\varphi_{r(j)}(x) \notin [\{b_{i, n}^s \mid \max(i, n) < f(s)\} \cup \mathcal{H}(\Gamma(c(j), r(j)), s)] - N_{e(j)}$

or

(2B) $\varphi_{r(j)}(x) \in \{b_{i, n}^s \mid \max(i, n) < f(s)\} - (\mathcal{H}(\Gamma(c(j), r(j)), s) \cup N_{e(j)})$

and if $b_{i, n}^s = \varphi_{r(j)}(x)$, then for all $b_{e, k}^s = \varphi_{r(j)}(y)$ where y has a $\Gamma(c(j), r(j))$ marker on it, $\max(i, n) > \max(e, k) + 1$.

If there is no such j , go to Case 0. If there is such a j , let $e(s+1)$

be the least such j and go to Case 1 if $e(s+1)$ satisfies condition (1) and go to Case 2 otherwise.

Case 0. For each i , consider the pair $x_i = b_{i,f(s)}^s$ and $y_i = b_{i,f(s)+1}^s$ and the markers that currently rest on x_i and y_i , if any. If x_i is not marked, put x_i into A^{s+1} . If x_i is marked and y_i is not marked, put y_i into A^{s+1} . Otherwise, suppose marker Φ_1 rests on x_i and marker Φ_2 rests on y_i . If Φ_2 has higher priority than Φ_1 , put x_i into A^{s+1} and if Φ_1 has higher priority than Φ_2 , put y_i into A^{s+1} . If Φ_1 and Φ_2 have the same priority, then Φ_1 and Φ_2 must either be $\lambda(n)$ markers or $\Gamma(i, n)$ markers for some n . In such a case, let $b_{a,m}^s = \varphi_n(x_i)$ and $b_{c,k}^s = \varphi_n(y_i)$. Put x_i into A^{s+1} if $\varphi_n(x_i)$ is in

$$\mathcal{H}(\lambda(n), s)(\mathcal{H}(\Gamma(i, n), s))$$

and $\varphi_n(y_i)$ is not and put y_i in A^{s+1} if $\varphi_n(y_i)$ is in

$$\mathcal{H}(\lambda(n), s)(\mathcal{H}(\Gamma(i, n), s))$$

and $\varphi_n(x_i)$ is not. Finally, if $\varphi_n(x_i), \varphi_n(y_i) \in \mathcal{H}(\lambda(n), s)(\mathcal{H}(\Gamma(i, n), s))$ or $\varphi_n(x_i), \varphi_n(y_i) \notin \mathcal{H}(\lambda(n), s)(\mathcal{H}(\Gamma(i, n), s))$, put x_i into A^{s+1} if $\max(a, m) \leq \max(c, k)$ and put y_i into A^{s+1} if $\max(a, m) > \max(c, k)$.

Case 1. Let $e = e(s+1)$ and z be the least x corresponding to e such that $\varphi_e(x)$ satisfies condition (1A) if there is a pair $(y, \varphi_e(y))$ satisfying condition (1A) or $\varphi_e(x)$ satisfies condition (1B) if there is no pair $(y, \varphi_e(y))$ satisfying condition (1A).

(A) If $\varphi_e(z)$ satisfies condition (1A), place a new $\lambda(e)$ marker on z and remove any marker that was on z at stage s and all $\lambda(e)$ markers that were active at stage s . Then put $\varphi_e(z)$ into A^{s+1} if it is not already in A^s . For each i , also put either $b_{i,f(s)}^s$ or $b_{i,f(s)+1}^s$ into A^{s+1} according to the instructions in Case 0. (Note: our choice of z ensures that $z \notin A^{s+1}$ so that requirement Q_n will be satisfied at stage $s+1$.)

(B) If $\varphi_e(z)$ satisfies condition (1B), place a new $\lambda(e)$ marker on z and remove any marker that was on z at stage s . Then, for each i , put either $b_{i,f(s)}^s$ or $b_{i,f(s)+1}^s$ into A^{s+1} according to the instructions in case 0.

Case 2. Let $e = e(s+1)$ and let z be the least x corresponding to e such that $\varphi_{r(e)}(x)$ satisfies condition (2A) if there is pair $(y, \varphi_{r(e)}(y))$ satisfying condition (2A) or $\varphi_{r(e)}(x)$ satisfies condition (2B) if there is no pair $(y, \varphi_{r(e)}(y))$ satisfying condition (2A).

