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APPLICATIONS OF APPROXIMATION THEORY TO DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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APPLICATIONS OF APPROXIMATION THEORY TO DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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The authors utilize fundamental approximation theory concepts to establish an existence theorem for an initial value problem with a deviating argument. The techniques used to establish this theorem result in an algorithm to compute polynomial approximations to the solution of the initial value problem.

1. Introduction. In recent years several authors [2, 5, 8, 9, 10, 12] have considered best approximating in some sense the solutions to various types of nonlinear differential and integro-differential equations. These papers have generally been devoted to examining questions involving the existence of best approximations and with proving that appropriate sequences of approximations converge to a solution of the differential or integro-differential equation. The best approximation problems of the above references are ordinarily nonlinear and consequently computational questions are not readily resolved.

More recently Allinger [1], Henry [1, 6] and Wiggins [6] have developed alternative approximation theory approaches that are computationally adaptive.

In this regard [6] deals with nonlinear initial value problems without deviating arguments, whereas [1] primarily considers linear initial value problems with deviating arguments. The approximation problems of [6] and [1] are basically different. The goal of the present paper is to extend the results of [6] to arbitrary order and to the more difficult deviating argument case. The theory developed in the present paper will not contain the linear theory of [1]; in a subsequent paper [7] the authors will extend the results in [1] and relate those extensions to the fundamental results of the present paper.

2. The initial value problem. Consider the scalar initial value problem

$$(1) \quad \begin{aligned} x^{(n)}(t) - \hat{f}(t, x(t), \dots, x^{(n-1)}(t), x(h(t)), \dots, x^{(n-1)}(h(t))) &= 0, \\ x^{(i)}(0) &= \alpha_i, \quad i = 0, 1, \dots, n-1, \quad t \in J = [-\gamma, \tau], \end{aligned}$$

where $\gamma \geq 0$, $\tau \geq 0$, and $\gamma + \tau > 0$. This initial value problem is

considered in [6] for $h(t) = t$ and $n = 2$. A function x is a solution to initial value problem (IVP) (1) on J if the $x^{(i)}(t)$, $i = 0, \dots, n$ are defined for $t \in J \cup h(J)$ and if x satisfies IVP (1) for $t \in J$. In (1) it is assumed that $\hat{f}: J \times R^{2n} \rightarrow R$ is continuous, and that $h \in C(J)$. Other hypotheses will be imposed later. We note that (1) generalizes not only the initial value problem considered in [6], but also the initial value problem with deviating argument of [1].

From a theoretical viewpoint, IVP (1) can be examined more efficiently in an equivalent vector setting. The scalar form will be utilized in the computations.

The standard transformation

$$(2) \quad X(t) = [x_1(t), \dots, x_n(t)]^T = [x(t), \dots, x^{(n-1)}(t)]^T$$

converts IVP (1) into a companion vector IVP

$$(1A) \quad \begin{aligned} \dot{X}(t) &= \bar{A}X(t) + F(t, X(t), X(h(t))), \\ X(0) &= A, \quad t \in J, \end{aligned}$$

where $A = (\alpha_0, \dots, \alpha_{n-1})^T$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

and

$$(3) \quad F(t, X(t), X(h(t))) = [0, \dots, 0, f_h[x](t)]^T,$$

where for notational convenience

$$(4) \quad f_h[x](t) = \hat{f}(t, x(t), \dots, x^{(n-1)}(t), x(h(t)), \dots, x^{(n-1)}(h(t))).$$

The conditions on \hat{f} imply that $F: J \times R^{2n} \rightarrow R^n$ may be assumed to be continuous. Let Π_k denote the set of scalar polynomials of degree at most k . Define

$$(5) \quad \begin{aligned} \mathcal{P}_k &= \{p \in \Pi_k: p(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1} + b_n t^n \\ &+ \dots + b_k t^k, t \in J, \text{ where } c_i = \alpha_i / i! \text{ and} \\ &b_i \in R, \quad (i = n, n+1, \dots, k)\}. \end{aligned}$$

Thus \mathcal{P}_k is the set of all scalar polynomials of degree at most k that satisfy the initial conditions of (1). Let $\hat{\Pi}_k$ denote the set of functions $P: J \rightarrow R^n$ such that each component of P is an element of Π_k . Corresponding to (5), define

$$(6) \quad \hat{\mathcal{P}}_k = \{P \in \hat{H}_k: P = [p, \dot{p}, \dots, p^{(n-1)}]^T, \quad p \in \mathcal{P}_k\}.$$

