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This paper is concerned with calculating centers of regular self-injective rings, particularly those obtained by completions with respect to rank functions, and those obtained as factor rings of other regular self-injective rings. Sufficient conditions are developed under which the completion of a regular ring R has the same center as R. For any regular self-injective ring R of Type I_f, it is shown that the center of any factor ring of R is a factor ring of the center of R. These results are used to distinguish among the simple regular self-injective rings of Type II_f by their possible centers.

All rings in this paper are associative with unit, and all ring maps are assumed to preserve the unit.

1. Introduction. The class of regular, right self-injective rings may be divided into subclasses using the theory of types as in [6, Chapters 5-7]. In particular, any indecomposable, regular, right self-injective ring must be one of Types I_f, I_∞, II_f, II_∞, or III [6, Corollary 7.6]. The indecomposable, regular, right self-injective rings of Types I_f and I_∞ are easy enough to describe, since those of Type I_f are the simple artinian rings, while those of Type I_∞ are the endomorphism rings of infinite-dimensional right vector spaces [6, Theorem 5.4].

In the remaining cases, however, very little is known. The center suggests itself as a reasonable invariant with which to distinguish among different rings of the same type, particularly in the indecomposable case, where the center is a field. In this paper, we develop some techniques for calculating centers, and we apply these techniques to the standard Type II examples (described below). In particular, we show that any field can be the center of a simple, regular, right and left self-injective ring of Type II, and that the standard Type II examples can be distinguished by means of their centers.

Both of the standard Type II examples are built up from certain sequences of simple artinian rings, one by the completion of a direct limit, the other by a factor ring of the direct product. The second of these is the easiest to describe, as follows. Let $R_1, R_2, \cdots$ be simple artinian rings whose composition series lengths are unbounded, and set $R = \Pi R_n$. If M is any maximal two-sided ideal of R which contains $\bigoplus R_n$, then it follows from [6, Corollary

11.10] that R/M is a simple, regular, right and left self-injective ring of Type II.

To describe the other example, we outline the completion process as developed in [8, 4, 5]. A *pseudo-rank function* on a regular ring R [4, p. 269] is a map $N: R \to [0, 1]$ such that

(a) $N(1) = 1$.

(b) $N(xy) \leqq N(x), N(y)$ for all $x, y \in R$.

(c) $N(e + f) = N(e) + N(f)$ for all orthogonal idempotents $e, f \in R$.

A *rank function* on R [13, p. 231] is a pseudo-rank function N such that

(d) $N(x) > 0$ for all nonzero $x \in R$.

It follows from (b) and (c) that $N(x + y) \leqq N(x) + N(y)$ for all $x, y \in R$ [13, Corollary, p. 231].

Any pseudo-rank function N on R induces a pseudo-metric δ on R according to the rule $\delta(x, y) = N(x - y)$, and the ring operations on R are uniformly continuous with respect to δ [13, pp. 231, 232]. Thus the (Hausdorff) completion of R with respect to δ is a ring $\bar{R}$, which we refer to as *the N-completion of R*. Note that the kernel of the natural map $R \to \bar{R}$ equals the kernel of N. According to [8, Theorem 3.7], $\bar{R}$ is a regular ring, N extends continuously to a rank function $\bar{N}$ on $\bar{R}$, and $\bar{R}$ is complete with respect to $\bar{N}$. Also, $\bar{R}$ is right and left self-injective by [4, Corollary 15].

Given a simple regular ring R with a rank function N, the N-completion $\bar{R}$ of R need not be a simple ring or even indecomposable, as [4, Examples B, C] show. Necessary and sufficient conditions for $\bar{R}$ to be simple are given in [4, Corollary 20]. In particular, [4, Corollary 21] says that if N is unique then $\bar{R}$ is simple.

The remaining Type II example is now constructed as in the following proposition.

PROPOSITION 1.1. *Let $R_1 \to R_2 \to \cdots$ be a sequence of simple artinian rings and ring maps such that the composition series lengths of the R_n are unbounded, and set $R = \varinjlim R_n$. Then there exists a unique rank function N on R, and the N-completion of R is a simple, regular, right and left self-injective ring of Type II.*

Proof. According to [4, Proposition 2], there is a unique rank function on each R_n. Inasmuch as rank functions on R are induced by compatible rank functions on the R_n, it follows that there is a unique rank function N on R.

Let $\bar{R}$ denote the N-completion of R. Then $\bar{R}$ is regular by

[8, Theorem 3.7] and [4, Corollary 15] says that $\bar{R}$ is right and left self-injective. Since N is unique, [4, Corollary 21] says that $\bar{R}$ is simple.

Inasmuch as the composition series lengths of the R_n are unbounded, we infer that there exists an infinite sequence of nonzero orthogonal idempotents in R. Since R is simple, the natural map $R \to \bar{R}$ is injective, hence $\bar{R}$ contains an infinite sequence of nonzero orthogonal idempotents. Thus $\bar{R}$ is not artinian. Consequently, we conclude as in [6, pp. 33, 34] that $\bar{R}$ is Type II.

We conclude this section by showing that any field can be the center of a regular, right self-injective ring of Type III. This result was proved by Handelman after seeing a preliminary version of this paper.

PROPOSITION 1.2 (Handelman). *Let F be any field. Then there exists a simple, regular, right self-injective ring R of Type* III *such that* $\text{center}(R) \cong F$.