(A) If $\varphi_{r(e)}(z)$ satisfies condition (2A), place a new $\Gamma(c(e), r(e))$ marker on z and remove any marker that was on z at stage s and

all $\Gamma(c(e), r(e))$ markers that were active at stage s . Then put $\varphi_e(z)$ into A^{s+1} if $\varphi_e(z) \notin \mathcal{H}(\Gamma(c(e), r(e), s)) \cup N_{c(e)} \cup A^s$. For each i , put either $b_{i, f(s)}^s$ or $b_{i, f(s)+1}^s$ into A^{s+1} according to the instructions in Case 0. (Note: our choice of z ensures that $z \notin A^{s+1}$ so that requirement $R(c(e), r(e))$ will be satisfied at stage $s+1$.)

(B) If $\varphi_{r(e)}(z)$ satisfies condition (2B), place a new $\Gamma(c(e), r(e))$ marker on z and remove any marker that was on z at stage s . Then for each i , put either $b_{i, f(s)}^s$ or $b_{i, f(s)+1}^s$ into A^{s+1} according to the instructions in case 0.

This completes the definition of A^{s+1} . It is possible that for some j and n , requirement $Q(n)(R(j, n))$ was not satisfied at stage s but there is now some $x \in \overline{A^{s+1}}$ with a $\lambda(n)(\Gamma(j, n))$ marker on it and $\varphi_n(x) \in A^{s+1}$ because $\varphi_n(x) \in \bigcup_i \{b_{i, f(s)}^s, b_{i, f(s)+1}^s\}$ and $\varphi_n(x)$ was forced into A^{s+1} . In such a case, we keep the $\lambda(n)(\Gamma(j, n))$ marker on the least such x and remove all other $\lambda(n)(\Gamma(j, n))$ markers that were active at stage s . Finally, some of the $\Delta(c(i), r(i))$ markers for $i \leq s$ may have been removed. Inductively we place new $\Delta(c(i), r(i))$ markers for $i \leq s+1$ as follows: having placed $\Delta(c(j), r(j))$ markers for $j < i$, place $\Delta(c(i), r(i))$ on the least $x \in \alpha_{c(i)}^{s+1} \cap I_{r(i)}$ which is unmarked if $\Delta(c(i), r(i))$ was removed during stage $s+1$ and otherwise leave $\Delta(c(i), r(i))$ where it is. (This is possible since $A^{s+1} \cap N_{c(i)}$ is finite and $N_{c(i)}$ is dense in Q .)

This completes the description of stage $s+1$. It is easy to check that each stage is completely effective and that conditions (a)–(e) hold at each stage. We let $A = \bigcup_s A^s$ so that A is r.e. We now prove a sequence of lemmas that will complete the proof of the theorem.

LEMMA 1. For all i and n , $\lim_s b_{i,n}^s$ exists.

Proof. $b_{i,n}^s \neq b_{i,n}^{s+1}$ only if $f(s) \leq \max(i, n)$. Since $f(s) \leq \max(i, n)$ only finitely often, $\lim_s b_{i,n}^s$ exists.

LEMMA 2. $A \leq_T \beta$.

Proof. It follows from our construction that for all x , $x = b_{i,n}^s$ and $x = b_{j,k}^{s+1}$ only if $i = j$ and $k \leq n$. Thus to decide if $x \in A$, first find i and n such that $x = b_{i,n}^0$. Then recursively in β , find a stage t such that $\forall s (s \geq t \rightarrow f(s) > \max(i, n))$. Since for any j and k , $b_{j,k}^s \neq b_{j,k}^{s+1}$ only if $f(s) \leq \max(j, k)$, it follows that $\forall k \forall s (k \leq n \ \& \ s \geq t \rightarrow b_{i,k}^s = b_{i,k})$. Thus $x \in A$ iff $x \notin \{b_{i,0}^t, \dots, b_{i,n}^t\} = \{b_{i,0}, \dots, b_{i,n}\}$. Therefore, $A \leq_T \beta$.

Since for each i , $\alpha_i = \bar{A} \cap N_i \leq_T A$, we have that $\alpha_i \leq_T \beta$. Thus to prove that for each i , $\alpha_i \equiv_T \beta$, we need only show that for each i , $\beta \leq_T \alpha_i$.

LEMMA 3. *For each i , $\beta \leq_T \alpha_i$.*

Proof. We note that for each i , $\alpha_i = \{b_{i,0}, b_{i,1}, \dots\}$ and $b_{i,0} < b_{i,1} < \dots$ since for all s , $b_{i,0}^s < b_{i,1}^s < \dots$. To decide if $x \in \beta$, first find, recursively in α_i , a stage t such that $\forall k (k \leq x + 1 \rightarrow b_{j,k}^t = b_{i,k})$. Since for any pair (j, n) and stage s , $b_{j,n}^s \neq b_{j,n}^{s+1}$ only if there is a $k \leq n$ such that $b_{j,k}^s \in A^{s+1}$, it follows that $\forall s \forall k (k \leq x + 1 \ \& \ s \geq t \rightarrow b_{i,k}^s = b_{i,k})$. Since at each stage $s + 1$, we put either $b_{i,f(s)}^s$ or $b_{i,f(s)+1}^s$ into A^{s+1} , it follows that $\forall s (s \geq t \rightarrow f(s) > x)$. Thus, $x \in \beta$ if $x \in \beta^t$ and hence $\alpha_i \geq_T \beta$.

