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A commutative hypergroup K is, roughly speaking, a space in which the product of two elements is a probability measure. Such spaces have been studied by Dunkl, Jewett, and Spector. Examples include locally compact abelian groups and double-coset spaces. K has a Haar measure m (Spector). It is shown that for several important classes of hypergroups the structure space of $L^1(m)$ is a hypergroup $\hat{K}$. For such spaces, $L^1(m)$ is shown to be regular, in fact, super-regular, and to have good approximate units. A Wiener-Tauberian theorem is given. Points in the center of $\hat{K}$ are shown to be strong Ditkin sets. Examples (due essentially to Reiter and Naimark) show that not all points in $\hat{K}$ need be spectral sets.

1. Introduction. The purpose of this paper is to determine to what extent results for the group algebra of a locally compact abelian group carry over to commutative hypergroups. The theory of topological hypergroups was initiated by Dunkl [3], Jewett [6], and Spector [12] and has recently received a good deal of attention from harmonic analysts. Throughout the paper, K will denote a commutative locally compact hypergroup such that $K^\wedge$ is a hypergroup under pointwise operations. Being commutative, K admits a Haar measure m, as shown by Spector [13]. The convolution algebra $L^1(m) = L^1(K)$ can be identified with the pointwise algebra $A(K^\wedge)$ of Fourier transforms on $K^\wedge$. The main reference will be Jewett [6] who calls hypergroups "convos." A survey of the subject appears in [10].

In §2 we establish some basic facts about $A(K^\wedge)$. $A(K^\wedge)$ is shown to be a regular algebra of functions on $K^\wedge$; in fact, $A(K^\wedge)$ is super-regular (2.9). It is shown that $A(K^\wedge)$ has some useful approximate units. A Wiener-Tauberian theorem is given. Some results on spectral synthesis are given in §3. The main result asserts that points in the center of $K^\wedge$ are strong Ditkin sets. Several examples are discussed in §4. In particular, it is observed that, in general, points of $K^\wedge$ need not be spectral sets. It is also observed that there exists nondiscrete $K^\wedge$ such that every closed subset is a Calderón set.

1.1. As remarked above, we assume throughout that

(H_1) $K^\wedge$ is a hypergroup under pointwise multiplication.

In (3.5)–(3.13) we impose another hypothesis which we now discuss.

The hypergroup $K^\wedge$ need not be the structure space of $A(K^\wedge)$. In fact, the structure space for $A(K^\wedge) \approx L^1(K)$ is the space $\mathscr{B}_b(K)$ of all bounded continuous multiplicative functions on K [6, 6.3]. We always have $K^\wedge \subseteq \mathscr{B}_b(K)$, but in (3.5)-(3.13) we assume that

(H_2) $K^\wedge = \mathscr{B}_b(K)$.

We devote the remainder of §1 to showing that (H_1) and (H_2) hold for several important classes of hypergroups.

1.2. To start with, observe that (H_1) and (H_2) hold if K is a locally compact abelian group. Now let G be a locally compact abelian group and let B be a subgroup of the automorphism group of G having compact closure. Then the space G_B of B^--orbits is a hypergroup. Property (H_1) holds for G_B because $(G_B)^\wedge$ is isomorphic with $(G^\wedge)_B$ where $G^\wedge$ denotes the character group of G. Property (H_2) also holds for G_B. See [11] for a more detailed discussion.

1.3. Compact hypergroups always satisfy (H_2) [3, 3.5] but (H_1) can even fail for three-element hypergroups [3, 3.8], [6, 9.1C].

PROPOSITION 1.4. *Let G be a Z-group, i.e. a group such that G/Z is compact where Z denotes the center of G. The hypergroup K of conjugacy classes of G satisfies (H_1) and (H_2). For each $x \in K$ let $x^\sim$ be defined on $K^\wedge$ by $x^\sim(\psi) = \overline{\psi(x)}$. Then $\mathfrak{X}_b(K^\wedge) = K^{\wedge\wedge} = \{x^\sim : x \in K\}$. In particular, $K^{\wedge\wedge}$ is isomorphic with K and (H_1) and (H_2) hold for $K^\wedge$.*

Proof. As observed in [11], $K^\wedge$ can be identified with the space $\mathfrak{X}$ of normalized characters on G. It is shown that $\mathfrak{X}$ is a hypergroup in [11, 5.5] and so K satisfies (H_1). A theorem of Hulanicki (see [16, 4.12]) shows that K also satisfies (H_2).

As noted in [6, 12.4], $x \to x^\sim$ is a homeomorphism of K onto a closed subset of $K^{\wedge\wedge}$. Now consider $\chi \in \mathfrak{X}_b(K^\wedge)$. The fact that $\chi = x^\sim$ for some $x \in K$ follows directly from Theorem 2 in Kaniuth and Steiner [15]. To translate into their notation, observe that $E(G) = \mathfrak{X} \cong K^\wedge$ and that $\mathfrak{R}(G) = K$. For $\xi \in L^1(K^\wedge)$, let $h(\xi) = \int_{K^\wedge} \xi(\psi)\chi(\bar{\psi})d\psi$. Then h is a nonzero multiplicative linear functional on $L^1(K^\wedge)$ and the theorem of Kaniuth and Steiner shows that $h(\xi) = \int_{K^\wedge} \xi(\psi)\psi(x)d\psi$ for some $x \in K$ and all $\xi \in L^1(K^\wedge)$. Then $\chi(\psi) = \bar{\psi}(x) = x^\sim(\psi)$ for $\psi \in K^\wedge$.

An interesting hypergroup for which both (H_1) and (H_2) fail is discussed in (4.8.)

2. The algebra $A(K\hat{})$. We begin by setting down some results that are straightforward consequences of basic results in [6]. As stated in §1, we assume throughout that K is a commutative hypergroup and that $K\hat{}$ is a hypergroup under pointwise operations. The Plancherel measure on $K\hat{}$ will be denoted by π. Since $K\hat{}$ is a hypergroup, π is the Haar measure for $K\hat{}$; see [6, 7.3, 12.4]. In view of [6, 7.3I] the Fourier transform on $L^1(m) \cap L^2(m)$ extends to an isometry of $L^2(m)$ onto $L^2(\pi)$.