Proof. Set $S_n = M_{2^n}(F)$ for all $n = 1, 2 \cdots$. Map each $S_n \to S_{n+1}$ along the diagonal, i.e., map $x \mapsto \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$, and set $S = \varinjlim S_n$. Note that S is a simple regular ring, and that $\text{center}(S) \cong F$.

Now let R be the maximal right quotient ring of S. Since S is a simple regular ring, we see that R is a simple, regular, right self-injective ring. Given any $x \in \text{center}(R)$, we see that $J = \{s \in S \mid xs \in S\}$ is a nonzero two-sided ideal of S, whence $J = S$ and so $x \in S$. Consequently, $\text{center}(R) = \text{center}(S) \cong F$.

In [7, Example (e), pp. 831, 832], it is shown that R is directly infinite. Since R is simple, it follows that R contains no nonzero directly finite idempotents. Therefore R is Type III.

2. Completions. This section is concerned with calculating the center of the completion $\bar{R}$ of a regular ring R with respect to a pseudo-rank function, and in particular with conditions which ensure that the center of $\bar{R}$ coincides with the center of R. As our methods deal with direct limits of semisimple (artinian) rings, we begin with the case where R equals the ring $M_n(D)$ of all $n \times n$ matrices over a division ring D. Our method, which is an extension of [9, Theorem 5], involves comparing the ranks of additive commutators $xy - yx$ with the ranks of differences $x - z$, where $z \in \text{center}(R)$. (By the *rank* of a matrix $x \in M_n(D)$, we mean the number of linearly independent rows (or columns) of x.)

In order to construct additive commutators in $M_n(D)$ with suitably large ranks, we use the rational canonical form for matrices in $M_n(D)$, which works as well over division rings as over fields [10, p. 50]. Specifically, any matrix in $M_n(D)$ is similar to a block diagonal matrix where each block is a *companion matrix*, i.e., a matrix of the form

$$\left(\begin{array}{c|cccc} 0 & 1 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ \hline * & * & * & \cdots & * \end{array}\right).$$

LEMMA 2.1. *Let D be a division ring, let $n \geqq 2$ be an integer, and let $x \in M_n(D)$ be a companion matrix. Then there exists $y \in M_n(D)$ such that $xy - yx$ is invertible.*

Proof. If n is even, set

$$y = \left(\begin{array}{cccccccc|c} 0 & 0 & & & \cdots & & & 0 & 0 \\ \hline 1 & & & & & & & & 0 \\ & 0 & & & & & & & 0 \\ & & 1 & & & & & & \\ & & & 0 & & & & & \vdots \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 0 & & \\ & & & & & & & 1 & 0 \end{array}\right).$$

In this case, we compute that $xy - yx$ has the form

$$\left(\begin{array}{ccccc|c} 1 & & & & & 0 \\ & 0 & & & & 0 \\ & & 1 & & & \\ & & & 0 & & \vdots \\ & & & & \ddots & \\ & & & & 1 & 0 \\ \hline * & * & & \cdots & * & 0 \end{array}\right) - \left(\begin{array}{c|ccccc} 0 & 0 & 0 & & \cdots & 0 \\ \hline 0 & 1 & & & & \\ 0 & & 0 & & & \\ & & & 1 & & \\ \vdots & & & & 0 & \\ & & & & & \ddots \\ 0 & & & & & 1 \end{array}\right) = \left(\begin{array}{ccccc|c} 1 & & & & & 0 \\ & -1 & & & & 0 \\ & & 1 & & & \\ & & & -1 & & \vdots \\ & & & & \ddots & \\ & & & & 1 & 0 \\ \hline * & * & & \cdots & * & -1 \end{array}\right)$$

which is clearly invertible.

If n is odd, set

$$y = \left(\begin{array}{ccccccccc|c}
1 & & & & & & & & & 0 \\
 & 0 & & & & & & & & 0 \\
 & & 1 & & & & & & & \\
 & & & 0 & & & & & & \vdots \\
 & & & & \ddots & & & & & \\
 & & & & & & 1 & & & \\
 & & & & & & & 0 & & 0 \\
\hline
1 & 0 & 0 & \cdots & & & 0 & 1 & & 0
\end{array}\right).$$

In this case, we compute that $xy - yx$ has the form

$$\left(\begin{array}{c|cccccc|c}
0 & 0 & & & & & & 0 \\
0 & & 1 & & & & & 0 \\
 & & & 0 & & & & \\
\vdots & & & & 1 & & & \vdots \\
 & & & & & \ddots & & \\
0 & & & & & & 0 & 0 \\
\hline
1 & 0 & 0 & \cdots & & 0 & 1 & 0 \\
* & * & * & \cdots & & * & * & 0
\end{array}\right) - \left(\begin{array}{c|cccccc|c}
0 & 1 & & & & & & 0 \\
0 & & 0 & & & & & 0 \\
 & & & 1 & & & & \\
\vdots & & & & 0 & & & \vdots \\
 & & & & & \ddots & & \\
0 & & & & & & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & \cdots & & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & & 0 & 1
\end{array}\right) = \left(\begin{array}{c|cccccc|c}
0 & -1 & & & & & & 0 \\
0 & & 1 & & & & & 0 \\
 & & & -1 & & & & \\
\vdots & & & & 1 & & & \vdots \\
 & & & & & \ddots & & \\
0 & & & & & & -1 & 0 \\
\hline
1 & 0 & 0 & \cdots & & 0 & 1 & 0 \\
* & * & * & \cdots & & * & * & -1
\end{array}\right)$$

which we see is invertible (rearrange the columns in the order $2, 3, \cdots, n-1, 1, n$ to obtain a triangular matrix).