LEMMA 4. *For each n , the requirements $D(c(n), r(n))$, $Q(n)$, and $R(c(n), r(n))$ are met.*

Proof. We proceed by induction. Fix $n \geq 0$ and assume that for all $i < n$, the requirements $D(c(i), r(i))$, $Q(i)$, and $R(c(i), r(i))$ are met and there is a stage $t > n$ and an integer p such that: (a) For all $s \geq t$ and $j < n$, no new $\Delta(c(j), r(j))$, $\lambda(j)$, or $\Gamma(c(j), r(j))$ marker becomes active or old $\Delta(c(j), r(j))$, $\lambda(j)$, or $\Gamma(c(j), r(j))$ marker is removed at stage s , (b) If $b_{i,k}^s \in \mathcal{H}(\Delta(c(n), r(n)), t)$, then $\max(i, k) < p$, (c) $\forall s (s \geq t \rightarrow f(s) > p)$, and (d) $\forall s (s \geq t \rightarrow e(s) \geq n)$. Thus by stage t all $\Delta(c(i), r(i))$, $\lambda(i)$, and $\Gamma(c(i), r(i))$ markers with $i < n$ rest on elements that never move after stage t .

First, we consider the requirement $D(c(n), r(n))$. Suppose that at stage $t + 1$, $\Delta(c(n), r(n))$ rests on $x \in \alpha_{c(n)}^{t+1} \cap I_{r(n)}$. We claim that for all $s \geq t + 1$, $\Delta(c(n), r(n))$ rests on x and thus $x \in \alpha_{c(n)} \cap I_{r(n)}$. For assume $s \geq t + 1$, $x = b_{c(n),j}^s$ for some j , and $\Delta(c(n), r(n))$ rests on x at stage s . Then at stage $s + 1$, if $e(s + 1)$ is defined, $e(s + 1) \geq n$ so that $x \neq z$, $x \neq \varphi_{e(s+1)}(z)$ for z as defined in Case 1 and $x \neq z$, $x \neq \varphi_{r(e(s+1))}(z)$ for z as defined in Case 2. Thus the only way x could be put into A^{s+1} is if $j \in \{f(s), f(s) + 1\}$. By our choice of t , $f(s) > p$ and thus the $y \in \{b_{c(n),f(s)}^s, b_{c(n),f(s)+1}^s\} - \{x\}$ is not in $\mathcal{H}(\Delta(c(n), r(n)), s)$. Hence $\Delta(c(n), r(n))$ must have a higher priority than the marker on y , if any, and hence y and not x would be placed into A^{s+1} . It follows that after stage $t + 1$ no new $\Delta(c(n), r(n))$ marker is ever introduced so that $\forall s (s \geq t + 1 \rightarrow \mathcal{H}(\lambda(n), s) = \mathcal{H}(\lambda(n), s + 1))$. Let $x = b_{c(n),k}$ and choose $t_1 > t$ and $p_1 > p$ such that $\max(c(n), k) < p_1$ and $\forall s (s \geq t_1 \rightarrow f(s) \geq p_1)$.

Now consider the requirement $Q(n)$. First we show that if $Q(n)$ is ever satisfied for some $s > t_1$, then requirement $Q(n)$ is met and

there is a stage t_2 and an integer p_2 such that (a') for all $s \geq t_2$, $i \leq n$, and $j < n$, no new $\Delta(c(i), r(i))$, $\lambda(i)$, or $\Gamma(c(j), r(j))$ marker becomes active or old $\Delta(c(i), r(i))$, $\lambda(i)$, or $\Gamma(c(j), r(j))$ marker is removed at stage s , (b') if $b_{i,k}^s \in \mathcal{H}(\Gamma(c(n), r(n)), t_2)$, then $\max(i, k) < p_2$, (c') $\forall s (s \geq t_2 \rightarrow f(s) > p_2)$, and (d') $\forall s (s \geq t_2 \rightarrow e(s) > n \vee (e(s) = n \text{ and we are in Case 2 at stage } s))$.