Hence for each $P \in \hat{\mathcal{P}}_k$, $P(0) = A$. Let $\bar{Z}(t)$ be the fundamental matrix solution to the matrix initial value problem

$$(7) \quad \dot{\bar{X}}(t) = \bar{A}\bar{X}(t), \quad \bar{X}(0) = I, \quad t \in J.$$

Hereafter, for $X = [x_1, \dots, x_n] \in R^n$,

$$|X| = \max_{1 \leq i \leq n} (|x_i|);$$

if X is a continuous mapping from J to R^n , then

$$|X|_J = \max_J \max_{1 \leq i \leq n} (|x_i(t)|).$$

Because of the nature of the matrix \bar{A} , the solution $\bar{Z}A$ to the vector initial value problem

$$(8) \quad \dot{X}(t) = \bar{A}X(t), \quad X(0) = A, \quad t \in J,$$

is an element of $\hat{\mathcal{P}}_k$. Due to the continuity of F , there exists a positive constant M depending on N such that $|F(t, X, Y)| \leq M$ for all $(t, X, Y) \in J \times R^{2n}$ with $\max(|X|, |Y|) \leq N$. Let $\bar{Z}(t) = (z_{ij}(t))$, $t \in J$. Define

$$\alpha = |\bar{Z}|_J = \max_J \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |z_{ij}(t)| \right).$$

Then $J_1 = [-\gamma_1, \tau_1] \subseteq J$, γ_1 and τ_1 nonnegative with $\gamma_1 + \tau_1 > 0$, and an appropriate deviating argument g are determined as follows:

H_1 . Let $N \geq 1 + |\bar{Z}A|_J$. Choose J_1 to insure that for all $t \in J_1$, $2|t|M\alpha \leq 1$.

H_2 . Assume an interval $J_2 = [-\gamma_2, \tau_2]$, γ_2 and τ_2 nonnegative with $\gamma_2 + \tau_2 > 0$, satisfying $J_2 \subseteq J_1$ and $h(J_2) \subseteq J_1$, exists. Define $g \in C(J_1)$ by $g(t) = h(t)$, $t \in J_2$, $g(J_1) \subseteq J_1$.

REMARK. If $h(0) = 0$, a condition frequently required of the deviating argument (see [11]), then the existence of a J_2 and g satisfying H_2 is assured.

Assuming hypotheses H_1 and H_2 , an alternative to initial value problem (1A) is

$$(1B) \quad \begin{aligned} \dot{X}(t) &= \bar{A}X(t) + F(t, X(t), X(g(t))), \\ X(0) &= A, \quad t \in J_1. \end{aligned}$$

3. **An existence theorem.** We are now in a position to define an approximating scheme that results in a constructive existence theorem for $IVP(1B)$ and hence for (1A) and (1).

Let

$$(9) \quad S_k = \{P \in \hat{\mathcal{S}}_k : |P - \bar{Z}A| \leq 2|t| M\alpha, \quad t \in J_1\}.$$

Then S_k is a compact, convex subset of $\hat{\mathcal{S}}_k$. For $x \in C^{n-1}(J_1)$, let

$$(10) \quad \hat{F}[X](t) = F(t, X(t), X(g(t))),$$

where again $X(t)$ is given by (2). If $P \in S_k$, consider the best approximation problem

$$(11) \quad \inf_{V \in \hat{\mathcal{S}}_k} |\dot{V} - \bar{A}V - \hat{F}[P]|_{J_1}.$$

Because of the nature of \bar{A} and $\hat{\mathcal{S}}_k$, this minimization problem reduces to the scalar best approximation problem

$$(12) \quad \inf_{v \in \Pi_{k-n}} \|v - f[p]\|_{J_1},$$

where p is the first component of P and where $f[p] = f_g[p]$. Since $f: C^{n-1}(J_1) \rightarrow C(J_1)$, it is well known from classical approximation theory that there exists a $\bar{v} \in \Pi_{n-k}$ such that

$$(13) \quad \inf_{v \in \Pi_{k-n}} \|v - f[p]\|_{J_1} = \|\bar{v} - f[p]\|_{J_1}.$$