LEMMA 2.2. *Let D be a division ring, and let n be a positive integer. Let $x \in M_n(D)$ be a diagonal matrix with diagonal entries $a_1, b_1, a_2, b_2, \cdots, a_t, b_t, a_{t+1}, a_{t+1}, \cdots, a_{t+1}$ such that $a_i \neq b_i$ for all $i=1, \cdots, t$.*

(a) *If $a_{t+1} \notin$ center(D), then there exists $y \in M_n(D)$ such that $xy - yx$ is invertible.*

(b) *If $a_{t+1} \in$ center(D), then there exists $y \in M_n(D)$ such that* rank$(xy - yx) = 2t$.

Proof. It clearly suffices to show that if $z = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_2(D)$ with $a \neq b$, then there exists $w \in M_2(D)$ such that $zw - wz$ is invertible. For this we need only take $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, since then $zw - wz =$

$\begin{bmatrix} 0 & a-b \\ b-a & 0 \end{bmatrix}$.

PROPOSITION 2.3. *Let D be a division ring, let n be a positive integer, and let $x \in M_n(D)$. Then there exist matrices $y \in M_n(D)$ and $z \in \text{center}(M_n(D))$ such that* $\text{rank}(x - z) \leqq \text{rank}(xy - yx)$.

Proof. By a change of basis, we may assume that x is in rational canonical form. Then, by a permutation of the basis, we may put x in the form $\begin{bmatrix} x' & 0 \\ 0 & x'' \end{bmatrix}$, where x' is an $n' \times n'$ diagonal matrix and x'' is an $n'' \times n''$ block diagonal matrix whose diagonal blocks are companion matrices of degree at least 2. In view of Lemma 2.1, there exists an $n'' \times n''$ matrix y'' over D such that $x''y'' - y''x''$ is invertible.

We may arrange the diagonal entries of x' in the order $a_1, b_1, a_2, b_2, \cdots, a_t, b_t, a_{t+1}, a_{t+1}, \cdots, a_{t+1}$ with $a_i \neq b_i$ for all $i = 1, \cdots, t$. If $a_{t+1} \notin \text{center}(D)$, then by Lemma 2.2 there exists an $n' \times n'$ matrix y' over D such that $x'y' - y'x'$ is invertible. In this case, set $y = \begin{bmatrix} y' & 0 \\ 0 & y'' \end{bmatrix}$ in $M_n(D)$, so that $xy - yx$ is invertible. Setting $z = 0$, we obtain $\text{rank}(x - z) \leqq n = \text{rank}(xy - yx)$, as desired.

Now assume that $a_{t+1} \in \text{center}(D)$, and define $z \in \text{center}(M_n(D))$ to be the diagonal matrix with all diagonal entries equal to a_{t+1}. Note that $\text{rank}(x - z) \leqq 2t + n''$. According to Lemma 2.2, there exists an $n' \times n'$ matrix y' over D such that $\text{rank}(x'y' - y'x') = 2t$. Setting $y = \begin{bmatrix} y' & 0 \\ 0 & y'' \end{bmatrix}$ in $M_n(D)$, we conclude that $\text{rank}(xy - yx) = 2t + n'' \geqq \text{rank}(x - z)$, as desired.

DEFINITION. Given modules A and B, we write $A \lesssim B$ to mean that A is isomorphic to a submodule of B. Given elements x and y in a regular ring R, we note that $xR \lesssim yR$ if and only if $x = ayb$ for some $a, b \in R$.

COROLLARY 2.4. *Let R be a semisimple ring, and let $x \in R$. Then there exist elements $y \in R$ and $z \in \text{center}(R)$ such that $(x-z)R \lesssim (xy - yx)R$.*

Proof. It suffices to consider the case when R is simple artinian, hence we may assume that $R = M_n(D)$ for some positive integer n and some division ring D. According to Proposition 2.3, there exist $y \in R$ and $z \in \text{center}(R)$ such that $\text{rank}(x - z) \leqq \text{rank}(xy - yx)$. As a result, we see that the composition series length of $(x - z)R$ is less than or equal to the composition series length of $(xy - yx)R$. Consequently, we conclude that $(x - z)R \lesssim (xy - yx)R$.

COROLLARY 2.5. *Let R be a directed union of semisimple subrings R_i, and assume that* $\text{center}(R_i) \subseteqq \text{center}(R_j)$ *whenever* $R_i \subseteqq R_j$. *Given any $x \in R$, there exist elements $y \in R$ and $z \in \text{center}(R)$ such that $(x - z)R \lesssim (xy - yx)R$.*

Proof. We have $x \in R_k$ for some k, hence by Corollary 2.4 there exist $y \in R_k$ and $z \in \text{center}(R_k)$ such that $(x-z)R_k \lesssim (xy-yx)R_k$. Since $\text{center}(R_i) \subseteqq \text{center}(R_j)$ whenever $R_i \subseteqq R_j$, we see that $z \in \text{center}(R)$. Inasmuch as R is a flat left R_k-module, we conclude that $(x - z)R \lesssim (xy - yx)R$.