Suppose $u > t_1$ and $Q(n)$ is satisfied at stage u . Thus there is an $x \in \bar{A}^u$ with a $\lambda(n)$ marker on it such that $\varphi_n^u(x) \downarrow$ and $\varphi_n(x) \in A^u$. We claim that x can never be put into A and the marker $\lambda(n)$ is never removed from x so that $Q(n)$ remains satisfied for all $s \geq u$. For suppose $s \geq u$, $x \in \bar{A}^s$, and x has a $\lambda(n)$ marker on it so that $Q(n)$ is satisfied at stage s . If $e(s+1)$ is defined, then either $e(s+1) > n$ or $e(s+1) = n$ and we are in Case 2 at stage $s+1$. Hence marker $\lambda(n)$ is not removed from x for the sake of a higher priority requirement and thus the only way x can be put into A^{s+1} is if $x = b_{i,k}^s$ for some $k \in \{f(s), f(s)+1\}$. By our choice of $s \geq u > t_1$, $f(s) > p_1$ and thus the $y \in \{b_{i,f(s)}^s, b_{i,f(s)+1}^s\} - \{x\}$ is not in $\mathcal{H}(\lambda(n), s)$. Thus $\lambda(n)$ must have a higher priority than the marker on y , if any, and hence y and not x would be placed into A^{s+1} . Thus it follows that after stage u , no new $\lambda(n)$ marker is ever introduced so that $\forall s (s \geq u \rightarrow \mathcal{H}(\Gamma(c(n), r(n)), s) = \mathcal{H}(\Gamma(c(n), r(n)), u))$. We have also shown that $x \in \bar{A}$ so that if $x = b_{i,k}$ we need only choose $p_2 > \max(p_1, i, k)$ and $t_2 \geq u$ such that $\forall s (s \geq t_2 \rightarrow f(s) \geq p_2 \text{ and } b_{i,k}^s = b_{i,k})$ and then p_2 and t_2 will satisfy conditions (a')-(d').

Now consider the case where there is no stage $s \geq t_1$ such that $Q(n)$ is satisfied at stage s . We claim that under this assumption, there are only finitely many $s \geq t_1$ such that $e(s) = n$ and we are in Case 1 at stage s . For suppose there are infinitely many such s ; we will show that β is recursive, contradicting our choice of β . First we shall prove by induction that if $u \geq t_1$ and there is an $x \in \bar{A}^u$ with a $\lambda(n)$ marker on it at stage u such that $\varphi_n(x) = b_{i,k}^u \notin \mathcal{H}(\lambda(n), u)$, then for all $s \geq u$, there is a $y \in \bar{A}^s$ with a $\lambda(n)$ marker on it at stage s such that $\varphi_n(y) = b_{j,l}^s \notin \mathcal{H}(\lambda(n), s)$ and $\max(j, l) \geq \max(i, k)$. Let $s \geq u$ and assume there is a y with the properties above. Now either $y \notin \{b_{i,f(s)}^s, b_{i,f(s)+1}^s\}$ for any i or if $y \in \{b_{i,f(s)}^s, b_{i,f(s)+1}^s\}$, then since $f(s) > p_1$ the $y' \in \{b_{i,f(s)}^s, b_{i,f(s)+1}^s\} - \{y\}$ does not have a higher priority marker than $\lambda(n)$ on it. Thus at stage $s+1$, it cannot be that $f(s) \leq \max(j, l)$ because then $(y, \varphi_n(y))$ would be a pair which could satisfy $Q(n)$ and hence our choice of $s \geq u > t_1$ would imply that $e(s+1) = n$ and that we are in Case 1 at stage $s+1$. In such a case, $Q(n)$ would be satisfied at stage $s+1$ which we assumed is not the case. Thus $f(s) > \max(j, l)$ and $\varphi_n(y) = b_{j,l}^s = b_{j,l}^{s+1}$. Since $e(s+1) \geq n$, it follows that if $e(s+1)$ is defined, then $y \neq z, y \neq$

$\varphi_{e(s+1)}(z)$ if we are in Case 1 and $y \neq z$, $y \neq \varphi_{r(e(s+1))}(z)$ if we are in Case 2 at stage $s+1$. Thus the only way y could be put into A^{s+1} is if $y \in \{b_{i,f(s)}^s, b_{i,f(s)+1}^s\}$ for some i .

Since $f(s) > p_i$, $\lambda(n)$ is the highest priority marker that could rest on either $b_{i,f(s)}^s$ or $b_{i,f(s)+1}^s$. Thus the only way y could be put into A^{s+1} is if the $y' \in \{b_{i,f(s)}^s, b_{i,f(s)+1}^s\} - \{y\}$ also has a $\lambda(n)$ marker on it and $\varphi_n(y') = b_{a,m}^s \notin \mathcal{H}(\lambda(n), s)$ and $\max(a, m) \geq \max(j, l)$. Moreover, it must be the case that $f(s) > \max(a, m)$ and hence $b_{a,m}^s = b_{a,m}^{s+1}$. Thus either y or y' is in \bar{A}^{s+1} and has a $\lambda(n)$ marker on it at stage $s+1$. Since $\mathcal{H}(\lambda(n), s) = \mathcal{H}(\lambda(n), s+1)$, we can conclude that $\varphi_n(y), \varphi_n(y') \in \bar{A}^{s+1} - \mathcal{H}(\lambda(n), s+1)$ and hence either $(y, \varphi_n(y))$ or $(y', \varphi_n(y'))$ satisfies the required properties at stage $s+1$.