LEMMA 2.1. *If f and g are in $L^2(m)$, then $f\hat{}$ and $g\hat{}$ are in $L^2(\pi)$ and $f\hat{} * g\hat{} = (fg)\hat{}$. Moreover, $(\varphi, \psi) \to \varphi * \psi$ is a jointly continuous mapping from $L^2(\pi) \times L^2(\pi)$ onto $A(K\hat{})$.*

LEMMA 2.2. *If f belongs to $L^1(m)$ or $L^2(m)$ and if $\gamma \in K\hat{}$, then $(f\overline{\gamma})\hat{} = (f\hat{})_\gamma$. If f belongs to $L^1(m)$ and $y \in K$, then $(f_y)\hat{}(\gamma) = \gamma(y) f\hat{}(\gamma)$ for all $\gamma \in K\hat{}$.*

The next lemma follows from (2.2B) and (5.4H) in [6].

LEMMA 2.3. *For f in $L^1(m)$, the mapping $y \to f_y$ is continuous from K into $L^1(m)$.*

LEMMA 2.4. *If f, g are in $L^1(m)$ and $y \in K$, then*

$$(f * g)_y = f * (g_y) = (f_y) * g \, .$$

The regularity of $A(K\hat{})$ is a consequence of the next lemma, whose statement and proof will be familiar.

LEMMA 2.5. *Let E be a compact subset of $K\hat{}$ and let V be a symmetric set such that $\pi(V) > 0$ and such that its closure V^- is compact. Then there is a function φ in $A(K\hat{})$ such that $0 \leqq \varphi \leqq 1$, $\varphi(\gamma) = 1$ for $\gamma \in E$ and $\varphi(\gamma) = 0$ for $\gamma \notin E * V * V$. Also, we have $\|\varphi\|_A^2 \leqq \pi(E * V)/\pi(V)$.*

Proof. Let ξ_V and ξ_{E*V} denote the characteristic functions of V and $E * V$, respectively, and let $\varphi = \pi(V)^{-1} \xi_V * \xi_{E*V}$. Lemma 2.1 shows that φ belongs to $A(K\hat{})$. For $\gamma \in K\hat{}$, we have

$$\varphi(\gamma) = \pi(V)^{-1} \int_V (p_\gamma * p_{\gamma'})(E * V) d\pi(\gamma') \, ;$$

p_γ denotes the point mass at γ. Since each $p_\gamma * p_{\gamma'}$ is a probability measure, we see that $0 \leqq \varphi \leqq 1$. If $\gamma \in E$, then supp $(p_\gamma * p_{\gamma'}) \subseteq E * V$ for all $\gamma' \in V$, from which it follows that $\varphi(\gamma) = 1$. If $\varphi(\gamma) \neq 0$, then $(p_\gamma * p_{\gamma'})(E * V) > 0$ for some $\gamma' \in V$. From (4.1B) in [6] it follows that

$\gamma \in \bar{\gamma}' * E * V \subseteq E * V * V$. The norm inequality is easily verified.

The subscript "00" on a family of functions restricts the family to its members having compact support.

THEOREM 2.6. *$A(K\hat{})$ is a regular Banach algebra of functions on $K\hat{}$. Moreover, $A_{00}(K\hat{})$ is dense in $A(K\hat{})$.*

Proof. Lemma 2.5 establishes the regularity. Lemma 2.1 implies that $C_{00}(K\hat{}) * C_{c0}(K\hat{})$ is dense in $A(K\hat{})$ and [6, 3.2B] implies that $C_{0c}(K\hat{}) * C_{00}(K\hat{}) \subseteq A_{00}(K\hat{})$.

We now obtain two useful approximate units for $L^1(m)$, i.e., for $A(K\hat{})$. We use the notation "$\check{}$" to signify involution.

LEMMA 2.7. *Let f be in $L^1(m)$ and $\varepsilon > 0$. There is a neighborhood V of the identity e in K such that for every nonnegative Borel function u supported by V and satisfying $\int_K u(\check{x})dm(x) = 1$, we have $||f - f * u||_1 < \varepsilon$.*

Proof. By Lemma 2.3 there is a symmetric neighborhood V of e such that $||f_y - f||_1 < \varepsilon$ for $y \in V$. If u is as indicated, then

$$f * u(x) - f(x) = \int_K [f_y(x) - f(x)]u(\check{y})dm(y)$$

for $x \in K$, and therefore

$$\begin{aligned} ||f * u - f||_1 &\leqq \int_K \int_K |f_y(x) - f(x)|\, u(\check{y})dm(y)dm(x) \\ &= \int_K ||f_y - f||_1 u(\check{y})dm(y) < \varepsilon . \end{aligned}$$

THEOREM 2.8. *$A(K\hat{})$ has an approximate unit $\{\varphi_\alpha\}$ such that each φ_α belongs to $A_{00}(K\hat{})$ and $||\varphi_\alpha||_A = 1$ for all α. If K is metrizable, $\{\varphi_\alpha\}$ can be chosen as a sequence.*

Proof. Let $\mathscr{U}$ be a basis of neighborhoods at e consisting of compact symmetric sets. We direct the net by $D = \{(U, \delta): U \in \mathscr{U}, 0 < \delta < 1\}$ where $(U_1, \delta_1) \geqq (U_2, \delta_2)$ signifies $U_1 \subseteq U_2$ and $\delta_1 \leqq \delta_2$. For $U \in \mathscr{U}$, let $f_U = m(U)^{-1}\xi_U$. For $\alpha = (U, \delta)$, use Theorem 2.6 to select ψ_α in $A_{00}(K\hat{})$ such that $||\hat{f}_U - \psi_\alpha||_A < \delta$. Finally, define $\varphi_\alpha = ||\psi_\alpha||_A^{-1}\psi_\alpha$. Some routine estimates and an application of Lemma 2.7 show that $\{\varphi_\alpha\}$ is an approximate unit for $A(K\hat{})$.

Before obtaining our second approximate unit, we use Theorem 2.8 to show that $A(K\hat{})$ is super-regular.

THEOREM 2.9. *Let E be a compact subset of $K^\wedge$ and let $\varepsilon > 0$. Then there is a function ψ in $A_{00}(K^\wedge)$ such that $\psi(\gamma) = 1$ for $\gamma \in E$ and $\|\psi\|_A < 1 + \varepsilon$.*

Proof. By Lemma 2.5 there exists φ in $A_{00}(K^\wedge)$ such that $\varphi = 1$ on E. By (2.8) there exists φ_0 in $A_{00}(K^\wedge)$ such that $\|\varphi_0\|_A = 1$ and $\|\varphi - \varphi\varphi_0\|_A < \varepsilon$. Now let $\psi = \varphi + \varphi_0 - \varphi\varphi_0$.

