

Pacific Journal of Mathematics

RINGS WITH QUIVERS THAT ARE TREES

KENT RALPH FULLER AND JOEL K. HAACK

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Associated with each artinian ring R are two diagrams called the left and right quivers of R . We generalize a well-known theorem on hereditary serial rings by proving that if these quivers have no closed paths then R is a factor ring of a certain ring of matrices over a division ring. It follows that the categories of finitely generated left and right R -modules are Morita dual to one another. Applying our theorem and theorems of Gabriel and Dlab and Ringel, we show how to write explicit matrix representations of all hereditary algebras of finite module type.

A *quiver* is, in the terminology of Gabriel [8], [9], a finite set of points (vertices) connected by arrows. Given an artinian ring R and a basic set of primitive idempotents e_1, \dots, e_n of R (see, for example, [1, §27]), one forms $\mathcal{Q}({}_R R)$ the *left quiver of R* (see [11]): The vertices of $\mathcal{Q}({}_R R)$ are v_1, \dots, v_n , one for each basic idempotent, with n_{ij} arrows from v_i to v_j iff Re_j/Je_j appears exactly n_{ij} times in a direct decomposition of the semisimple left R -module Je_i/J^2e_i . The right quiver $\mathcal{Q}(R_R)$ is formed similarly, with vertices v'_1, \dots, v'_n and n'_{ij} arrows from v'_i to v'_j iff e_jR/e_jJ appears exactly n'_{ij} times in a direct decomposition of e_iJ/e_iJ^2 . Note that $n'_{ij} \neq 0$ iff $n_{ji} \neq 0$. Also, R is indecomposable iff $\mathcal{Q}({}_R R)$ is connected, i.e., there is a nonoriented path from v_i to v_j for every $i, j = 1, \dots, n$.

A quiver \mathcal{Q} is called a *tree* in case it is connected and contains no cycles, i.e., in case it has a unique nonoriented path from v_i to v_j , for every i, j . Let \mathcal{Q} be such a quiver. Then the vertices of \mathcal{Q} are partially ordered by \leq , where $v_i \leq v_j$ iff there is an oriented path from v_j to v_i (or $i = j$), and we can relabel the vertices so that $v_i \not\leq v_j$ implies $i \leq j$. Having done this, we see that for any ring D , the set of matrices

$$T = \{[d_{ij}] \mid d_{ij} \in D, d_{ij} = 0 \text{ if } v_i \not\leq v_j\}$$

is a subring of the ring of upper triangular matrices over D . Moreover, if D is a division ring, then $\mathcal{Q}({}_T T) = \mathcal{Q}$, $\mathcal{Q}(T_T)$ is the dual quiver of \mathcal{Q} , and T is the unique basic *tic tac toe ring* (in the sense of Mitchell [12, §10.8]) over D with left quiver \mathcal{Q} .

Murase [14] showed that an indecomposable artinian ring whose quivers are of the form

$$v_1 \longleftarrow v_2 \longleftarrow v_3 \cdots v_{n-1} \longleftarrow v_n$$

is a factor ring of a block upper triangular matrix ring (i.e., of one whose basic ring is an upper triangular matrix ring) over a division ring. (Goldie [10] proved a similar result.) A ring with such a quiver is a serial ring, and an indecomposable hereditary artinian ring is serial iff it has quivers of this form. We extend this result, showing that any artinian ring whose quivers are trees is a factor ring of a tic tac toe ring over a division ring. As an application we also prove that such rings are self-dual in the sense that there is a Morita duality between their categories of finitely generated left and right modules.

Before proceeding to the proofs we note that, by the work of Gabriel [8], [9], and Dlab and Ringel [4], an indecomposable hereditary algebra over an algebraically closed field is of finite module type iff its quivers are Dynkin diagrams of type A_n , D_n , E_6 , E_7 , or E_8 . These diagrams are all trees, so the theorem we are about to prove allows one to apply Gabriel's argument [8] (see also [2], [11]) to show that any artinian ring with quivers of type A_n , D_n , E_6 , E_7 , or E_8 is a ring of finite module type.