We define $l^s = \max(\{\max(j, k) \mid \exists y(y \in \bar{A}^s \text{ and } y \text{ has a } \lambda(n) \text{ marker on it at stage } s \text{ and } \varphi_n(y) = b_{j,k}^s \notin \mathcal{H}(\lambda(n), s)\})$. The immediately preceding induction proved that if $s \geq t_1$ and l^s is defined, then $f(s) > l^s$ and l^{s+1} is defined and $l^{s+1} \geq l^s$. Thus if $s \geq t_1$ and l^s is defined, then $\forall u (u \geq s \rightarrow f(u) > l^u \geq l^s)$. Now suppose $s_1 \geq t_1$, $e(s_1) = n$, and we are in Case 1 at stage s_1 . If z is defined as in Case 1, then $\varphi_n(z)$ must satisfy clause (1B) of the definition of $e(s_1)$ so that $\varphi_n(z) \notin \mathcal{H}(\lambda(n), s_1 - 1) = \mathcal{H}(\lambda(n), s_1)$. Thus l^{s_1} must be defined. If $s_2 > s_1$ and $e(s_2) = n$ and we are in Case 1 at s_2 , then let z^* denote the z defined in Case 1 at stage s_2 . We know l^{s_2-1} is defined, $l^{s_2-1} \geq l^{s_1}$, and $\varphi_n(z^*)$ must satisfy clause (1B) of the definition of $e(s_2)$; thus $\varphi_n(z^*) = b_{a,m}^s \notin \mathcal{H}(\lambda(n), s_2 - 1)$ and $\max(a, m) > l^{s_2-1} + 1$. Then z^* has a $\lambda(n)$ marker on it at stage s_2 and $\varphi_n(z^*) = b_{e,g}^{s_2}$ where $\max(e, g) > l^{s_2-1}$ since no more than one element is removed from any one column. Thus $l^{s_2} > l^{s_2-1}$. It follows that if there are infinitely many $s \geq t_1$ such that $e(s) = n$ and we are in Case 1 at stage s , then we can find a recursive sequence of stages $t_1 \leq s_1 < s_2 < \dots$ such that $l^{s_1} < l^{s_2} < \dots$. But the existence of such a sequence would imply that β is recursive. For to decide if $x \in \beta$, we need only find a stage s_i such that $l^{s_i} \geq x$ and then we know $x \in \beta$ iff $x \in \beta^{s_i}$ since $\forall s (s > s_i \rightarrow f(s) > l^{s_i})$.

Thus we have shown that if $Q(n)$ is never satisfied at any stage $s \geq t_1$, then $e(s) = n$ and we are in Case 1 at stage s for only finitely many $s \geq t_2$. Since new $\lambda(n)$ markers can be introduced only at stage s' where $e(s) = n$ and we are in Case 1 at stage s , it follows that there are t_2 and p_2 which satisfy conditions (a')-(d'). However we must still check that if $Q(n)$ is never satisfied for any $s \geq t_1$, then requirement $Q(n)$ is met. Suppose requirement $Q(n)$ fails. Thus $\bar{A} \subseteq \delta\varphi_n$ and $\varphi_n \upharpoonright \bar{A}$ is a 1-1 map from \bar{A} into itself and $\{a \in \bar{A} \mid a \neq \varphi_n(a)\}$ is infinite. We have shown the existence of a stage t_2 such that for all $s \geq t_2$ either $e(s) > n$ or $e(s) = n$ and we are in Case 2 at stage s . But consider stage t_2 . Since $\mathcal{H}(\lambda(n), s) = \mathcal{H}(\lambda(n), t_1)$