THEOREM 2.6. *Let R be a directed union of semisimple subrings R_i, and assume that* $\text{center}(R_i) \subseteqq \text{center}(R_j)$ *whenever* $R_i \subseteqq R_j$. *Let N be a pseudo-rank function on R, let $\bar{R}$ denote the N-completion of R, and let $\phi: R \to \bar{R}$ be the natural map. Then* $\phi(\text{center}(R))$ *is dense in* $\text{center}(\bar{R})$.

Proof. Let $\bar{N}$ be the natural extension of N to $\bar{R}$. Let $x \in \text{center}(\bar{R})$, and let $\varepsilon > 0$ be a real number.

Choose $w \in R$ such that $\bar{N}(\phi(w) - x) < \varepsilon/3$. According to Corollary 2.5, there exist $y \in R$ and $z \in \text{center}(R)$ such that $(w - z)R \lesssim (wy - yw)R$. Then $w - z = a(wy - yw)b$ for some $a, b \in R$, from which we see that $N(w - z) \leqq N(wy - yw)$. Since x commutes with $\phi(y)$, we obtain

$$\begin{aligned} N(w - z) &\leqq N(wy - yw) = \bar{N}(\phi(w)\phi(y) - \phi(y)\phi(w)) \\ &= \bar{N}((\phi(w) - x)\phi(y) - \phi(y)(\phi(w) - x)) \\ &\leqq 2\bar{N}(\phi(w) - x) < 2\varepsilon/3 \,, \end{aligned}$$

and consequently

$$\begin{aligned} \bar{N}(\phi(z) - x) &\leqq \bar{N}(\phi(z) - \phi(w)) + \bar{N}(\phi(w) - x) \\ &< N(w - z) + (\varepsilon/3) < \varepsilon \,. \end{aligned}$$

Therefore $\phi(\text{center}(R))$ is dense in $\text{center}(\bar{R})$.

COROLLARY 2.7. *Let R be a directed union of semisimple subrings R_i, and assume that* $\text{center}(R_i) \subseteqq \text{center}(R_j)$ *whenever* $R_i \subseteqq R_j$. *Let N be a rank function on R, let $\bar{R}$ denote the N-completion of R, and assume that $\bar{R}$ is indecomposable (as a ring). Then the natural map* $\text{center}(R) \to \text{center}(\bar{R})$ *is an isomorphism.*

Proof. Since N is a rank function, we see that the natural map $\phi: R \to \bar{R}$ must be injective. Let $\bar{N}$ be the natural extension of N to $\bar{R}$. According to Theorem 2.6, $\phi(\text{center}(R))$ is dense in center$(\bar{R})$ in the $\bar{N}$-metric. Since $\bar{R}$ is an indecomposable regular ring, center$(\bar{R})$ is a field. Then for any $x \in \text{center}(\bar{R})$, either $x = 0$ or x is invertible, hence either $\bar{N}(x) = 0$ or $\bar{N}(x) = 1$. Thus the $\bar{N}$-metric on center$(\bar{R})$ is discrete, hence we conclude that $\phi(\text{center}(R)) = \text{center}(\bar{R})$. Therefore ϕ restricts to an isomorphism of center(R) onto center$(\bar{R})$.

Corollary 2.7 does not remain valid without the compatibility condition on the centers of the R_i, as Example 2.10 shows.

The calculation of the center of the completion of a direct limit of semisimple rings with respect to a rank function was first performed by Alexander in the following special case [1, Theorem 12.8], using a fairly involved procedure. Specializing this case to the situation where D is a field of characteristic zero, Handelman developed a relatively short proof in [9, Proposition 6].

THEOREM 2.8. *Let D be a division ring, let $n(1) < n(2) < \cdots$ be positive integers such that $n(k) \mid n(k+1)$ for all k, and set $R_k = M_{n(k)}(D)$ for all k. Map each $R_k \to R_{k+1}$ by the obvious block diagonal map, and set $R = \varinjlim R_k$. Then*

(a) *There exists a unique rank function N on R.*

(b) *The N-completion $\bar{R}$ of R is a simple, regular, right and left self-injective ring of Type* II.

(c) *The natural map* center$(D) \to$ center$(\bar{R})$ *is an isomorphism.*

Proof. (a) and (b) are given by Proposition 1.1.

(c) It is clear that the natural map center$(D) \to$ center(R) is an isomorphism. Since the maps center$(R_k) \to$ center(R_{k+1}) are isomorphisms for all k, we conclude from Corollary 2.7 that the natural map center$(R) \to$ center$(\bar{R})$ is an isomorphism. Thus the natural map center$(D) \to$ center$(\bar{R})$ must be an isomorphism.

COROLLARY 2.9. *Let F be any field. Then there exists a simple, regular, right and left self-injective ring $\bar{R}$ of Type* II *such that* center$(\bar{R}) \cong F$.

Proof. Set $D = F$ and $n(k) = 2^k$ for all k, and construct $\bar{R}$ as in Theorem 2.8.

We close this section with an example which shows that Theorem 2.6 and Corollary 2.7 may fail if we do not assume that center$(R_i) \subseteq$ center(R_j) whenever $R_i \subseteq R_j$.

EXAMPLE 2.10. There exists a simple regular ring R such that

(a) R is the direct limit of a sequence $R_1 \to R_2 \to \cdots$ of simple artinian rings and ring maps.

(b) There exists a unique rank function N on R.

(c) The N-completion $\bar{R}$ of R is a simple, regular, right and left self-injective ring of Type II.