LEMMA 1. *Let R be an artinian ring with e_1, \dots, e_n a basic set of primitive idempotents. If Re_i/Je_i is isomorphic to a direct summand of $J^k e_j/J^{k+1} e_j$, then in $\mathcal{Q}({}_R R)$ there is an oriented path from v_j to v_i of length k . Moreover, if in addition R is hereditary, then the converse is also true.*

Proof. We induct on k . The cases $k = 0$ and $k = 1$ follow immediately from the definition of a quiver. Now let Re_i/Je_i be isomorphic to a direct summand of $J^k e_j/J^{k+1} e_j$. Let

$$\bigoplus_{r=1}^t Re_{j_r} \xrightarrow{f} J^{k-1} e_j \longrightarrow 0$$

be a projective cover. Then f induces an epimorphism

$$\bigoplus_{r=1}^t (Je_{j_r}/J^2 e_{j_r}) \xrightarrow{\bar{f}} J^k e_j/J^{k+1} e_j \longrightarrow 0.$$

Since $\bigoplus_{r=1}^t (Je_{j_r}/J^2 e_{j_r})$ and $J^k e_j/J^{k+1} e_j$ are semisimple R/J -modules, \bar{f} splits. Thus there exists an r with Re_i/Je_i isomorphic to a direct summand of $Je_{j_r}/J^2 e_{j_r}$. By induction, there is an oriented path of length $(k-1)$ from v_j to v_{j_r} and one of length 1 from v_{j_r} to v_i , which combine to give the desired path of length k .

For the moreover part, suppose we have an oriented path

$$v_i = v_{i_k} \longleftarrow v_{i_{k-1}} \cdots v_{i_1} \longleftarrow v_{i_0} = v_j.$$

Assume that Re_{i_m}/Je_{i_m} is a direct summand of $J^m e_j/J^{m+1} e_j$ ($m < k$).

Then since R is hereditary, $J^m e_j \cong Re_{i_m} \oplus M$, some M . Thus

$$J^{m+1} e_j / J^{m+2} e_j \cong Je_{i_m} / J^2 e_{i_m} \oplus JM / J^2 M,$$

and we are done since $Re_{i_{m+1}} / Je_{i_{m+1}}$ is a direct summand of $Je_{i_m} / J^2 e_{i_m}$.

Now we are ready to prove the promised result.

THEOREM 2. *If the left and right quivers of an artinian ring R are trees, then there is an indecomposable tic tac toe ring T over a division ring D such that R is isomorphic to a factor ring of T . Moreover, $\mathcal{Q}(R) = \mathcal{Q}(T)$; and R is hereditary iff $R \cong T$.*

Proof. It is easy to see that a ring is Morita equivalent to an upper triangular tic tac toe ring over a division ring D iff it is isomorphic to a (block-upper-triangular) tic tac toe ring over D . Thus we may assume that R is basic.

Suppose that $\mathcal{Q} = \mathcal{Q}(R)$ and $\mathcal{Q}(R_R)$ are trees, and correspondingly, relabel the vertices of \mathcal{Q} and the idempotents of R as in the earlier discussion. In particular then, v_1 is minimal with respect to the partial order \leq , and hence no arrows leave v_1 .

Note that for each basic idempotent e_i , $e_i Re_i$ is a division ring since $e_i Je_i = 0$ by Lemma 1. For each pair of idempotents e_p and e_q with an arrow from v_q to v_p in \mathcal{Q} (and hence one arrow from v'_p to v'_q in $\mathcal{Q}(R_R)$), we have a left $e_p Re_p$ -right $e_q Re_q$ -bimodule $e_p Je_q$ with $\dim(e_p Re_p e_p Je_q) = 1 = \dim(e_p Je_q e_q Re_q)$. So we may choose $e_{pq} \in e_p Je_q$ with $e_{pq} \neq 0$ and define a division ring isomorphism $\sigma_{pq}: e_p Re_p \rightarrow e_q Re_q$ via $x e_{pq} = e_{pq} \sigma_{pq}(x)$ for $x \in e_p Re_p$. Since \mathcal{Q} is connected, we have $e_p Re_p \cong e_r Re_r$ for all primitive idempotents e_p and e_r . Define $e_{ii} = e_i$ and for each $v_i \leq v_j$ with oriented path

$$v_i = v_{i_0} \longleftarrow v_{i_1} \cdots v_{i_{k-1}} \longleftarrow v_{i_k} = v_j,$$

define

$$e_{ij} = e_i e_{i_0 i_1} \cdots e_{i_{k-1} i_k} e_j.$$

For $v_p \leftarrow v_q$ in \mathcal{Q} , define $\gamma_{pq} = \sigma_{pq}$ and $\gamma_{qp} = \sigma_{pq}^{-1}$. Now let $v_1 = v_{i_0}, v_{i_1}, \dots, v_{i_k} = v_j$ be the vertices of a nonoriented path from v_1 to v_j for $j \neq 1$. Define

$$\sigma_{1j} = \gamma_{i_{k-1} i_k} \circ \cdots \circ \gamma_{i_1 i_2} \circ \sigma_{i_0 i_1} \quad \text{for } j = 2, \dots, n.$$

Define $\sigma_{11} = 1_{e_1 Re_1}$. Let

$$D = \left\{ \sum_{j=1}^n \sigma_{1j}(x) \mid x \in e_1 Re_1 \right\}.$$

Then $D \cong e_j Re_j$ and D is a division subring of R .