for all $s \geq t_1$, there must be an $x \in \bar{A}$ such that $x \neq \varphi_n(x)$ and $\varphi_n(x) = b_{j,k} \notin \mathcal{H}(\lambda(n), t_2)$ and if l^2 is defined, then $\max(j, k) > l^2 + 1$. Now suppose $s > t_2$ is a stage such that $\varphi_n^s(x) \downarrow$. Then $\varphi_n^s(x) = b_{j,m}^s$ for some $m > k$ and either $x \notin \{b_{i,f(s)}^s, b_{i,f(s)+1}^s\}$ for any i or if $x \in \{b_{i,(s)}^s, b_{i,f(s)+1}^s\}$, then since $f(s) > p_1$, the $y \in \{b_{i,f(s)}^s, b_{i,f(s)+1}^s\} - \{x\}$ does not have a higher priority marker than $\lambda(n)$ on it. Thus the pair $(x, \varphi_n(x))$ would be candidates to satisfy Case 1 of the definition of $e(s)$ for n unless l^{s-1} is defined and $\max(j, m) \not\leq l^{s-1} + 1$. Therefore, since our choice of t_2 precludes us from being in Case 1 with $e(s) = n$ at stage s , it must be the case that $\max(j, m) \not\leq l^{s-1} + 1$. Now if l^2 was defined, then $l^{s-1} > l^2$. Thus we must conclude there is a stage $s' \geq t_3$ such that either $l^{s'-1}$ was undefined and $l^{s'}$ is defined or $l^{s'-1}$ is defined and $l^{s'} > l^{s'-1}$. But both of these cases imply that we are in Case 1 with $e(s') = n$ a stage s' which contradicts our choice of t_2 . Thus requirement $Q(n)$ must be met.

We have shown requirement $Q(n)$ must have been met and there are t_2 and p_2 satisfying conditions (a')-(d'). The argument for requirement $R(c(n), r(n))$ is almost exactly the same as the one for requirement $Q(n)$. Namely, we can show that if there is an $s \geq t_2$ such that $R(c(n), r(n))$ is satisfied at stage s , then requirement $R(c(n), r(n))$ is met and there is a stage t_3 and an integer p_3 such that (a'') for all $s \geq t_3$ and $j \leq n$, no new $\Delta(c(j), r(j))$, $\lambda(j)$, or $\Gamma(c(j), r(j))$ marker becomes active or old $\Delta(c(j), r(j))$, $\lambda(j)$, or $\Gamma(c(j), r(j))$ marker is removed at stage s , (b'') if $b_{i,k}^{t_3} \in \mathcal{H}(\Delta(c(n+1), r(n+1)), t_3)$, then $\max(i, k) < p_3$, (c'') $\forall s(s \geq t_3 \rightarrow f(s) > p_3)$, and (d'') $\forall s(s \geq t_3 \rightarrow e(s) \geq n+1)$. If there is no stage $s \geq t_3$ such that $R(c(n), r(n))$ is satisfied at stage s , then we can argue that the assumption that there are infinitely many $s \geq t_3$ such that we are in Case 2 with $e(s) = n$ at stage s leads to the contradiction that β is recursive. Hence there can be only finitely many s such that we are in Case 2 with $e(s) = n$ at stage s and thus there are t_3 and p_3 satisfying conditions (a'')-(d''). Finally, we can argue that existence of t_3 and p_3 implies that requirement $R(c(n), r(n))$ is met. These arguments complete the induction step for n .

THEOREM 2. *Let β be any recursively enumerable set which is not recursive and let $P = (N, \leq^*)$ be a recursive partial ordering. Then there is a collection co-r.e. bi-dense subsets of Q with property \mathcal{P} , each Turing equivalent to β , such that under $\prec_c, \prec_i, \prec_e$, this collection is order isomorphic to P .*

Proof. Since P is a recursive partial ordering, $R_i = \{j \in N \mid i \leq^* j\}$ is a recursive set for each i . Let M be a map from N into the set of all subsets of N defined by $M(i) = \bigcup_{j \in R_i} \alpha_j$. It easily follows that

for each i , $M(i)$ is a co-r.e. bi-dense subset of Q which has property \mathcal{P} and is Turing equivalent to β . We shall prove that M is an order preserving map from (N, \leq^*) onto $\{M(i) \mid i \in N\}$ under either $<_i$, $<_c$, $<_e$. First we show M is 1-1. If $M(i) = M(k)$, then it must be the case that $R_i = R_k$. Thus $i \in R_k = \{j \in N \mid j \leq^* k\}$ and hence $i \leq^* k$. Similarly $k \leq^* i$ so that $k = i$. Now suppose $i \leq^* k$ and $i \neq k$; we show that $M(i) <_i M(k)$, $M(i) <_c M(k)$, and $M(i) <_e M(k)$. R_i is strictly contained in R_k since $k \in R_k - R_i$. Thus $M(i) \subset M(k)$. Moreover if $W = \bigcup_{j \in R_i} N_j$ and $\bar{W} = \bigcup_{j \in \bar{R}_i} N_j$ where N_j are the sets defined in Theorem 1, then W and \bar{W} are recursive sets. Also $W \cap M(k) = W \cap \bigcap_{j \in R_k} \alpha_j = \bigcap_{j \in R_i} \alpha_j = M(i)$ and $\bar{W} \cap M(k) = \bar{W} \cap \bigcup_{j \in R_k} \alpha_j = \bigcup_{j \in R_k - R_i} \alpha_j = M(k) - M(i)$. Thus W and \bar{W} witness that $M(i) <_i M(k)$. It follows immediately from the definitions of $<_i$, $<_c$, and $<_e$ that $\forall \alpha, \beta \subseteq Q (\alpha <_i \beta \rightarrow \alpha <_c \beta \rightarrow \alpha <_e \beta)$. Thus we also have $M(i) <_c M(k)$ and $M(i) <_e M(k)$. Now suppose $i \not\leq^* k$. Thus $i \notin R_k$ so that $\alpha_i \cap M(k) = \alpha_i \cap \bigcup_{j \in R_k} \alpha_j = \emptyset$. We claim that $M(i) <_e M(k)$. For if $M(i) <_e M(k)$, then there is a partial recursive function φ such that $M(i) \subseteq \delta\varphi$ and $\varphi \upharpoonright M(i)$ is a 1-1 map from $M(i)$ into $M(k)$. But then $\alpha_i \subseteq M(i)$ and $M(k) \subseteq \bigcup_{j \neq i} \alpha_j$ imply that $\varphi \upharpoonright \alpha_i$ is a 1-1 map from α_i into $\bigcup_{j \neq i} \alpha_j$ and thus $\alpha_i <_e \bigcup_{j \neq i} \alpha_j$. But our construction in Theorem 1 ensured $\alpha_i \not<_e \bigcup_{j \neq i} \alpha_j$. Thus $M(i) \not<_e M(k)$ and hence $M(i) \not<_c M(k)$ and $M(i) \not<_i M(k)$. Thus M is an order preserving map as claimed.