(d) center$(R) \cong \boldsymbol{R}$ but center$(\bar{R}) \cong \boldsymbol{C}$.

Proof. (a) Set $s(n) = 2^{n(n+1)/2}$ and $R_n = M_{s(n)}(\boldsymbol{C})$ for all $n = 1, 2, \cdots$. Define $\boldsymbol{R}$-algebra maps $\phi_n: R_n \to R_{n+1}$ according to the rule

$$\phi_n(x) = \begin{pmatrix} x & & & & & & \\ & x & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & x & \\ & & & & & & \bar{x} \end{pmatrix},$$

where $\bar{x}$ denotes the conjugate (not transposed) of the matrix x. Set $R = \varinjlim R_n$, and for all n let $\psi_n: R_n \to R$ be the natural map.

(b) For each n, there is a unique rank function P_n on R_n, given by the rule $P_n(x) = \text{rank}(x)/s(n)$. Observing that $P_{n+1}\phi_n = P_n$ for all n, we see that there is a rank function N on R such that $N\psi_n = P_n$ for all n. Since the P_n are unique, N must be unique.

(c) is given by Proposition 1.1.

(d) It is clear that center$(R) \cong \boldsymbol{R}$. Since N is a rank function, we may identify R with its image in $\bar{R}$. Let $\bar{N}$ be the natural extension of N to $\bar{R}$.

For all n, let $\theta_n: \boldsymbol{C} \to R_n$ be the natural isomorphism of $\boldsymbol{C}$ onto the center of R_n. Given any $x \in \boldsymbol{C}$, we claim that the sequence $\{\psi_n\theta_n(x)\} \subseteq R$ is Cauchy with respect to N. For each n, we see that $\theta_{n+1}(x) - \phi_n\theta_n(x)$ is a diagonal matrix with 0 for the first $s(n+1) - s(n)$ diagonal entries and $x - \bar{x}$ for the remaining $s(n)$ diagonal entries, whence rank $(\theta_{n+1}(x) - \phi_n\theta_n(x)) \leqq s(n)$. Consequently,

$$\begin{aligned} N(\psi_{n+1}\theta_{n+1}(x) - \psi_n\theta_n(x)) &= N\psi_{n+1}(\theta_{n+1}(x) - \phi_n\theta_n(x)) \\ &= P_{n+1}(\theta_{n+1}(x) - \phi_n\theta_n(x)) \leqq s(n)/s(n+1) = 1/2^{n+1}. \end{aligned}$$

As a result, we see for all $k > n$ that

$$\begin{aligned} N(\psi_k\theta_k(x) - \psi_n\theta_n(x)) &\leqq \sum_{j=n}^{k-1} N(\psi_{j+1}\theta_{j+1}(x) - \psi_j\theta_j(x)) \\ &\leqq \sum_{j=n}^{k-1} 1/2^{j+1} < 1/2^n. \end{aligned}$$

Therefore $\{\psi_n\theta_n(x)\}$ is indeed Cauchy with respect to N, hence there is a unique element $\gamma(x)\varepsilon\bar{R}$ such that $\psi_n\theta_n(x)\to\gamma(x)$ in the $\bar{N}$-metric. Inasmuch as $N(\psi_n\theta_n(x)-\psi_k\theta_k(x))<1/2^n$ for all $k>n$, we see that $\bar{N}(\psi_n\theta_n(x)-\gamma(x))\leqq 1/2^n$ for all n.

We now have a map $\gamma: \boldsymbol{C}\to\bar{R}$, and it is clear that γ is an injective ring map. We claim that $\gamma(\boldsymbol{C})=\text{center}(\bar{R})$.

In order to prove that $\gamma(\boldsymbol{C})\subseteqq\text{center}(\bar{R})$, it suffices to show that for any $x\in\boldsymbol{C}$, $\gamma(x)$ commutes with any $y\in R$. Choose k such that $y\in\psi_k(R_k)$. Given any $n\geqq k$, we have $y=\psi_n(z_n)$ for some $z_n\in R_n$. Since $\theta_n(x)$ commutes with z_n, we see that $\psi_n\theta_n(x)$ commutes with y. Taking limits, we find that $\gamma(x)$ commutes with y, as desired.

Finally, consider any $w\in\text{center}(\bar{R})$. There exists $x\in R_n$ for some n such that $\bar{N}(\psi_n(x)-w)<1/8$. According to Corollary 2.4, there exist elements $y\in R_n$ and $z\in\text{center}(R_n)$ such that $(x-z)R_n\lesssim(xy-yx)R_n$, whence $P_n(x-z)\leqq P_n(xy-yx)$. Since w commutes with $\psi_n(y)$, we obtain

$$\begin{aligned}P_n(x-z)\leqq P_n(xy-yx)&=N(\psi_n(x)\psi_n(y)-\psi_n(y)\psi_n(x))\\&=\bar{N}((\psi_n(x)-w)\psi_n(y)-\psi_n(y)(\psi_n(x)-w))\\&\leqq 2\bar{N}(\psi_n(x)-w)<1/4\,.\end{aligned}$$

In addition, we have $z=\theta_n(t)$ for some $t\in\boldsymbol{C}$, hence

$$\bar{N}(\psi_n(x)-\psi_n\theta_n(t))=N\psi_n(x-z)=P_n(x-z)<1/4\,.$$

Recalling that $\bar{N}(\psi_n\theta_n(t)-\gamma(t))\leqq 1/2^n$, we find that

$$\begin{aligned}\bar{N}(w-\gamma(t))\leqq\bar{N}(w-\psi_n(x))+\bar{N}(\psi_n(x)-\psi_n\theta_n(t))+\bar{N}(\psi_n\theta_n(t)-\gamma(t))\\<(1/8)+(1/4)+(1/2^n)<1\,.\end{aligned}$$

Consequently, $w-\gamma(t)$ is a noninvertible element of the field center $(\bar{R})$, hence $w-\gamma(t)=0$. Thus $w\in\gamma(\boldsymbol{C})$.