THEOREM 2.10. *$A(K^\wedge)$ has an approximate unit $\{\varphi_\alpha\}$ with the following properties. If $\varphi_\alpha = \hat{f}_\alpha$ for $f_\alpha \in L^1(m)$, then each f_α belongs to $C_{00}^+(K)$, each φ_α belongs to $L^1(\pi)$, $\varphi_\alpha \geqq 0$, and $\|\varphi_\alpha\|_A = 1$ for all α. If K is metrizable, $\{\varphi_\alpha\}$ can be chosen as a sequence.*

Proof. Our directed set will be a basis $\mathscr{U}$ of compact symmetric neighborhoods of e. For each U in $\mathscr{U}$, select V in $\mathscr{U}$ satisfying $V * V \subseteqq U$. Let $g_V = m(V)^{-1}\xi_V$, $f_U = g_V * g_V$, and $\varphi_U = \hat{f}_U$. Then f_U is easily seen to be in $C_{00}^+(K)$. Since $\hat{g}_V$ belongs to $L^2(\pi)$, $\varphi_U = (\hat{g}_V)^2$ belongs to $L^1(\pi)$. Since V is symmetric and g_V is real-valued, $\hat{g}_V$ is real-valued and so $\varphi_U \geqq 0$. Since $\hat{g}_V(1) = 1$, we have

$$1 = \varphi_U(1) \leqq \|\varphi_U\|_u \leqq \|\varphi_U\|_A = \|f_U\|_1 \leqq \|g_V\|_1 \|g_V\|_1 = 1 ,$$

i.e., $\|\varphi_U\|_A = 1$. Finally, Lemma 2.7 implies that $\{\varphi_U\}$ is an approximate unit for $A(K^\wedge)$.

THEOREM 2.11. *A subset I of $L^1(K)$ is a closed ideal if and only if it is a closed translation-invariant subspace.*

Proof. Let I be a closed ideal, $f \in I$, $y \in K$. If $\{g_\alpha\}$ is an approximate unit for $L^1(K)$, then $f_y = \lim_\alpha (f_y) * g_\alpha$. But by (2.4), each $(f_y) * g_\alpha = f * (g_\alpha)_y$ is in I and so $f_y \in I$.

Now suppose that I is a closed translation-invariant subspace and let $f \in I$, $g \in L^1(K)$, $\varepsilon > 0$. Select $h \in C_{00}(K)$ so that $\|f\|_1 \|g - h\|_1 < \varepsilon/2$. Use Lemma 2.3 to partition the compact set $\operatorname{supp}(h) \cup \operatorname{Supp}(h)^\vee$ into Borel sets $\{B_j\}_{j=1}^n$ so that

$$x, y \in B_j \quad \text{imply} \quad \|h^\vee\|_1 \|f_y - f_x\|_1 < \varepsilon/2 .$$

Select $x_j \in B_j$ and set $c_j = \int_{B_j} h(\check{x}) dm(x)$. For $x \in K$, we have

$$(h * f)(x) - \sum_{j=1}^n c_j (f_{x_j})(x) = \sum_{j=1}^n \int_{B_j} [f_y(x) - f_{x_j}(x)] h(\check{y}) dm(y) ,$$

from which it follows that $\|h * f - \sum_{j=1}^n c_j (f_{x_j})\|_1 < \varepsilon/2$. Hence $\|g * f - \sum_{j=1}^n c_j (f_{x_j})\|_1 < \varepsilon$. Thus $g * f \in I$ since I is translation-invariant and closed.

Here is a Wiener-Tauberian theorem for hypergroups.

THEOREM 2.12. *If f belongs to $L^1(K)$ and if $\hat{f}$ vanishes nowhere on $K^\wedge$, then the closed translation-invariant subspace of $L^1(K)$ generated by f is $L^1(K)$ itself.*

We omit the direct proof which is similar to that for the group case. If $K^\wedge$ is the structure space $\mathscr{M}_b(K)$ for $L^1(K)$, then this theorem follows from the abstract Tauberian theorem [5, 39.27] in view of Theorem 2.6. See the discussion in (1.1).

3. Synthesis in $A(K^\wedge)$. As before, K denotes a commutative hypergroup such that $K^\wedge$ is also a hypergroup. The center $Z(K)$ of K consists of all x in K such that supp $(p_x * p_y)$ is a singleton for each $y \in K$. The center $Z(K^\wedge)$ of $K^\wedge$ is the set of ψ in $K^\wedge$ such that $|\psi| = 1$. A detailed study of centers appears in [11]. The main result in this section, and in this paper, asserts that each point in $Z(K^\wedge)$ is a strong Ditkin set for the algebra $A(K^\wedge)$; see (3.3) and (3.8). We begin with two lemmas.

LEMMA 3.1. *Let F be a compact symmetric subset of K and $\varepsilon > 0$. If $H \subseteqq \{\gamma \in K^\wedge : |\gamma(y) - 1| < \varepsilon$ for all $y \in F\}$, $\pi(H) < \infty$, and if $\hat{g} = \xi_H$, then $\|g_x - g\|_2 \leqq \varepsilon\pi(H)^{1/2}$ for $x \in F$.*

Proof. By Lemma 2.2, we have

$$\|g_x - g\|_2^2 = \|\hat{g}_x - \hat{g}\|_2^2 = \int_{K^\wedge} |\gamma(x)\hat{g}(\gamma) - \hat{g}(\gamma)|^2 d\pi(\gamma)$$

$$= \int_H |\gamma(x) - 1|^2 d\pi(\gamma) \leqq \varepsilon^2\pi(H) .$$

LEMMA 3.2. *If F is a compact symmetric subset of K and $\delta > 0$, then there exists $g \in L^1(K)$ such that*

(i) $\|g\|_1 < 2$,

(ii) $\hat{g} = 1$ *on a neighborhood of* 1,

(iii) $\int_K |g(y * \check{x}) - g(y)|\, dm(y) \leqq \delta$ *for* $x \in F$.