Let $v_i \leq v_j$ via an oriented path of length k . Then $e_i Re_j = e_i J^k e_j$

by Lemma 1. It is then straightforward to verify that $De_{ij} = e_i J^k e_j$, using the equalities $De_p = e_p R e_p$ and $e_p R e_{pq} = e_p R e_p e_{pq} = e_p J e_q$ for $v_p \leftarrow v_q$. Hence we have shown that

$$(*) \quad R = \sum_{v_i \leq v_j} De_{ij}.$$

Next we claim that $de_{ij} = e_{ij}d$ for any $d \in D$ and $v_i \leq v_j$ in \mathcal{Q} . Suppose we have $v_p \leftarrow v_q$. Let $v_1 = v_{i_0}, v_{i_1}, \dots, v_{i_m} = v_q$ be the non-oriented path from v_1 to v_q . If $v_p = v_{i_{m-1}}$, then $\sigma_{1q} = \sigma_{pq} \circ \sigma_{1p}$, and

$$\begin{aligned} e_{pq} \left(\sum_{r=1}^n \sigma_{1r}(x) \right) &= e_{pq} \sigma_{1q}(x) = e_{pq} \sigma_{pq}(\sigma_{1p}(x)) = \sigma_{1p}(x) e_{pq} \\ &= \left(\sum_{r=1}^n \sigma_{1r}(x) \right) e_{pq}. \end{aligned}$$

If $v_p \neq v_{i_{m-1}}$, then $\sigma_{1p} = \sigma_{pq}^{-1} \circ \sigma_{1q}$, and

$$\begin{aligned} e_{pq} \left(\sum_{r=1}^n \sigma_{1r}(x) \right) &= e_{pq} \sigma_{1q}(x) = \sigma_{pq}^{-1}(\sigma_{1p}(x)) e_{pq} = \sigma_{1p}(x) e_{pq} \\ &= \left(\sum_{r=1}^n \sigma_{1r}(x) \right) e_{pq}. \end{aligned}$$

Now the claim follows by induction on the length of the path from v_j to v_i .

Let T be the tic tac toe ring

$$T = \{ \llbracket d_{ij} \rrbracket \mid d_{ij} \in D, d_{ij} = 0 \text{ if } v_i \not\leq v_j \}.$$

Define

$$\Phi: T \longrightarrow R \quad \text{via} \quad \Phi: \llbracket d_{ij} \rrbracket \longmapsto \sum_{v_i \leq v_j} d_{ij} e_{ij}.$$

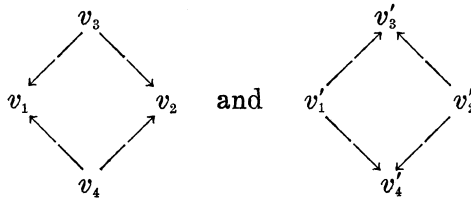
Since the elements of D commute with each e_{ij} , and since

$$e_{km} e_{pq} = \begin{cases} e_{kq} & \text{if } m = p \\ 0 & \text{if } m \neq p, \end{cases}$$

Φ is a ring homomorphism. Also Φ is onto by (*).

Clearly $\mathcal{Q}(T) = \mathcal{Q}(R)$. If R is hereditary, then for $v_i \leq v_j$ with oriented path of length k , $De_{ij} = e_i J^k e_j \neq 0$ by Lemma 1. So $e_{ij} \neq 0$ and Φ is an isomorphism. If T is a tic tac toe ring whose quivers are trees, then T is hereditary by [12, Theorem IX. 10.9].