Therefore $\gamma(\boldsymbol{C})=\text{center}(\bar{R})$ as claimed, so that $\boldsymbol{C}\cong\text{center}(\bar{R})$.

3. Factor rings. This section is concerned with calculating the centers of factor rings of regular, right self-injective rings.

DEFINITION. For any ring R, we use $B(R)$ to stand for the Boolean algebra of all central idempotents of R. If R is regular and right self-injective, then $B(R)$ is complete by [6, Proposition 4.1].

LEMMA 3.1. *Let R be a regular, right self-injective ring, and let $X\subseteqq R$. Let $Y\subseteqq B(R)$, and set $f=\vee Y$. If $xe=0$ for all $x\in X$ and all $e\in Y$, then $xf=0$ for all $x\in X$.*

Proof. Let J be the right annihilator of X, and note that $(R/J)_R$ is nonsingular. According to [6, Proposition 4.1], $\sum_{e\in Y} eR$ is an essential right R-submodule of fR. Inasmuch as $\sum_{e\in Y} eR \leqq J$, we conclude that $fR \leqq J$, so that $xf = 0$ for all $x \in X$.

LEMMA 3.2. *Let R be a regular, right self-injective ring. Given any $x \in R$, there exist elements $y \in R$ and $f \in B(R)$ such that $fx \in$ center(R) and $(xy - yx)e \neq 0$ for all nonzero $e \leqq 1 - f$ in $B(R)$.*

Proof. Set $Y = \{e \in B(R) \mid (xt - tx)e = 0$ for all $t \in R\}$ and $f = \vee Y$. According to Lemma 3.1, $(fx)t - t(fx) = (xt - tx)f = 0$ for all $t \in R$, hence $fx \in$ center(R).

Let W be the set of those $g \in B(R)$ for which there exists an element $t \in R$ such that $(xt - tx)e \neq 0$ for all nonzero $e \leqq g$ in $B(R)$. We claim that every nonzero $h \leqq 1 - f$ in $B(R)$ must lie above some nonzero member of W.

Now $h \nleqq f$ and so $h \notin Y$, hence $(xt - tx)h \neq 0$ for some $t \in R$. Setting $k = \vee \{e \in B(R) \mid (xt - tx)e = 0\}$, we see by Lemma 3.1 that $(xt - tx)k = 0$. Then $h \nleqq k$, hence $g = h - hk$ is a nonzero member of $B(R)$. It is clear that $g \leqq h$, and that $(xt - tx)e \neq 0$ for all nonzero $e \leqq g$ in $B(R)$, whence $g \in W$.

Thus every nonzero $h \leqq 1 - f$ in $B(R)$ lies above some nonzero member of W, as claimed. Consequently, we infer that there exist orthogonal idempotents $g_i \in W$ such that $\vee g_i = 1 - f$. For each i, there exists an element $y_i \in R$ such that $(xy_i - y_ix)e \neq 0$ for all nonzero $e \leqq g_i$ in $B(R)$. Since R is right self-injective, there exists $y \in R$ such that $yg_i = y_ig_i$ for all i. Given any nonzero $e \leqq 1-f$ in $B(R)$, we have $eg_i \neq 0$ for some i, whence $(xy-yx)eg_i = (xy_i - y_ix)eg_i \neq 0$. Therefore $(xy - yx)e \neq 0$ for all nonzero $e \leqq 1 - f$ in $B(R)$.

THEOREM 3.3. *Let R be a regular, right self-injective ring, and let P be a minimal prime ideal of R. Then the natural map* center$(R) \to$ center(R/P) *is surjective.*

Proof. Given any $x \in R$ such that $\bar{x} \in$ center(R/P), we must show that $\bar{x} = \bar{z}$ in R/P for some $z \in$ center(R). According to Lemma 3.2, there exist $y \in R$ and $f \in B(R)$ such that $fx \in$ center(R) and $(xy - yx)e \neq 0$ for all nonzero $e \leqq 1 - f$ in $B(R)$.

Since $\bar{x} \in$ center(R/P), we obtain $xy - yx \in P$. According to [3, Theorem 2.3], $P = [P \cap B(R)]R$, from which it follows that $xy - yx = g(xy - yx)$ for some $g \in P \cap B(R)$. Consequently, $(xy - yx)(1 - g) = 0$, which implies that $(1 - g)(1 - f) = 0$. Thus $1 - f = g(1 - f) \in P$, and so $x - fx \in P$.

Therefore we have $fx \in$ center(R) such that $\bar{x} = \overline{fx}$ in R/P.

DEFINITION. For any ring R, we use $BS(R)$ to stand for the set of all maximal ideals of the Boolean algebra $B(R)$.