Proof. Let $\varepsilon = \delta/12$ and let

$$\Phi = \{\gamma \in K^\wedge : |\gamma(y) - 1| < \varepsilon \quad \text{for} \quad y \in F\} .$$

Since $K^\wedge$ has the compact-open topology, Φ is a neighborhood of 1 and so Φ contains an open symmetric neighborhood H_1 of 1 having compact closure. Since π is a regular measure on $K^\wedge$, there exists a compact symmetric neighborhood H_2 of 1 such that

$H_2 \subseteq H_1$ and $\pi(H_2) > (1/4)\pi(H_1)$. By [6, 3.2D], there is a neighborhood $\mathcal{V}$ of 1 such that $\mathcal{V} * H_2 \subseteq H_1$. Now there exist g_1, g_2 in $L^2(K)$ such that $\hat{g}_i = \xi_{H_i}$, $i = 1, 2$. Finally we put $g = \pi(H_2)^{-1} g_1 g_2$. Observe that $\|g_i\|_2^2 = \pi(H_i)$ and so

$$\|g\|_1 = \pi(H_2)^{-1}\|g_1 g_2\|_1 \leqq \pi(H_2)^{-1}\|g_1\|_2\|g_2\|_2 = [\pi(H_1)/\pi(H_2)]^{1/2} < 2 .$$

To check (ii), consider $\psi \in \mathcal{V}$. Then

$$\pi(H_2)\hat{g}(\psi) = (g_1 g_2)\hat{}(\psi) = \hat{g}_1 * \hat{g}_2(\psi) = \int_{K\hat{}} \xi_{H_1}(\psi * \gamma)\xi_{H_2}(\overline{\gamma}) d\pi(\gamma)$$

$$= \int_{H_2} \xi_{H_1}(\psi * \gamma) d\pi(\gamma) = \int_{H_2}\int_{K\hat{}} \xi_{H_1} d(p_\psi * p_\gamma) d\pi(\gamma) .$$

For $\gamma \in H_2$, we have $\operatorname{supp}(p_\psi * p_\gamma) \subseteq \mathcal{V} * H_2 \subseteq H_1$ and so ξ_{H_1} is identically 1 on $\operatorname{supp}(p_\psi * p_\gamma)$. Therefore $\pi(H_2)\hat{g}(\psi) = \pi(H_2)$, i.e., $\hat{g}(\psi) = 1$. Thus $\hat{g} = 1$ on $\mathcal{V}$ and (ii) holds.

Since F is symmetric, it suffices to prove (iii) for x in place of $\check{x}$. Observe that

$$\begin{aligned} \pi(H_2)[g(y * x) - g(y)] &= (g_1 g_2)(y * x) - g_1(y) g_2(y) \\ &= \int_K [g_1(z) - g_1(y)][g_2(z) - g_2(y)] d(p_y * p_x)(z) \\ &\quad + g_1(y)[(g_2)_x(y) - g_2(y)] + g_2(y)[(g_1)_x(y) - g_1(y)] \\ &\equiv G_1(y) + G_2(y) + G_3(y) . \end{aligned} \tag{1}$$

Using Lemma 3.1, we obtain

$$\begin{aligned} \int_K |G_2|\, dm &= \|g_1[(g_2)_x - g_2]\|_1 \\ &\leqq \|g_1\|_2 \|(g_2)_x - g_2\|_2 \leqq \pi(H_1)^{1/2} \varepsilon \pi(H_2)^{1/2} . \end{aligned} \tag{2}$$

Similarly, we have

$$\int_K |G_3|\, dm \leqq \varepsilon \pi(H_1)^{1/2} \pi(H_2)^{1/2} . \tag{3}$$

Estimation of the integral of G_1 is more delicate. By Hölder's inequality, we have $|G_1(y)|^2 \leqq A_1(y) A_2(y)$ where

$$A_i(y) = \int_K |g_i(z) - g_i(y)|^2 d(p_y * p_x)(z) .$$

Now

$$\begin{aligned} A_i(y) &= \int_K [|g_i(z)|^2 - 2 \operatorname{Re} g_i(z)\overline{g_i(y)} + |g_i(y)|^2] d(p_y * p_x)(z) \\ &= (|g_i|^2)_x(y) - 2 \operatorname{Re} \overline{g_i(y)}(g_i)_x(y) + |g_i(y)|^2 \\ &= (|g_i|^2)_x(y) - |(g_i)_x(y)|^2 + |(g_i)_x(y) - g_i(y)|^2 . \end{aligned}$$

By [6, 3.3B], we have

$$\int_K (|g_i|^2)_x dm \leqq \int_K |g_i|^2 dm = ||g_i||_2^2$$

and so

$$\begin{aligned}\int_K [(|g_i|^2)_x - |(g_i)_x|^2] dm &\leqq ||g_i||_2^2 - ||(g_i)_x||_2^2 \\ &= [||g_i||_2 + ||(g_i)_x||_2][||g_i||_2 - ||(g_i)_x||_2] \\ &\leqq 2||g_i||_2 ||g_i - (g_i)_x||_2 .\end{aligned}$$

Also

$$\int_K |(g_i)_x - g_i|^2 dm = ||(g_i)_x - g_i||_2^2 \leqq 2||g_i||_2 ||(g_i)_x - g_i||_2$$

and so

$$\int_K A_i dm \leqq 4||g_i||_2 ||(g_i)_x - g_i||_2 \leqq 4\pi(H_i)^{1/2} \varepsilon \pi(H_i)^{1/2} .$$

By Hölder's inequality again, we obtain

$$\begin{aligned}(4) \qquad \int_K |G_1| dm &\leqq \int_K A_1^{1/2} A_2^{1/2} dm \leqq \left[\int_K A_1 dm\right]^{1/2} \left[\int_K A_2 dm\right]^{1/2} \\ &\leqq [16\varepsilon^2 \pi(H_1)\pi(H_2)]^{1/2} = 4\varepsilon\pi(H_1)^{1/2}\pi(H_2)^{1/2} .\end{aligned}$$

Combining (1)-(4), we find

$$\pi(H_2) \int_K |g(y * x) - g(y)| dm(y) \leqq 6\varepsilon\pi(H_1)^{1/2}\pi(H_2)^{1/2}$$

or

$$\int_K |g(y * x) - g(y)| dm(y) \leqq 6\varepsilon[\pi(H_1)/\pi(H_2)]^{1/2} < 12\varepsilon = \delta .$$

The next theorem shows that the identity 1 of $K^\wedge$ may be viewed as a strong Ditkin set for $A(K^\wedge)$ even if $K^\wedge$ is not the structure space for $A(K^\wedge)$; see (1.1).