COROLLARY 2.1. *Let β be any recursively enumerable set which is not recursive and let P be any countable partial ordering. Then there is a collection of co-r.e. bi-dense subsets of Q with property \mathcal{P} , each Turing equivalent to β , such that under $<_c$, $<_i$, or $<_e$, this collection is order isomorphic to P .*

Proof. It is a well known result of Mostowski [7] that there is an \aleph_0 -universal recursive partial ordering on N . Thus assume that $\langle N, \leq^* \rangle$ is an \aleph_0 -universal recursive partial ordering on N and let $P = \langle \mathcal{C}, \leq^{**} \rangle$ be any countable partial ordering. If $f: \mathcal{C} \rightarrow N$ be an order preserving map from P to $\langle N, \leq^* \rangle$, then $M \circ f$ is an order preserving map from P to $\{M(i) \mid i \in N\}$ under either $<_i$, $<_c$, or $<_e$. Thus $\{M(i) \mid i \in N\}$ is a collection which satisfies the properties required by the corollary.

COROLLARY 2.2. *Let α be any nonzero r.e. degree. Then $\langle \bar{B}(\alpha, Q), \leq_c \rangle$, $\langle \Lambda_\alpha, \leq_i \rangle$, and $\langle \Lambda_\alpha, \leq_e \rangle$ are all \aleph_0 -universal partial orderings.*

Proof. $\langle N, \leq^* \rangle$ be as in the proof of Corollary 2.1. Since $i \neq j$ implies either $i \not\leq^* j$ or $j \not\leq^* i$, it follows that either $M(i) \not<_e M(j)$ or

$M(j) \prec_e M(i)$. Thus $i \neq j$ implies $M(i)$ and $M(j)$ are in distinct equivalence classes $\text{mod } \sim_e$ and that the recursive equivalence types $\langle M(i) \rangle$ and $\langle M(j) \rangle$ are distinct. Also, since each $M(i)$ has property \mathcal{S} , each $M(i)$ is isolated and thus $M(i) \in A_z$.

3. Differences between the partial orderings. First we briefly discuss the differences between $<_i$, $<_e$, and $<_e$ on the co-r.e. subsets of Q . We noted earlier that $\forall \alpha, \beta \subseteq Q (\alpha <_i \beta \rightarrow \alpha <_e \beta \rightarrow \alpha <_e \beta)$. We show that none of the reverse implications hold. Let $\tilde{N} = \{\tilde{0}, \tilde{1}, \tilde{2}, \dots\}$ denote the natural numbers as they sit inside of Q . Since \tilde{N} is a recursive subset of Q , there is a 1-1 recursive function from Q onto \tilde{N} . Thus $Q \prec_e \tilde{N}$ but it is clearly the case that $Q \not\prec_i \tilde{N}$. Next consider the recursive sets $\tilde{E} = \{\tilde{0}, \tilde{2}, \tilde{4}, \dots\}$ and $\tilde{D} = \{\tilde{1}, \tilde{3}, \tilde{5}, \dots\}$. Clearly $\tilde{E} \prec_e \tilde{D}$ but $\tilde{E} \not\prec_i \tilde{D}$ since $\tilde{E} \not\subseteq \tilde{D}$.