One could apparently use an argument similar to the one in [4, Proposition 10.2] to show that the rings of Theorem 2 are factor rings of so-called tensor rings (see [5]). The same argument, however, shows that rings with quivers



are also tensor rings. But these need not be tic tac toe rings. Indeed, let φ be an automorphism of a division ring D which does not fix the center of D . Then the ring R_φ of matrices

$$\begin{pmatrix} a & 0 & x & m \\ & b & y & z \\ & & c & 0 \\ & & & d \end{pmatrix}$$

with all entries in D except $m \in {}_D M_D$ where ${}_D M = {}_D D$ and multiplication in M_D is given by $m \cdot d = m\varphi(d)$, is a tensor ring that is not a tic tac toe ring. In contrast, by Theorem 2 or originally by Murase in [14], the ring S_φ of matrices

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$$

with $a, b \in D$ and $m \in M$ is isomorphic to the ring of upper triangular 2×2 matrices over D . (A word of caution: Associativity is lost if one tries this trick for 3×3 upper triangular matrices.) The ring R_φ fails to behave similarly, for the center of R_φ is all scalar matrices cI with $\varphi(c) = c \in \text{center}(D)$.

Note also that the above example indicates that Theorem 2 does not extend to include rings whose quivers are not trees.

A gap in the Morita duality theory that begs to be filled is the nearly total lack of knowledge of which artinian rings (in addition to artin algebras and QF rings) are self-dual. The characterization of artinian rings whose quivers are trees given in Theorem 2 enables us to show that such rings are self-dual. We employ the following lemma whose proof is dual to that of [7, Lemma 4]. In what follows, $E(M)$ is the injective envelope of M and $\text{Soc}_k(M)$ is the k th term in the lower Loewy series of M .

LEMMA 3. *Let R be any ring. Then the following statements about a left R -module M are equivalent:*

(a) *M is distributive.*

(b) *For each simple left R -module T , the set of submodules $\{\ker \gamma \mid \gamma \in \text{Hom}_R(M, E(T))\}$ is linearly ordered.*

(c) For each simple left R -module T the right $\text{End}({}_R E(T))$ -module $\text{Hom}_R(M, E(T))$ is uniserial.

PROPOSITION 4. *If R is an artinian ring whose quivers are trees, then there is a Morita duality between the categories of finitely generated left and finitely generated right R -modules.*

Proof. Assume that R is indecomposable and basic with identity element a sum of orthogonal primitive idempotents $1_R = e_1 + \cdots + e_n$. Let $E_i = E(Re_i/Je_i)$ for $i = 1, \dots, n$, let $E = E_1 \oplus \cdots \oplus E_n$, and let $S = \text{End}({}_R E)$. Then $(\)^* = \text{Hom}_R({}_R _, {}_R E_S)$ defines a duality between the categories of finitely generated left R -modules and finitely generated right S -modules [13] and [7, Lemma 5]. Write $1_S = f_1 + \cdots + f_n$ where the f_i are the orthogonal primitive idempotents in S such that $Ef_i = E_i$. Let $N = J(S)$. We will show that the quivers of S are the same as the quivers of R .

From the results in [1, §24], we see that for $i = 1, \dots, n$,

$$\begin{aligned} (Re_i/Je_i)^* &\cong (\text{Soc } E_i)^* \cong f_i S/f_i N \\ (\text{Soc}_2(E_i)/\text{Soc}(E_i))^* &\cong f_i N/f_i N^2. \end{aligned}$$

So by [7, Lemma 5], $f_i N/f_i N^2$ is square free, and by [6, Theorem 2.4],

$$\begin{aligned} e_i R/e_i J &\text{ embeds in } e_j J/e_j J^2 \\ \text{iff } Re_i/Je_i &\text{ embeds in } \text{Soc}_2(E_j)/\text{Soc}(E_j) \\ \text{iff } f_i S/f_i N &\cong (Re_i/Je_i)^* \text{ embeds in } (\text{Soc}_2(E_j)/\text{Soc}(E_j))^* \\ &\cong f_j N/f_j N^2. \end{aligned}$$

Thus the right quiver of S is the same as the right quiver of R .

Now to see that the left quivers of R and S are the same we need only show that $\dim({}_{f_i S f_i} f_i S f_j) = 0$ or 1 for all i, j . But (writing maps on the right), $f_i S f_j \cong \text{Hom}_R(E_i, E_j) \cong \text{Hom}_{e_j R e_j}(e_j E_i, e_j E_j)$ by [6, Lemma 2.1]. Note that since the quivers of R are trees, Re_i and $e_i R$ are distributive R -modules for each $i = 1, \dots, n$ [3]. So by Lemma 3 and [7, Lemmas 4 and 5], ${}_{e_j R e_j} e_j E_i {}_{f_i S f_i} \cong \text{Hom}_R(Re_j, E_i)$ is left and right uniserial, so since $f_i S f_i$ is also a division ring, $e_j E_i$ is both left and right one-dimensional or zero. Now since ${}_{e_j R e_j} e_j E_j$ is also one-dimensional, it follows that ${}_{f_i S f_i} f_i S f_j$ is zero or one-dimensional. Note also that $f_i S f_j \neq 0$ iff $e_j E_i \neq 0$ iff $e_i Re_j \neq 0$. Thus R and S are isomorphic factor rings of tic tac toe rings with the same quivers over isomorphic division rings $e_i Re_i \cong f_i S f_i$.