LEMMA 3.4. *Let R be a regular, right self-injective ring which is Type I_n for some n. For any $M \in BS(R)$, R/MR is a simple artinian ring, and the natural map* $\mathrm{center}(R) \to \mathrm{center}(R/MR)$ *is surjective.*

Proof. According to [3, Theorem 2.3], MR is a minimal prime ideal of R, hence Theorem 3.3 says that the natural map $\mathrm{center}(R) \to \mathrm{center}(R/MR)$ is surjective. Now $R \cong M_n(S)$ for some abelian, regular, right self-injective ring S, hence $R/MR \cong M_n(S/Q)$ for some prime ideal Q of S. Inasmuch as S is strongly regular, S/Q is a division ring, whence R/MR is simple artinian.

PROPOSITION 3.5. *Let R be a regular, right self-injective ring of Type I_f. Given any $x \in R$, there exist elements $y \in R$ and $z \in$* $\mathrm{center}(R)$ *such that $(x - z)R \lesssim (xy - yx)R$.*

Proof. According to [6, Corollary 6.5], there exist regular, right self-injective rings $R_1, R_2, \cdots$ such that $R \cong \Pi R_n$ and each R_n is Type I_n. Consequently, there exist orthogonal central idempotents $e_1, e_2, \cdots \in B(R)$ such that $\vee e_n = 1$ and each e_nR is Type I_n.

Let X be the set of those $f \in B(R)$ for which there exist elements $y, a, b \in R$ and $z \in \mathrm{center}(R)$ such that $f(x-z) = fa(xy-yx)b$. We claim that any nonzero $g \in B(R)$ must lie above some nonzero member of X.

Now $ge_n \neq 0$ for some n, hence there exists $M \in BS(R)$ such that $ge_n \notin M$. Since $e_n \notin M$, we see that $M \cap e_nR \in BS(e_nR)$ and that $e_nR/(M \cap e_nR)e_nR \cong R/MR$. Consequently, it follows from Lemma 3.4 that R/MR is a simple artinian ring, and that the natural map $\mathrm{center}(R) \to \mathrm{center}(R/MR)$ is surjective. Applying Corollary 2.4 to the element $\bar{x} \in R/MR$, we obtain elements $y' \in R/MR$ and $z' \in \mathrm{center}(R/MR)$ such that $(\bar{x} - z')(R/MR) \lesssim (\bar{x}y' - y'\bar{x})(R/MR)$. Thus $\bar{x} - z' = a'(\bar{x}y' - y'\bar{x})b'$ for some $a', b' \in R/MR$. Now there exist elements $y, a, b \in R$ such that $\bar{y} = y'$, $\bar{a} = a'$, and $\bar{b} = b'$, and there exists $z \in \mathrm{center}(R)$ such that $\bar{z} = z'$.

Thus $x - z - a(xy - yx)b \in MR$, from which it follows that

$$x - z - a(xy - yx)b = h[x - z - a(xy - yx)b]$$

for some $h \in M$. Since $g \notin M$ and $h \in M$, we see that $f = g(1 - h)$ is not in M, whence f is a nonzero member of $B(R)$ such that $f \leqq g$. In addition, $f[x - z - a(xy - yx)b] = 0$, so that $f \in X$. Thus the claim holds.

Inasmuch as every nonzero member of $B(R)$ lies above a nonzero member of X, we infer that there exist orthogonal idempotents $f_i \in X$ such that $\vee f_i = 1$. For each i, there exist elements y_i, a_i, $b_i \in R$ and $z_i \in \mathrm{center}(R)$ such that $f_i(x - z_i) = f_i a_i(xy_i - y_i x)b_i$. Since R is right self-injective, there exist elements $y, a, b, z \in R$ such that $yf_i = y_i f_i$, $af_i = a_i f_i$, $bf_i = b_i f_i$, and $zf_i = z_i f_i$ for all i. Then $f_i(x - z) = f_i a(xy - yx)b$ for all i, hence we see by Lemma 3.1 that $x - z = a(xy - yx)b$. Thus $(x - z)R \lesssim (xy - yx)R$. Since $zf_i = z_i f_i \in \mathrm{center}(R)$ for all i, we conclude from Lemma 3.1 that $z \in \mathrm{center}(R)$.

THEOREM 3.6. *Let R be a regular, right self-injective ring of Type* I_f. *If K is any two-sided ideal of R, then the natural map* $\mathrm{center}(R) \to \mathrm{center}(R/K)$ *is surjective.*

Proof. Consider any $x \in R$ such that $\bar{x} \in \mathrm{center}(R/K)$. According to Proposition 3.5, there exist $y \in R$ and $z \in \mathrm{center}(R)$ such that $(x - z)R \lesssim (xy - yx)R$. Since R/K is a flat left R-module, it follows that $(\bar{x} - \bar{z})(R/K) \lesssim (\bar{x}\bar{y} - \bar{y}\bar{x})(R/K)$. Since $\bar{x} \in \mathrm{center}(R/K)$, we obtain $\bar{x}\bar{y} = \bar{y}\bar{x}$, and consequently $\bar{x} = \bar{z}$.

The centers of the other standard Type II examples may now be calculated using Theorem 3.6, as follows.

THEOREM 3.7. *Let $D_1, D_2, \cdots$ be division rings, let $n(1) < n(2) < \cdots$ be positive integers, and set $R_k = M_{n(k)}(D_k)$ and $F_k =$* $\mathrm{center}(R_k)$ *for all k. Set $R = \Pi R_k$, and let M be any maximal two-sided ideal of R which contains $\bigoplus R_k$. Then R/M is a simple, regular, right and left self-injective ring of Type* II, *and* $\mathrm{center}(R/M) \cong (\Pi F_k)/[M \cap (\Pi F_k)]$.