Integrate \bar{v} n times to obtain the k th degree polynomial $v_b \in \mathcal{S}_k$. If $V_b = (v_b, \dots, v_b^{(n-1)})$, then

$$(14) \quad \inf_{V \in \hat{\mathcal{S}}_k} |\dot{V} - \bar{A}V - \hat{F}[P]|_{J_1} = |\dot{V}_b - \bar{A}V_b - \hat{F}[P]|_{J_1}.$$

Define an operator T_k on S_k as follows: if $P \in S_k$, then

$$(15) \quad T_k[P](t) = V_b(t), \quad t \in J_1.$$

Thus if $P \in S_k$, $T_k[P]$ is the best approximation from $\hat{\mathcal{S}}_k$ to $\hat{F}[P]$ in the sense of (14). Note that since (13) has a unique solution v_b , V_b is also unique.

THEOREM 1. Assume that \hat{f} and h satisfy the continuity requirements described below (1), and that H_1 and H_2 are valid. Then T_k is a continuous mapping from S_k into S_k .

Proof. Suppose that $T_k[P] = V_b$. Let

$$(16) \quad \dot{V}_b(t) - \bar{A}V_b(t) - \hat{F}[P](t) = E(t).$$

Since $P \in S_k$, H_1 and H_2 imply that $|\hat{F}[P]|_{J_1} \leq M$, and consequently, since $\bar{Z}(t)A \in \hat{\mathcal{S}}_k$ and V_b is best in the sense of (14),

$$\begin{aligned} & |\dot{V}_b - \bar{A}V_b - \hat{F}[P]|_{J_1} \\ & \leq |\dot{\bar{Z}}A - \bar{A}\bar{Z}A - \hat{F}[P]|_{J_1} \\ & = |\hat{F}[P]|_{J_1} \leq M. \end{aligned}$$

This inequality and (16) imply that

$$(17) \quad |E|_{J_1} \leq M.$$

Now (16) holds if and only if

$$\begin{aligned} (18) \quad & V_b(t) - \bar{Z}(t)A \\ & = \int_0^t \bar{Z}(t)\bar{Z}^{-1}(s)\hat{F}[P](s)ds \\ & \quad + \int_0^t \bar{Z}(t)\bar{Z}^{-1}(s)E(s)ds. \end{aligned}$$

A variation of parameters argument readily verifies equality (18). Since \bar{A} is a constant matrix, $\bar{Z}(t)\bar{Z}^{-1}(s) = \bar{Z}(t-s)$; therefore (17) and (18) imply that

$$\begin{aligned} |V_b - \bar{Z}A| & \leq 2|t|M|\bar{Z}|_{J_1} \\ & = 2|t|M\alpha, \quad t \in J_1. \end{aligned}$$

Thus $V_b \in S_k$ and hence T_k maps S_k into S_k .

Establishing that T_k is continuous is somewhat more difficult and requires the use of fundamental concepts of approximation theory. Suppose that $P \in S_k$ and that \bar{P} is any other element of S_k ; if $T_k[P] = Q$ and $T_k[\bar{P}] = \bar{Q}$, then as in (13),

$$\inf_{v \in \Pi_{k-n}} \|v - f[p]\|_{J_1} = \|q^{(n)} - f[p]\|_{J_1}$$

and

$$\inf_{v \in \Pi_{k-n}} \|v - f[\bar{p}]\|_{J_1} = \|\bar{q}^{(n)} - f[\bar{p}]\|,$$

where p , \bar{p} , q , and \bar{q} are the first components of P , \bar{P} , Q , and \bar{Q} , respectively. The classical Freud theorem [3, p. 82] of approximation theory implies that

$$(19) \quad \|q^{(n)} - \bar{q}^{(n)}\|_{J_1} \leq \lambda_p \|f[p] - f[\bar{p}]\|_{J_1}.$$

Let $\varepsilon > 0$ be given. Since \hat{f} is uniformly continuous on compact subsets of $J \times R^{2n}$, there exists a $\delta > 0$ such that if $|P - \bar{P}|_{J_1} < \delta$,

$$(20) \quad \|f[p] - f[\bar{p}]\|_{J_1} < \frac{\varepsilon}{\lambda_p} \frac{1}{\max_{0 \leq i \leq n} [\max(\gamma_1, \tau_1)]^{n-i}},$$

$$i = 0, 1, \dots, n.$$

Inequalities (19) and (20) imply that

$$|q^{(i)}(t) - \bar{q}^{(i)}(t)| < \varepsilon \quad i = 0, 1, \dots, n-1, t \in J.$$