THEOREM 3.3. *There is a net $\{f_\alpha\}$ in $L^1(K)$ such that*
(i) $||f_\alpha||_1 < 3$ *for all* α,
(ii) *if* $f \in L^1(K)$ *and* $\hat{f}(1) = 0$, *then* $\lim_\alpha ||f - f * f_\alpha||_1 = 0$,
(iii) *each* $\hat{f}_\alpha$ *vanishes in a neighborhood of* 1 *in* $K^\wedge$ *and has compact support.*

Proof. Let $\{u_\beta\}$ be an approximate unit for $L^1(K)$ such that $||u_\beta||_1 = 1$ and $\hat{u}_\beta \in A_{n0}(K^\wedge)$ for all β; see (2.8). The net $\{f_\alpha\}$ will be directed by the set

$$D = \{(F, \delta, \beta): F \subseteq K \text{ compact symmetric, } \delta > 0\}$$

where $(F, \delta, \beta) \geqq (F', \delta', \beta')$ signifies $F \supseteq F'$, $\delta \leqq \delta'$, $\beta \geqq \beta'$. Given $\alpha = (F, \delta, \beta)$, select $g_{F,\delta}$ as in Lemma 3.2 and then define $f_\alpha = u_\beta - u_\beta * g_{F,\delta}$.

Properties (i) and (iii) are easy to verify. To verify (ii), consider $f \in L^1(K)$ such that $\hat{f}(1) = 0$, and let $\varepsilon > 0$. Select β_0 so that $\beta \geqq \beta_0$ implies $\|f - f * u_\beta\|_1 < \varepsilon/2$; select compact symmetric F_0 in K so that $\int_{K \setminus F_0} |f|\, dm < \varepsilon/16$; and select δ_0 so that $4\delta_0 \|f\|_1 < \varepsilon$. If $\alpha_0 = (F_0, \delta_0, \beta_0)$, routine estimates show that $\alpha \geqq \alpha_0$ implies $\|f - f * f_\alpha\|_1 < \varepsilon$.

REMARK 3.4. The proof of Theorem 3.3 follows the same pattern as in the group case (see [5, 39.28], for example). The main new difficulty is that the relation $(fg)_x = f_x g_x$, familiar for functions on groups, does not in general hold for functions on hypergroups.

Concepts such as spectral set, strong Ditkin set and Calderón set are normally defined for subsets of the structure space of a regular commutative Banach algebra. For these reasons, we adopt

AN ADDITIONAL HYPOTHESIS 3.5. *For the remainder of this section, we assume that* $K\hat{} = \mathscr{X}_b(K)$. *Thus* K *will be a commutative hypergroup such that* $K\hat{}$ *is a hypergroup and* $K\hat{}$ *is the structure space of* $L^1(K)$.

If K is compact, then $K\hat{}$ is discrete and all its subsets are Calderón sets. So our results below on synthesis are of special interest only if K is noncompact. In any case, $\{1\}$ is a strong Ditkin set for $L^1(K)$ by Theorem 3.3. Our next result and its corollary allow us to extend this result to points in $Z(K\hat{})$. It should be compared with results of Rieffel [9, §4.2].

THEOREM 3.6. *Let* h *be a nonzero bounded continuous function on* K, *and define* $J_h(g) = hg$ *for each* $g \in L^1(K)$. *If* $\gamma \in Z(K\hat{})$, *then* J_γ *is an isometric algebra isomorphism of* $L^1(K)$ *onto* $L^1(K)$. *Conversely, if* J_h *preserves the convolution of* $L^1(K)$, *then* h *belongs to* $Z(K\hat{})$.

Proof. The first statement is easily verified. For example, the identity $J_\gamma(f * g) = J_\gamma(f) * J_\gamma(g)$ is checked using the definition of convolution and the fact that γ is constant with value $\gamma(x)\gamma(y)$ on $\operatorname{supp}(p_x * p_y)$ for each $x, y \in K$ and $\gamma \in Z(K\hat{})$ [3, 2.2].

For the converse, we first consider $x, y \in K$ and show that $h(s) = h(x)h(y)$ for all $s \in \operatorname{supp}(p_x * p_y)$. Assume that $\beta = h(x)h(y) - h(s) \neq 0$

for some $s \in \operatorname{supp}(p_x * p_y)$ so that $[h(x)h(y) - h(s)]\overline{\beta} > 0$. There exist compact neighborhoods V and W of x and y, and $\eta > 0$, so that

$$\operatorname{Re}[h(v)h(w) - h(s)]\overline{\beta} \geqq \eta \qquad \text{for} \quad v \in V,\ w \in W .$$

Select a continuous nonnegative function f so that $f(x) > 0$ and $\operatorname{supp}(f) \subseteq V$, and let $g = \xi_W$. Since $x \in \operatorname{supp}(p_s * p_y^{\vee})$ we have $f(s * \check{y}) > 0$. By the continuity of $w \mapsto f(s * \check{w})$, we see that $f(s * \check{w}) > 0$ in a neighborhood of y and so

$$\int_W f(s * \check{w}) dm(w) > 0 . \tag{1}$$

Direct calculations show that

$$(hf) * (hg)(s) = \int_W \int_V h(v)f(v)d(p_s * p_w^{\vee})(v)h(w)dm(w)$$

and

$$h(f * g)(s) = h(s) \int_W \int_V f(v)d(p_s * p_w^{\vee})(v)dm(w) .$$

Since $(hf) * (hg) = h(f * g)$, we conclude that

$$0 = \int_W \int_V [h(v)h(w) - h(s)]f(v)d(p_s * p_w^{\vee})(v)dm(w) ,$$

contrary to the estimate

$$\operatorname{Re} \int_W \int_V [h(v)h(w) - h(s)]\overline{\beta}f(v)d(p_s * p_w^{\vee})(v)dm(w)$$
$$\geqq \eta \int_W \int_V f(v)d(p_s * p_w^{\vee})(v)dm(w) = \eta \int_W f(s * \check{w})dm(w) > 0 ;$$

see (1).

Thus $h(s) = h(x)h(y)$ for $s \in \operatorname{supp}(p_x * p_y)$ and hence $h(x * y) = h(x)h(y)$. Since $h(x * e) = h(x)h(e)$ and h is nonzero somewhere, we conclude that $h(e) = 1$. Since $h(x)h(\check{x}) = h(e) = 1$ and h is bounded above, $|h|$ is bounded away from zero. Hence $|h| = 1$ by [9, 4.2.3] and so h is in $Z(K\hat{})$.

COROLLARY 3.7. *Suppose that $\gamma \in Z(K\hat{})$ and that E is a spectral set (respectively, Calderón set or strong Ditkin set) in $K\hat{}$ for $A(K\hat{})$. Then γE has the same property.*

To check this corollary, use Lemma 2.2: $(J_\gamma(f))\hat{} = (\hat{f})_{\overline{\gamma}}$. The next theorem is immediate from Theorem 3.3 and Corollary 3.7.

THEOREM 3.8. *Points in the center $Z(K\hat{})$ of $K\hat{}$ are strong Ditkin sets for the algebra $A(K\hat{})$.*

COROLLARY 3.9. *Countable closed subsets of $Z(K^\wedge)$ are Calderón sets for $A(K^\wedge)$. See* [5, 39.39.c].