Finally, we give an example to show that \leq_i and \leq_e do not agree on A_z . We start with a few definitions. A set $\alpha \subseteq N$ is *cohesive* (*r-cohesive*) if α is infinite and there is no r.e. (recursive) set W such that $W \cap \alpha$ and $\bar{W} \cap \alpha$ are both infinite. (Note: it follows immediately that if α is cohesive or *r-cohesive*, then α is isolated.) A r.e. set β is *maximal* (*r-maximal*) if $\bar{\beta}$ is cohesive (*r-cohesive*). Given r.e. sets $B \subseteq A$ we say B is a *major subset* of A if $A - B$ is infinite and for any r.e. set W such that $W \cup A = N$, $N - (W \cup B)$ is finite. Lachlan proves in [6] that every nonrecursive r.e. set has a major subset and that a major subset of a maximal set is an *r-maximal* set. So let A be a maximal set and B be a major subset of A . Let $\alpha = \bar{A}$ and $\beta = \bar{B}$. Thus α is cohesive and β is *r-cohesive* so that $\langle \alpha \rangle, \langle \beta \rangle \in A_z$. Also $\alpha \subseteq \beta$ so the identity map shows that $\alpha <_e \beta$ and hence $\langle \alpha \rangle \leq_e \langle \beta \rangle$. We shall show that $\langle \alpha \rangle \not\leq_i \langle \beta \rangle$. Suppose $\langle \alpha \rangle \leq_i \langle \beta \rangle$. Then there are sets $\alpha' \in \langle \alpha \rangle$ and $\beta' \in \langle \beta \rangle$ such that $\alpha' <_i \beta'$. Thus $\alpha' \subseteq \beta'$ and there are r.e. sets W_1 and W_2 such that $W_1 \cap \beta' = \alpha'$ and $W_2 \cap \beta' = \beta' - \alpha'$. Also since $\alpha' \in \langle \alpha \rangle$ and $\beta' \in \langle \beta \rangle$, there are 1-1 partial recursive functions q and p such that $\alpha' \subseteq \delta q$ and $q \upharpoonright \alpha'$ is a 1-1 map from α' onto α and $\beta' \subseteq \delta p$ and $p \upharpoonright \beta'$ is a 1-1 map from β' onto β . It must be the case that $\beta' - \alpha'$ is infinite. For suppose $\beta' - \alpha'$ is finite. If $\alpha'' = p(\alpha')$, then $\beta - \alpha''$ is finite and hence $A \cap \alpha''$ and $\bar{A} \cap \alpha''$ are infinite since $A \cap \beta$ and $\bar{A} \cap \beta$ are infinite. Now $q \circ p^{-1} \upharpoonright \alpha''$ is a 1-1 map from α'' onto α . Let U be the r.e. set $A \cap \delta q \circ p^{-1}$. Then $q \circ p^{-1}(U)$ is a r.e. set such that $q \circ p^{-1}(U) \cap \alpha \supseteq q \circ p^{-1}(U \cap \alpha'')$ and $\overline{q \circ p^{-1}(U)} \cap \alpha \supseteq q \circ p^{-1}(\bar{U} \cap \alpha)$. Thus $q \circ p^{-1}(U) \cap \alpha$ and $\overline{q \circ p^{-1}(U)} \cap \alpha$ are both infinite which violates the fact that α is cohesive. Next, consider the r.e. sets $U_1 = W_1 \cap \delta p$ and $U_2 = W_2 \cap \delta p$. Then $p(U_1)$ and $p(U_2)$ are r.e. sets and $p(U_1) \cap \beta \supseteq p(U_1 \cap \beta') = p(\alpha')$ and $p(U_2) \cap \beta \supseteq p(U_2 \cap \beta') = p(\beta' - \alpha')$. Thus

$p(U_1) \cap \beta$ and $p(U_2) \cap \beta$ are both infinite. Now let $V_1 = B \cup p(U_1)$ and $V_2 = B \cup p(U_2)$. Note that $U_1 \cup U_2 \supseteq \delta p \supseteq \beta'$ and hence $p(U_1) \cup p(U_2) \supseteq \beta = N - B$ which implies $V_1 \cup V_2 = N$. From the enumerations of V_1 and V_2 , we can construct recursive sets R_1 and R_2 as follows. We put x in R_1 if x is enumerated in V_1 before it is enumerated in V_2 and put x in R_2 otherwise. Then $\bar{R}_1 = R_2$ and $R_1 \cap \beta = V_1 \cap \beta = p(U_1) \cap \beta$ and $R_2 \cap \beta = V_2 \cap \beta = p(U_2) \cap \beta$. Thus R_1 violates the fact that β is r -cohesive. Thus $\langle \alpha \rangle \not\leq_i \langle \beta \rangle$ and we have proved the following.

THEOREM 3. \leq_i and \leq_e do not agree on A_z .

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