Thus

$$(21) \quad |Q(t) - \bar{Q}(t)| < \varepsilon.$$

Since $T_k[P] = Q$ and $T_k[\bar{P}] = \bar{Q}$, the above analysis establishes that given $P \in S_k$ and $\varepsilon > 0$, there exists $\delta_o > 0$ such that if $\bar{P} \in S_k$ and $|P - \bar{P}|_{J_1} < \delta$, then

$$|T_k[P] - T_k[\bar{P}]|_{J_1} < \varepsilon.$$

Hence T_k is a continuous mapping from S_k into S_k and the proof is complete.

COROLLARY 1. *Assume the hypotheses of Theorem 1 are valid. Then the mapping T_k has a fixed point in S_k .*

Proof. Since S_k is a compact, convex subset of $\hat{\mathcal{S}}_k$, the result follows from the Schauder fixed point theorem [16, p. 25].

We summarize the results of Theorem 1 and Corollary 1. There exists a $P \in \hat{\mathcal{S}}_k$ such that if we seek the best approximation V_b to $\hat{F}[P]$ in the sense of (14), this best approximation is precisely $V_b = P$. In the equivalent scalar form, if we seek the best approximation \bar{v} from Π_{k-n} to $f[p]$ in the sense of (13), then $\bar{v} = p^{(n)}$.

The proof of Theorem 1 does not depend on k , and consequently the hypotheses of Theorem 1 imply that there exists a sequence

$$(22) \quad \{P_k(t)\}_{k=n}^{\infty}, \quad t \in J_1,$$

of fixed points, one for each operator T_k , $k \geq n$.

Define E_k , $k \geq n$, by

$$(23) \quad \dot{P}_k(t) - \bar{A}P_k(t) - \hat{F}[P_k](t) = E_k(t),$$

$t \in J_1$, where P_k is an element of the sequence (22). Since $P_k \in S_k$, $k \geq n$,

$$(24) \quad |P_k(t) - \bar{Z}(t)A| \leq 2|t|M\alpha, \quad t \in J_1.$$

Equation (23) and inequalities (17) and (24) imply that the sequences

$$(25) \quad \{|P_k|_{J_1}\}_{k=n}^{\infty} \quad \text{and} \quad \{|\dot{P}_k|_{J_1}\}_{k=n}^{\infty}$$

are bounded. Thus the sequence (22) is an equicontinuous and uniformly bounded family on J_1 . Let W be a cluster point of this sequence. Equality (23) is valid if and only if

$$(26) \quad \begin{aligned} P_k(t) - \bar{Z}(t)A &= \int_0^t \bar{Z}(t)\bar{Z}^{-1}(s)\hat{F}[P_k](s)ds \\ &+ \int_0^t \bar{Z}(t)\bar{Z}^{-1}(s)E_k(s)ds. \end{aligned}$$

Thus if

$$(27) \quad \lim_{k \rightarrow \infty} |E_k|_{J_1} = 0,$$

then (26) implies that W is a solution to IVP (1B). Thus W is a solution to IVP (1A) on J_2 , and the first entry w of W satisfies IVP (1) on J_2 . These observations constitute part of the conclusions of the next theorem.

THEOREM 2. *Suppose the hypotheses of Theorem 1 are satisfied. Then (27) is valid. If W is a cluster point of (22) on J_1 , then W is a solution of IVP (1B) on J_1 and W is a solution of IVP (1A) on J_2 .*

Proof. Based on the remarks preceeding the statement of Theorem 2, only (27) needs to be established. Let

$$(28) \quad P_k = (p_k, \dots, p_k^{(n-1)}).$$

Then since P_k is a fixed point of T_k ,

$$\inf_{v \in \Pi_{k-n}} \|v - f[p_k]\|_{J_1} = \|p_k^{(n)} - f[p_k]\|_{J_1}.$$

If

$$e_{k-n}(f[p_k]) = \|p_k^{(n)} - f[p_k]\|_{J_1},$$

then

$$(29) \quad e_{k-n}(f[p_k]) = |E_k|_{J_1}.$$

Jackson's theorem [13, p. 22] implies that

$$(30) \quad e_{k-n}(f[p_k]) \leq 6\omega_k\left(\frac{L}{k-n}\right),$$

where L is half the length of the interval J_1 , and ω_k is the modulus of continuity of $f[p_k]$ on J_1 . Let $\varepsilon > 0$ be given. Since the sequences (25) are bounded, the sequences