Theorem 3.8 is best possible in the sense that the points in $K^\wedge \backslash Z(K^\wedge)$ need not even be spectral sets; see (4.5), (4.7) and (4.8). Note, however, that Ditkin's condition holds at ∞ by Theorem 2.8.

A careful reading of (39.24), (39.39) and (39.42) in [5] shows that the following general result holds.

THEOREM 3.10. *Let $\mathfrak{A}$ be a regular Banach algebra in $C_0(X)$, where X is the structure space of $\mathfrak{A}$. Suppose that $\mathfrak{A}$ satisfies Ditkin's condition at ∞, and let X_0 denote the set of x in X at which $\mathfrak{A}$ satisfies Ditkin's condition.*

(i) *If E is a closed subset of X such that $\partial E \subseteqq X_0$ and such that ∂E contains no nonvoid perfect sets, then E is a Calderón set for $\mathfrak{A}$.*

(ii) *If E is a closed nonspectral subset of X_0, then there exists a continuum of closed ideals in $\mathfrak{A}$ with zero-set E.*

COROLLARY 3.11. *If E is a closed subset of $K^\wedge$ such that $\partial E \subseteqq Z(K^\wedge)$ and ∂E contains no nonvoid perfect sets, then E is a Calderón set for $A(K^\wedge)$.*

COROLLARY 3.12. *Suppose that $K^\wedge$ is discrete at points of $K^\wedge \backslash Z(K^\wedge)$. If E is closed in $K^\wedge$ and $E \cap Z(K^\wedge)$ contains no nonvoid perfect sets, then E is a Calderón set. In particular, if $K^\wedge \backslash Z(K^\wedge)$ is discrete and $Z(K^\wedge)$ is countable, then every closed subset of $K^\wedge$ is a Calderón set.*

See Example 4.6.

COROLLARY 3.13. *If E is a closed nonspectral set in $Z(K^\wedge)$, then there exists a continuum of closed ideals in $A(K^\wedge)$ with zero-set E.*

Corollary 3.13 can fail if E is a closed subset of $K^\wedge$; see (4.5).

4. **Examples.** In this section we give examples to show that points in $K^\wedge$ need not be spectral sets. We begin with a general discussion.

Let A be a Banach algebra of continuous functions on a locally compact Hausdorff space X such that X can be considered as a subset of the structure space of A. The zero-set $Z(I)$ for a closed ideal I in A is $\{x \in X: f(x) = 0 \text{ for all } f \in I\}$. A closed subset E of X will

be called a spectral set if it is not the zero-set of two distinct closed ideals in A. For a subset S of the conjugate space A^* of A, we write $I(S)$ for the set $\{f \in A: \varphi(f) = 0 \text{ for all } \varphi \in S\}$. The set $I(S)$ need not be an ideal in A, but it will be under certain conditions on S.

Derivations 4.1. A set $D = \{d_0, d_1, \cdots, d_n\}$ in A^* is often called a system of derivations if Leibniz rule

$$d_k(fg) = \sum_{j=0}^{k} \binom{k}{j} d_j(f) d_{k-j}(g)$$

holds for $f, g \in A$, $0 \leqq k \leqq n$. Note that for $k = 0$ this requirement simply asserts that d_0 is a multiplicative linear functional on A. If d_0 is point evaluation at x_0 in X, we call D a system of derivations at x_0, and we call d_1 a point derivation at x_0.

It is easy to show that D is a system of derivations in A^*, then $I(D)$ is a closed ideal in A. In particular, if d_1 is a point derivation at x_0 in X and if d_0 denotes the evaluation map at x_0, then $I(\{d_0, d_1\})$ is a closed ideal in A. If its zero-set is $\{x_0\}$, then $\{x_0\}$ is not a spectral set. For a partial converse, see Atzmon [1, 4.5].

In order to characterize spectral sets in X, we introduce a generalization of a "set of multiplicative linear functionals."

Collectionwise multiplicative sets 4.2. A subset S of A^* is called collectionwise multiplicative if, for $\varphi \in S$ and $f, g \in A$, there exist ψ_1 and ψ_2 in S (depending on φ, f, and g) such that $\varphi(fg) = \psi_1(f)\psi_2(g)$ [2]. For a collectionwise multiplicative set S in A^*, $I(S)$ is a closed ideal in A. In fact, it can be shown that a closed subset S of X is spectral if and only if for every collectionwise multiplicative set T in A^* with $Z(I(T)) = S$ we have $I(T) = I(S)$.

There are various ways to obtain collectionwise multiplicative sets. If $D = \{d_0, d_1, \cdots, d_n\}$ is a system of derivations, then the absolutely convex hull of $\{2^n d_0, 2^{n-1} d_1, \cdots, 2d_{n-1}, d_n\}$ is collectionwise multiplicative. Clearly the union of collectionwise multiplicative sets is collectionwise multiplicative, and so is its absolutely convex closed hull. More generally, suppose that S is a collectionwise multiplicative set, that $a \geqq 1$, and that $T \subseteq A^*$ has the property that if $\varphi \in T$ and $f, g \in A$, then there exist $\psi \in S$ and complex numbers $\alpha, \beta, \gamma, \delta$ such that $\varphi(fg) = \alpha\varphi(f)\varphi(g) + \beta\varphi(f)\psi(g) + \gamma\psi(f)\varphi(g) + \delta\psi(f)\psi(g)$ and $|\alpha| + |\beta| + |\gamma| + |\delta| \leqq a$. Then $a\langle T \cup S\rangle$ is collectionwise multiplicative where $\langle T \cup S\rangle$ is the absolutely convex closed hull of $T \cup S$.

Point derivations in $M(K)$ 4.3. Now we give a general method to obtain point derivations for the algebra $M(K)$ which can be considered as a Banach algebra of bounded continuous functions on $\mathscr{C}_b(K)$ and $M(K)$ can be regarded as a Banach algebra of functions on $K^\wedge$ by [6, 6.3, 7.3E].

Let K be a commutative hypergroup such that $K^\wedge$ is a hypergroup. Let $\{\chi_\alpha\}$ be a net in $\mathscr{C}_b(K)$, $\{a_\alpha\}$ a net of complex numbers, f a nonzero bounded continuous function on K, and $\chi \in \mathscr{C}_b(K)$ such that $a_\alpha(\chi_\alpha - \chi) \to f$ and $\chi_\alpha \to \chi$ uniformly on compact subsets of K. Define $\Phi_f(\mu) = \int_K f d\mu$ for $\mu \in M(K)$, so that $\mathscr{C}_b(K)$ is embedded in the structure space of $M(K)$ via the mapping $\chi \to \Phi_\chi$. (Note that our Φ_χ is the same as Jewett's $F_{\bar{\chi}}$ [6, 6.3].) For f described above, Φ_f is a point derivation for $M(K)$ at $\chi \in \mathscr{C}_b(K)$.