Regarding algebras of finite module type, we conclude with

REMARK 5. Let R be an indecomposable hereditary artin algebra of finite module type which does not satisfy the hypotheses of Theorem

2. Then according to Dlab and Ringel [4], [5] the quivers of R or of its opposite ring are Dynkin diagrams of one of the types

$$\begin{array}{ll} B_n: v_n \longrightarrow \cdots \longrightarrow v_2 \longleftarrow v_1 & v'_n \longrightarrow \cdots \longrightarrow v'_2 \rightrightarrows v'_1 \\ C_n: v_1 \longleftarrow v_2 \longrightarrow \cdots \longrightarrow v_n & v'_1 \rightrightarrows v'_2 \longrightarrow \cdots \longrightarrow v'_n \\ F'_4: v_1 \longrightarrow v_2 \longleftarrow v_3 \longrightarrow v_4 & v'_1 \longrightarrow v'_2 \rightrightarrows v'_3 \longrightarrow v'_4 \\ G_2: v_1 \longleftarrow v_2 & v'_1 \rightrightarrows v'_2. \end{array}$$

Using an argument similar to that in the proof of Theorem 2, one can show that if R is an artinian ring with quivers of one of the above types, then R is a factor ring of a generalized tic tac toe ring; that is, R is isomorphic to a factor of a matrix ring with some of the entries from a division subring C of a division ring D and the other nonzero entries from D . (For example, if R is hereditary with quivers

$$\begin{aligned} \mathcal{Q}({}_R R) &= v_3 \longrightarrow v_1 \longleftarrow v_2 \longleftarrow v_4 \\ \mathcal{Q}(R_R) &= v'_3 \longleftarrow v'_1 \rightrightarrows v'_2 \longrightarrow v'_4 \end{aligned}$$

then R is isomorphic to a ring T of matrices

$$\begin{bmatrix} D & D & D & D \\ & C & 0 & C \\ & & D & 0 \\ & & & C \end{bmatrix}$$

with $\dim(D_C) = 2$.) To show this, assume that R is basic and that $\mathcal{Q}({}_R R)$ is a tree. Arrange the right and left quivers of R so that the multiple arrows point to the right. Let v'_α be the vertex of $\mathcal{Q}(R_R)$ at the tails of the multiple arrows, and let v'_β be the vertex of $\mathcal{Q}(R_R)$ at the heads of the multiple arrows. Let \mathcal{Q}_α be the subquiver of $\mathcal{Q}({}_R R)$ containing v_α and the arrows and vertices to the left of v_α , and let \mathcal{Q}_β be the subquiver of $\mathcal{Q}({}_R R)$ containing v_β and the arrows and vertices to the right of v_β . Notice that $\dim({}_{e_\alpha R e_\alpha} e_\alpha J e_\beta) = 1$ since $\mathcal{Q}({}_R R)$ is a tree. For $v_p \leftarrow v_q$ in $\mathcal{Q}({}_R R)$, let e_{pq} generate ${}_{e_p R e_p} e_p J e_q$, and define e_{ji} as before for $v_i \leq v_j$. Define $\sigma_{\alpha j}$ for $v_j \in \mathcal{Q}_\alpha$ and $\sigma_{\beta j}$ for $v_j \in \mathcal{Q}_\beta$ as in the proof of Theorem 2. Let

$$C' = \left\{ \sum_{v_j \in \mathcal{Q}_\beta} \sigma_{\beta j}(x) \mid x \in e_\beta R e_\beta \right\}.$$

Let

$$D = \left\{ \sum_{v_j \in \mathcal{Q}_\alpha} \sigma_{\alpha j}(x) \mid x \in e_\alpha R e_\alpha \right\}.$$

Define $\theta: C' \rightarrow D$ via $e_{\alpha\beta}e_{\beta}c' = \theta(c')e_{\alpha}e_{\alpha\beta}$ for $c' \in C'$. Then $C = \text{im } \theta \cong C'$. Now let

$$T = \{ \llbracket d_{ij} \rrbracket \mid d_{ij} \in D, d_{ij} = 0 \text{ if } v_i \not\leq v_j, \\ \text{and } d_{ij} \in C \text{ if } v_i \in \mathcal{Q}_{\beta} \}.$$