$$\{p_k^{(i)}\}_{k=n}^{\infty}, \quad i = 0, \dots, n-1,$$

are uniformly bounded and equicontinuous families on J_1 . Thus the uniform continuity of \hat{f} on compact subsets of $J_1 \times R^{2n}$ implies that the sequence $\{f[p_k]\}_{k=n}^\infty$ is an equicontinuous family on J_1 . Consequently given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|t - s| \leq \delta$ implies that $|f[p_k](t) - f[p_k](s)| \leq \varepsilon$, independent of k . Therefore $\omega_k(\delta) \leq \varepsilon$, independent of k . If K is large enough to insure that for all $k \geq K$, $L/(k - n) \leq \delta$, then the monotonicity of ω_k [13, p. 14] implies for all $k \geq K$,

$$\omega_k\left(\frac{L}{k - n}\right) \leq \omega_k(\delta) \leq \varepsilon.$$

Thus for all $k \geq K$, (29) and (30) imply that $|E_k|_{J_1} \leq 6\varepsilon$. Consequently $\lim_{k \rightarrow \infty} |E_k|_{J_1} = 0$, completing the proof.

4. Computations. We now describe an algorithm based on the theory of section three. For k sufficiently large, initially choose $P_{k_0}(t) = \bar{Z}(t)A$. At the $l + 1$ step solve via the second algorithm of Remes [3, p. 97] the best approximation problem

$$(31) \quad \inf_{V \in \mathcal{V}_k} |\dot{V} - \bar{A}V - \hat{F}[P_{k,l}]|_{J_1} = \inf_{v \in \Pi_{k-n}} \|v - f[p_{k,l}]\|_{J_1}.$$

Let the solution of this minimizing problem be $P_{k,l+1}$ and continue. We again note that $p_{k,l+1}^{(n)}$ is then the minimizing polynomial to the scalar problem in (31). This procedure yields a unique sequence

$$(32) \quad \{P_{k,l}\}_{l=0}^\infty$$

with a corresponding sequence of first elements of (32),

$$(33) \quad \{p_{k,l}\}_{l=0}^\infty.$$

The sequences (32) and (33) have cluster points P_k and p_k , respectively, which are approximate solutions of degree k to (1A) and (1) on J_2 .

The actual calculations are carried out in the scalar setting. Furthermore, the computations are effected for any interval $J_1 \subseteq J$ for which $h(J_1) \subseteq J_1$. If no such interval exists, the construction of an appropriate g would be necessary (see H_2). If convergence does not occur, then J_1 is reduced (see H_1).

We note that a cluster point of (32) need not be a fixed point of T_k in the sense of Corollary 1. However, an additional checkable condition insures that a cluster point of (32) is a fixed point of T_k . In particular, entries of sequence (33) satisfy

$$\inf_{v \in \Pi_{k-n}} \|v - f[p_{k,l}]\|_{J_1} = \|p_{k,l+1}^{(n)} - f[p_{k,l}]\|_{J_1}.$$

In calculating $p_{k,l+1}$, an alternating set (see [3, p. 75]) consisting of $k - n + 2$ arises. If $X_l = \{t_{1,l}, \dots, t_{k-n+2,l}\}$ is an alternating set, and if

$$e_{k,l+1}(t) = p_{k,l+1}^{(n)}(t) - f[p_{k,l}](t),$$

then

$$\begin{aligned} e_{k,l+1}(t_{i,l}) &= -e_{k,l+1}(t_{i+1,l}), \\ i &= 1, \dots, n - k + 1. \end{aligned}$$

Let $X = \{t_1, \dots, t_{n-k+2}\}$ be a cluster point of $\{X_l\}_{l=0}^\infty$, and let

$$(34) \quad e_k(t) = p_k^{(n)}(t) - f[p_k](t)$$

where p_k is the first entry of the cluster point P_k of (32) (and hence a cluster point of (33)). Without loss of generality we can assume that the subsequences of $\{X_l\}$ and of (33) that converge to X and p_k , respectively, involve the same subscripts.

H_3 . Suppose for $t_i, t_{i+1} \in X$ that $e_k(t_i) = -e_k(t_{i+1})$, $i = 1, \dots, k - n + 1$, where e_k is defined in (34).