We check that Φ_f is a point derivation at χ, noting first that Φ_f is clearly a nonzero continuous linear functional on $M(K)$. If $x, y \in K$, then $a_\alpha(\chi_\alpha - \chi) \to f$ uniformly on the (compact) support of $p_x * p_y$ and so

$$\begin{aligned} f(x*y) &= \int_K f d(p_x * p_y) = \lim_\alpha a_\alpha \int_K [\chi_\alpha - \chi] d(p_x * p_y) \\ &= \lim_\alpha a_\alpha [\chi_\alpha(x)\chi_\alpha(y) - \chi(x)\chi(y)] \\ &= \lim_\alpha \{a_\alpha[\chi_\alpha(x) - \chi(x)]\chi_\alpha(y) + \chi(x)a_\alpha[\chi_\alpha(y) - \chi(y)]\} \\ &= f(x)\chi(y) + \chi(x)f(y) . \end{aligned}$$

Now for $\mu, \nu \in M(K)$, we have

$$\begin{aligned} \Phi_f(\mu * \nu) &= \int_K f d(\mu * \nu) = \int_K \int_K f(x*y) d\mu(x) d\nu(y) \\ &= \int_K \int_K f(x)\chi(y) d\mu(x) d\nu(y) + \int_K \int_K \chi(x) f(y) d\mu(x) d\nu(y) \\ &= \Phi_f(\mu)\Phi_\chi(\nu) + \Phi_\chi(\mu)\Phi_f(\nu) , \end{aligned}$$

i.e., Φ_f is a point derivation for $M(K)$ at χ. Note that Φ_f can, of course, also be regarded as a point derivation for $L^1(K)$ at χ.

For a subset $\mathscr{F}$ of the space $C_b(K)$ of bounded continuous functions on K, we write $I(\mathscr{F})$ for the following closed subspace of $L^1(K)$: $\{g \in L^1(K): \Phi_f(g) = 0 \text{ for all } f \in \mathscr{F}\}$. The following simple observation will help us determine when our ideals are distinct.

LEMMA 4.4. *If $\mathscr{F}_1$ and $\mathscr{F}_2$ are distinct finite subsets of a linearly independent subset $\mathscr{F}$ of $C_b(K)$, then $I(\mathscr{F}_1) \neq I(\mathscr{F}_2)$.*

Proof. $\{\Phi_f: f \in \mathscr{F}\}$ is a linearly independent set of linear func-

tionals on $L^1(K)$. The lemma follows from an elementary fact concerning linear functionals [5, E. 12].

EXAMPLE 4.5. Let $G = \boldsymbol{R}^n$ and let B be the group of rotations in G. The study of rotation-invariant functions and measures on $\boldsymbol{R}^n$ can be viewed as a study of functions and measures on the hypergroup $K = G_B$; see (1.2). As a set, K is identified with $\boldsymbol{R}^+ = [0, \infty)$; see the discussion and references in [10, §3]. The hypergroup $K\hat{}$ is isomorphic with K and so $L^1(K)$ and $A(K)$ are isometrically isomorphic. The center of K is $\{0\}$ and this is a Calderón set by Theorem 3.8. The remaining points of K are nonspectral sets if $n \geqq 3$. This follows from the work of Reiter [7] which shows the connection between radial functions on $\boldsymbol{R}^n$ and L. Schwartz's original example of a nonsynthesis set. Varopoulos [14] has determined completely the closed ideals of $A(K)$ with zero-set $\{\rho\}$ in $(0, \infty)$.

Let us briefly cast these known results of Reiter and Varopoulos in terms of the notations in 4.1 and 4.2. Let $A^\infty(K)$ be the space of infinitely differentiable functions in $A(K)$. Fix ρ in $(0, \infty)$ and let δ^0 denote the point evaluation functional on $A(K)$: $\delta^0(f) = f(\rho)$ for $f \in A(K)$. Similarly, let δ^k denote the kth-derivative evaluated at ρ: $\delta^k(f) = f^{(k)}(\rho)$ for $f \in A^\infty(K)$. For $n \geqq 3$, δ^1 is continuous in the topology of $A(K)$ [8, Ch. 2, 6.3(4)] and can be defined on all of $A(K)$. Accordingly δ^1 is a point derivation at ρ and $\{\rho\}$ is not a spectral set as discussed in (4.1). Varopoulos [14, p. 384] shows that δ^k is continuous on $A(K)$ if and only if $k \leqq K_n$ where K_n is the greatest integer less than or equal to $(1/2)(n-1)$. For $k \leqq K_n$, the set $D_k = \{\delta^0, \delta^1, \cdots, \delta^k\}$ is a system of derivations at ρ and $I(D_k)$ is a closed ideal in $A(K)$. These ideals are distinct and they all have zero-set $\{\rho\}$. Varopoulos shows that these are the only closed ideals with zero-set $\{\rho\}$. In particular, all points in K are spectral sets if $n = 2$.

EXAMPLE 4.6. Let G be the group Δ_p of p-adic integers and let B denote the group of units in G acting on G. Then G_B is a compact hypergroup identified with the one-point compactification Z_+^* of $Z_+ = \{0, 1, 2, \cdots\}$. The hypergroup $K = G_B\hat{}$ is identified with Z_+ and $G_B = K\hat{}$; see [4] for more details. Since the center of G_B is $\{\infty\}$, Corollary 3.12 shows that every closed subset of G_B is a Calderón set. For a class of hypergroups that includes G_B, Dunkl and Ramirez [4, 10.6] prove that every subset is a spectral set.

EXAMPLE 4.7. As in [6, 15.4], let F be the hypergroup of conjugacy classes of the compact group $SU(2)$. Then $K = F\hat{}$ is a commutative discrete hypergroup and Proposition 1.4 shows that $K\hat{}$ is

a hypergroup isomorphic with F and that $\mathscr{X}_b(K) = K\hat{}$. We may take $\{0, 1, 2, \cdots\}$ as a model for K and $[0, \pi]$ as a model for $K\hat{} = F$. Specifically, $K\hat{} = \{\chi_\theta : \theta \in [0, \pi]\}$ where $\chi_\theta(n) = \sin(n+1)\theta/[(n+1)\sin\theta]$ for $\theta \in (0, \pi)$, $\chi_0(n) = 1$ and $\chi_\pi(n) = (-1)^n$ for $n = 0, 1, 2, \cdots$. Since $Z(K\hat{}) = \{\chi_0, \chi_\pi\}$, the sets $\{\chi_0\}$, $\{\chi_\pi\}$ are strong Ditkin sets.