Proof. Since each R_k is a regular, right self-injective ring of Type $\mathrm{I}_{n(k)}$, [6, Corollary 11.10] says that R/M is a simple, regular, right and left self-injective ring of Type II. According to [6, Corollary 6.5], R is Type I_f, hence Theorem 3.6 says that the natural map $\mathrm{center}(R) \to \mathrm{center}(R/M)$ is surjective. Observing that $\mathrm{center}(R) = \Pi F_k$, we conclude that $\mathrm{center}(R/M) \cong (\Pi F_k)/[M \cap (\Pi F_k)]$.

Unlike the Type II examples obtained from completions, the rings R/M in Theorem 3.7 do not have completely arbitrary centers, as the following corollary (of Theorem 3.6) shows.

COROLLARY 3.8. *Let $R_1, R_2, \cdots$ be simple artinian rings, set $R = \Pi R_k$, and let M be any maximal two-sided ideal of R which* *contains* $\bigoplus R_k$. *Then* center(R/M) *is either finite or uncountable.*

Proof. Set $F_k = \text{center}(R_k)$ for all k, and note that center$(R) = \Pi F_k$. Note also that $P = M \cap (\Pi F_k)$ is a prime ideal of ΠF_k which contains $\bigoplus F_k$. By [6, Corollary 6.5], R is Type I_f, hence Theorem 3.6 says that the natural map center$(R) \to$ center(R/M) is surjective. Thus center$(R/M) \cong (\Pi F_k)/P$.

Since ΠF_k is a commutative regular ring, all of its prime ideals are maximal, hence it follows that P is a minimal prime ideal of ΠF_k. Consequently, [3, Proposition 3.3] says that there exists an ultrafilter $\mathscr{F}$ on the index set $N = \{1, 2, \cdots\}$ such that

$$P = \{x \in \Pi F_k \mid \{k \in N \mid x_k = 0\} \in \mathscr{F}\} .$$

Thus $(\Pi F_k)/P$ is an ultraproduct of the F_k. Since $\bigoplus F_k \subseteq P$, we see that $\mathscr{F}$ is a nonprincipal ultrafilter on N.

Now if the set $\{k \in N \mid F_k \text{ is finite}\}$ belongs to $\mathscr{F}$, then we see by [2, Lemmas 3.7, 3.11] that $(\Pi F_k)/P$ is either finite or uncountable. On the other hand, if the set $\{k \in N \mid F_k \text{ is infinite}\}$ belongs to $\mathscr{F}$, then we see by [2, Corollaries 1.10, 3.14] that $(\Pi F_k)/P$ is uncountable. Therefore in all cases center$(R/M) \cong (\Pi F_k)/P$ is either finite or uncountable.

The following example, and the basic idea for its proof, was suggested by Handelman in correspondence.

EXAMPLE 3.9. There exists a simple regular ring R such that

(a) R is the direct limit of a sequence $R_1 \to R_2 \to \cdots$ of simple artinian rings and ring maps.

(b) There exists a unique rank function N on R.

(c) The N-completion $\bar{R}$ of R is a simple, regular, right and left self-injective ring.

(d) There do not exist simple artinian rings $S_1, S_2, \cdots$ such that $\bar{R}$ is isomorphic to a factor ring of ΠS_n.

Proof. Set $D = \boldsymbol{Q}$ and $n(k) = 2^k$ for all $k = 1, 2, \cdots$, and construct R as in Theorem 2.8. Then (a), (b), (c) hold, and center $(\bar{R}) \cong \boldsymbol{Q}$.

(d) Suppose that there do exist simple artinian rings $S_1, S_2, \cdots$ such that $\bar{R} \cong (\Pi S_n)/M$ for some maximal two-sided ideal M of ΠS_n. Since $\bar{R}$ is Type II, we see that $\bar{R} \not\cong S_n$ for all n, whence $\bigoplus S_n \subseteq M$. Then Corollary 3.8 says that the center of $(\Pi S_n)/M$ is either finite or uncountable. Since center$(\bar{R})$ is countably infinite, this is impossible.

In view of Example 3.9, we ask whether the simple, regular, right and left self-injective rings R/M of Theorem 3.7 can be obtained as in Theorem 2.8. More generally, can every simple, regular, right and left self-injective ring of Type II be obtained as in Theorem 2.8? The following example of von Neumann [12] indicates that the answer to this second question is probably negative.

Set $D = \boldsymbol{C}$, choose positive integers $n(1) < n(2) < \cdots$ such that $n(k) \mid n(k+1)$ for all k, and construct $\bar{R}$ as in Theorem 2.8. Let S be the "regular ring" of a complex W^*-factor of Type II_f (denoted $U(M)$ in [12, § 5]). As indicated in [12, § 5], S is a continuous regular ring. It is not hard to check that S is simple, and [11, Theorem 7.9] says that S is right and left self-injective. By [11, Theorem 5.1], S is directly finite, from which we infer that S is Type II. However, [12, Theorem E] says that $\bar{R} \not\cong S$.

This example shows that there exists a simple, regular, right and left self-injective ring S of Type II such that S cannot be obtained as in Theorem 2.8 in case D is a field (since then $D \cong \boldsymbol{C}$). It still might be possible to obtain S as in Theorem 2.8 with some noncommutative D, but this seems unlikely.

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