We note that this condition is weaker than assuming the alternation theorem.

Assuming H_3 is valid, P_k is a fixed point of T_k , and consequently the theory of §3 applies. If (33) actually converges, H_3 is satisfied.

The following examples employ the algorithm described above. A Fortran IV program using double precision arithmetic is utilized in the calculations. The calculations for Examples 1 and 2 were executed on the Xerox Sigma 7 computer, located at Montana State University. The calculations for Example 3 were effected on the IBM 370, at The College of Charleston.

EXAMPLE 1.

$$\begin{aligned} \ddot{x}(t) - 2(2 - t)\dot{x}(t)x^2(t) &= 0, \quad J = [-.7, .7] \\ x(0) &= 1/2, \quad \dot{x}(0) = 1/4 \end{aligned}$$

The algorithm is employed over all of J , since $g(t) = t$. For $k = 5$ approximation is from

$$\mathcal{P}_5 = \{p(t): p(t) = .5 + .25t + b_2t^2 + b_3t^3 + b_4t^4 + b_5t^5\}.$$

Let $B_l = (.5, .25, b_{2l}, b_{3l}, b_{4l}, b_{5l})$ be the coefficients of the l -th iterate $p_{k,l}$ of sequence (33). Then

$$\begin{aligned} B_0 &= (.5, .25, 0, 0, 0, 0) \\ B_1 &= (.5, .25, .125000, .020833, -.005208, -.001562) \\ B_{15} &= (.5, .25, .119879, .057090, .042496, .023350). \end{aligned}$$

If the algorithm is terminated with B_{15} (a stopping criteria is employed here), then the cluster point of (33) is

$$p_5(t) = .5 + .25t + .119879t^2 + .570909t^3 + .424964t^4 + .233505t^5.$$

The actual solution is $x(t) = 1/(2 - t)$, and $\|x - p_5\|_J = .001779$.

EXAMPLE 2.

$$\begin{aligned}\ddot{x}(t) + 4x(t/2)\dot{x}(t/2) &= 0, \quad J = [-1, 1], \\ x(0) &= 0, \quad \dot{x}(0) = 2.\end{aligned}$$

We observe that $h(J) \subseteq J$. Approximation is again from \mathcal{P}_5 , and

$$\begin{aligned}B_0 &= (0, 2, 0, 0, 0, 0), \\ B_9 &= (0, 2, 0, -1.290025, 0, .204262).\end{aligned}$$

Then B_9 and B_{10} are equal through six decimal places. Thus

$$p_5(t) = 2t - 1.290025t^3 + .204262t^5.$$

The actual solution is $x(t) = \sin 2t$ and $\|x - p_5\|_J = .006865$.

EXAMPLE 3. $h(t) = 1/2(e^t - 1)$

$$\begin{aligned}\ddot{x}(t) + 4[2x(h(t)) + 1]^{-2} &= 0, \quad t \in [-.175, 1] \\ x(0) &= 0, \quad \dot{x}(0) = 2\end{aligned}$$

In this example $h(J) = [-.08, .859] \subseteq J$. For approximation from \mathcal{P}_7 , the alternating set is

$$X = \{-.1750, -.1211, .0509, .3313, .6519, .9054, 1.0000\}.$$

The actual solution is $x(t) = \ln(2t + 1)$, and

$$\begin{aligned}p_7(t) &= 2t - 1.9688t^2 + 2.8151t^3 - 4.7085t^4 \\ &\quad + .58950t^5 - 4.0266t^6 + 1.0974t^7.\end{aligned}$$

Hypothesis H_3 is satisfied by e_7 , and $\|e_7\| = .08519$. Finally, $\|x - p_7\| = .00496$.

5. Conclusions. This paper utilizes approximation theory techniques to obtain uniform approximate solutions to variants of *IVP* (1). The primary result of the paper is an existence theorem for *IVP* (1). An algorithm based on this existence theorem is used to compute approximate solutions to *IVP* (1). This paper extends results of the authors [6] to higher order initial value problems

with deviating arguments. The present paper is closely related to [1], but results in an approximation problem and a companion algorithm that differ from those given in [1] (for additional literature involving other deviating argument problems, see the references of [1]). In a subsequent paper [7] the authors intend to present a theory that parallels the theory of the present paper but contains the linear theory of [1]. The results of the present paper do substantially improve the nonlinear results contained in [1].

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