We now show that for $\theta \in (0, \pi)$, $\{\chi_\theta\}$ is not a spectral set. Let

$$g_\theta(n-1) = \frac{d}{d\theta}(\chi_\theta(n-1)) = \frac{n\cos n\theta \sin\theta - \sin n\theta \cos\theta}{n\sin^2\theta}$$

for $n \geqq 1$. Then g_θ is a bounded (continuous) function on K and hence Φ_{g_θ} is a continuous linear functional on $M(K)$. Since $\chi_{\theta+\delta} \to \chi_\theta$ and $\delta^{-1}[\chi_{\theta+\delta} - \chi_\theta] \to g_\theta$ uniformly on compact subsets of K as $\delta \to 0$, Φ_{g_θ} is a point derivation at χ_θ by (4.3). Since $\{\chi_\theta, g_\theta\}$ is linearly independent, the closed ideals $I(\{\chi_\theta\})$ and $I(\{\chi_\theta, g_\theta\})$ are distinct by (4.4). Each of these ideals has zero-set $\{\chi_\theta\}$ and so $\{\chi_\theta\}$ is not a spectral set.

NAIMARK'S EXAMPLE 4.8. We assume familiarity with Jewett's treatment of Naimark's example $K = [0, \infty)$ [6, 9.5]. If $\operatorname{supp}(\pi)$ denotes the support of the Plancherel measure on $K\hat{}$, then $\operatorname{supp}(\pi) \subsetneqq K\hat{} \subsetneqq \mathscr{X}_b(K)$ and $1 \notin \operatorname{supp}(\pi)$. In particular, $K\hat{}$ is not a hypergroup. We will show that many points of $\mathscr{X}_b(K)$ are not spectral sets for $L^1(K)$ and, in fact, we will show that they are the zero-sets of infinitely many closed ideals.

Fix complex numbers a, b with $a = b^2$, $a \neq 0$, and $0 \leqq \operatorname{Im}(b) < 1$. For $n = 0, 1, 2, \cdots$, let

$$f_{2n}(x) = x^{2n}\sin bx/\sinh x \quad \text{and} \quad f_{2n+1}(x) = x^{2n+1}\cos bx/\sinh x\,.$$

Note that $b^{-1}f_0 = \chi_a$, an element of $\mathscr{X}_b(K)$. Each f_m is a bounded continuous function on K. So each Φ_{f_m} is a continuous linear functional on $M(K)$. Straightforward calculations show that

$$bf_m(x*y) = \sum_{j=0}^{m}\binom{m}{j}(-1)^{j(m+1)}f_j(x)f_{m-j}(y) + m(-1)^m f_{m-1}(x*y)$$

for $x, y \in K$. It follows that

$$b\Phi_{f_m}(\mu*\nu) = \sum_{j=0}^{m}\binom{m}{j}(-1)^{j(m+1)}\Phi_{f_j}(\mu)\Phi_{f_{m-j}}(\nu) + m(-1)^m\Phi_{f_{m-1}}(\mu*\nu)$$

for $\mu, \nu \in M(K)$. Hence for any $m \geqq 0$, $I_m = I(\{f_0, f_1, \cdots, f_m\})$ is a closed ideal in $L^1(K)$. Since $\{f_0, f_1, \cdots\}$ is a linearly independent set of functions on K, Lemma 4.4 shows that the ideals $I_0, I_1, \cdots$ are

all distinct. Moreover, each of these ideals has zero-set $\{\chi_a\}$ in $\mathscr{X}_b(K)$.

Similar computations show that $\{\chi_0\}$ is the zero-set of infinitely many closed ideals in $L^1(K)$. The auxiliary functions f_n in this case are defined by $f_n(x) = x^{2n+1}/\sinh x$. Note that $f_0 = \chi_0$.

We do not know the status of the identity χ_{-1} or the other points of $\mathscr{X}_b(K)$ not covered by the preceding discussion. However, we observe that no point of $\operatorname{supp}(\pi) = \{\chi_t : t \in [0, \infty)\}$ is a spectral set.

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References

1. A. Atzmon, *Spectral synthesis in regular Banach algebras*, Israel J. Math., **8** (1970), 197-212.
2. A. K. Chilana, *Topological algebras with a given dual*, Proc. Amer. Math. Soc., **42** (1974), 192-197.
3. C. F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc., **179** (1973), 331-348.
4. C. F. Dunkl and D. E. Ramirez, *A family of countable compact P_*-hypergroups*, Trans. Amer. Math. Soc., **202** (1975), 339-356.
5. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*. I and II, Springer-Verlag, 1963 and 1970.
6. R. I. Jewett, *Spaces with an abstract convolution of measures*, Advances in Math., **18** (1975), 1-101.
7. H. Reiter, *Contributions to harmonic analysis*. IV, Math. Annalen **135** (1958), 467-476.
8. ———, *Classical harmonic analysis and locally compact groups*, Oxford Mathematical Mongraphs, 1968.
9. M. A. Rieffel, *A characterization of commutative group algebras and measure algebras*, Trans. Amer. Math. Soc., **116** (1965), 32-65.
10. K. A. Ross, *Hypergroups and centers of measure algebras*, Ist. Naz. Alta Mat. (Symposia Math.), volume XXII, (1977), 189-203.
11. ———, *Centers of hypergroups*, Trans. Amer. Math. Soc., to appear.
12. R. Spector, *Apercu de la theorie des hypergroupes*, Lecture Notes in Mathematics #497. (Analyse Harmonique sur les Groupes de Lie, Sem. Nancy-Strasburg 1973-1975), Springer-Verlag.
13. ———, *Measures invariantes sur les hypergroupes*, to appear.
14. N. Th. Varopoulos, *Spectral synthesis on spheres*, Proc. Camb. Phil. Soc., **62** (1966), 379-387.
15. E. Kaniuth and D. Steiner, *On complete regularity of group algebras*, Math. Ann., **204** (1973), 305-329.
16. R. D. Mosak, *The L^1- and C^*-algebras of $[FIA]^-_B$ groups, and their representations*, Trans. Amer. Math. Soc., **163** (1972), 277-310.

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