Then T is a ring, since if $v_k \in \mathcal{Q}_{\alpha}$ and $v_i \in \mathcal{Q}_{\beta}$, then $v_i \not\leq v_k$ (and hence $d_{ik} = 0$). So in any nonzero product $d_{ij}d_{jk}$, we must have $v_i \leq v_j$ and $v_j \leq v_k$, giving $v_i \leq v_k$, and thus either $v_i \in \mathcal{Q}_{\alpha}$, or both $v_i, v_j \in \mathcal{Q}_{\beta}$ and $d_{ij}d_{jk} \in C$. Now define

$\Phi: T \rightarrow R$ by

$$\Phi: \llbracket d_{ij} \rrbracket \longmapsto \sum_{v_i \leq v_j, v_i \in \mathcal{Q}_{\beta}} \theta^{-1}(d_{ij})e_{ij} + \sum_{v_i \leq v_j, v_i \in \mathcal{Q}_{\alpha}} d_{ij}e_{ij}.$$

The map Φ is clearly additive and onto. To show that Φ preserves the multiplication, we need only add to the proof of Theorem 2 that for $d \in D$ and $c \in C$,

$$de_{\alpha\beta}\theta^{-1}(c)e_{\beta k} = dce_{\alpha\beta}e_{\beta k} = dce_{\alpha k},$$

which is immediate by the definition of θ .

ACKNOWLEDGMENT. We wish to thank E. L. Green whose talks at The University of Iowa this fall were a principal source of inspiration for this work. Also we acknowledge that the first part of Lemma 1 was already known to him.

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Received January 18, 1977.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).

8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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Stephanie Brewster Brewer Taylor Alexander, <i>Local and global convexity in complete Riemannian manifolds</i>	283
Claudi Alsina i Català, <i>On countable products and algebraic convexifications of probabilistic metric spaces</i>	291
Joel David Berman and George Grätzer, <i>Uniform representations of congruence schemes</i>	301
Ajit Kaur Chilana and Kenneth Allen Ross, <i>Spectral synthesis in hypergroups</i>	313
David Mordecai Cohen and Howard Leonard Resnikoff, <i>Hermitian quadratic forms and Hermitian modular forms</i>	329
Frank Rimi DeMeyer, <i>Metabelian groups with an irreducible projective representation of large degree</i>	339
Robert Ellis, <i>The Furstenberg structure theorem</i>	345
Heinz W. Engl, <i>Random fixed point theorems for multivalued mappings</i>	351
William Andrew Ettl, <i>On arc length sharpenings</i>	361
Kent Ralph Fuller and Joel K. Haack, <i>Rings with quivers that are trees</i>	371
Kenneth R. Goodearl, <i>Centers of regular self-injective rings</i>	381
John Gregory, <i>Numerical algorithms for oscillation vectors of second order differential equations including the Euler-Lagrange equation for symmetric tridiagonal matrices</i>	397
Branko Grünbaum and Geoffrey Shephard, <i>Isotoxal tilings</i>	407
Myron Stanley Henry and Kenneth Leroy Wiggins, <i>Applications of approximation theory to differential equations with deviating arguments</i>	431
Mark Jungerman, <i>The non-orientable genus of the n-cube</i>	443
Robert Richard Kallman, <i>Only trivial Borel measures on S_∞ are quasi-invariant under automorphisms</i>	453
Joyce Longman and Michael Rich, <i>Scalar dependent algebras in the alternative sense</i>	463
Richard A. Mollin, <i>The Schur group of a field of characteristic zero</i>	471
David Pokrass, <i>Some radical properties of rings with $(a, b, c) = (c, a, b)$</i>	479
Margaret Shay and Paul Ruel Young, <i>Characterizing the orders changed by program translators</i>	485
Jerrold Norman Siegel, <i>On the structure of $B_\infty(F)$, F a stable space</i>	491
Surjeet Singh, <i>(hnp)-rings over which every module admits a basic submodule</i>	509
A. K. Snyder, <i>Universal interpolating sets and the Nevanlinna-Pick property in Banach spaces of functions</i>	513
Jeffrey D. Vaaler, <i>On the metric theory of Diophantine approximation</i>	527
Roger P. Ware, <i>When are Witt rings group rings? II</i>	541