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The structure of those bounded (hnp)-rings over which every module admits a basic submodule, is determined. It is shown that such rings are precisely the block lower triangular matrix rings over $D\backslash M$ where D is a discrete valuation ring with M as its maximal ideal.

In [12], the author generalized some well known results on decomposability of torsion abelian groups to torsion modules over bounded (hnp)-rings. Let R be a bounded (hnp)-ring and M be a (right) R-module. A submodule N of M is called a *basic* submodule of M if it satisfies the following conditions:

(i) N is decomposable in the sense that it is a direct sum of uniserial modules and finitely generated uniform torsion free modules.

(ii) N is a pure submodule of M.

(iii) M/N is a divisible module.

The following result has been proved by the author (see [9] for details):

THEOREM 1. *Any torsion module M over a bounded (hnp)-ring has a basic submodule and any two basic submodules of M are isomorphic.*

In general an R-module need not have a basic submodule. However Marubayashi [8, Theorem (3.6)] showed that every module over a g-discrete valuation ring has a basic submodule. In this paper we determine the structure of those bounded (hnp)-rings, over which every (right) module admits a basic submodule (Theorems 3 and 4).

As defined by Marubayashi [8, p. 432], a prime, right as well as left principal ideal ring R, such that its Jacobson radical $J(R)$ is the only maximal ideal, and idempotents modules $J(R)$ can be lifted, is called a g-discrete valuation ring; further if $R/J(R)$ is a division ring, then R is called a discrete valuation ring. In view of [8, Lemma (3.1)] and [7, Lemma (2.1)], g-discrete valuation rings are precisely the matrix rings over discrete valuation rings. Modules considered will be unital right modules and the notations and terminology of [12, 13] will be used without comment.

Henceforth in all lemmas, R is a bounded (hnp)-ring over which every module admits a basic submodule. Further Q stands for the classical quotient ring of R.

LEMMA 1. *A submodule N of a torsion free module M over an (hnp)-ring S is pure if and only if M/N is torsion free.*

Proof. Necessity. Let for some $x \in M$ and a regular element b in S, $xb = y \in N$. As N is pure, for some $z \in N$, $xb = zb$. This in turn gives $x = z \in N$. This proves that M/N is torsion free.

SUFFICIENCY. Let M/N be torsion-free. Consider a finite system of equations $\sum_i x_i r_{ij} = s_j$, $s_j \in N$, having a solution $\{x_i\}$ in M. If $K = \sum x_i S + N$, then K/N being finitely generated and torsion free, is projective. Hence $K = K_1 \oplus N$. This gives that the above system of equations have a solution in N. Hence N is pure in M.

LEMMA 2. *If U is a uniform torsion free right R-module, then either U is finitely generated, or divisible.*

Proof. Since by Lemma 1, 0 and U are only pure submodules of U, so 0 or U is the basic submodule of U. Hence U is divisible or finitely generated.

LEMMA 3. *Every over-ring of R different from Q is finitely generated as an R-module.*

Proof. Consider an over ring S of R such that $S \neq Q$. Now $S = \oplus \sum U_i$, U_i are uniform as right S-modules, since S is an (hnp)-ring [6]. If any U_i is divisible as a right R-module, then $S = Q$, otherwise by Lemma 2, S_R is finitely generated.

Let L be any ring and J be an ideal of L. Let n be a positive integer and $(k_1, k_2, \cdots, k_r)$ be an ordered r-tuple of positive integers such that $k_1 + k_2 + \cdots + k_r = n$. In the notations of Reiner [10, Chapter 8], we can form a block matrix ring of the type:

$$\begin{bmatrix} (L) & (J) \cdots (J) \\ (L) & (L) \cdots (J) \\ (L) & (L) \cdots (L) \end{bmatrix} (k_1, k_2, \cdots, k_r) .$$

In the terminology of Robson [11], any such matrix ring is said to be a block lower triangular matrix ring over $L \backslash J$.

THEOREM 2. *Let R be a bounded (hnp)-ring over which every module admits a basic submodule. Then there exists a discrete valuation ring D with maximal ideal M such that R is a block lower triangular matrix ring over $D \backslash M$.*

Proof. First of all we show that R has only one maximal invertible ideal. Let A be a maximal invertible ideal of R. If A is not the only maximal invertible ideal, then in the notations of [13] $R < R_A < Q$. There exists a non unit regular element a in R such that a is a unit modulo A. Then $U_n a^{-n} R \subset R_A$ and $U_n a^{-n} R$ is not finitely generated as a right R-module. This contradicts Lemma 3. Hence A is the only maximal invertible ideal of R and $R = R_A$. Then $J(R) = A$. This then gives R has only finitely many idempotent ideals. Let B be a minimal nonzero idempotent ideal of R. Then $O_l(B) = \{x \in Q: xB \subset B\}$ is a Dedekind prime ring [3, Proposition (1.8)]. As for R, every torsion free uniform $O_l(B)$-module is either finitely generated or divisible. As a consequence $O_l(B)$ has only one maximal ideal P and $O_l(B) = O_l(B)_P$. So by [7, Lemma (2.1)] $O_l(B) = D_n$ for some discrete valuation ring D. By Jacobson [5, p. 120], R is equivalent to $O_l(B)$. Hence by Robson [11, Theorem (6.3) and Corollary (2.8)], R is a block lower triangular matrix ring over $D\backslash M$, where D is a discrete valuation ring with M as its maximal ideal.

It is clear that any non block lower triangular matrix ring over $D\backslash M$ where D is a discrete valuation ring with M is its maximal ideal, is equivalent to D_n. So to prove the converse of the above theorem it is enough to prove the following:

THEOREM 3. *Let R be a bounded (hnp)-ring such that R is equivalent to S, for some g-discrete valuation ring S, which is an overring of R, then every R-module admits a basic submodule.*

Proof. First of all we show that any uniform torsion free R-module U is either divisible or finitely generated. Suppose U is not divisible. Now $S = D_n$. There exist regular elements a and b in R such that $aSb \subset R$. Since S is bounded there exists a nonzero ideal $\mathscr{I}$ of S such that $\mathscr{I} \subset Sb$. Then $a\mathscr{I} \subset R$ and the fact that S_S is embeddable in $a\mathscr{I}$ gives that S_R is finitely generated. Similarly ${}_RS$ is finitely generated. So using [3, Theorem (1.6)], we get $S = O_l(A) = A^* = AA^*$ for some idempotent ideal A of R. We can suppose that $U \subset Q$, the classical quotient ring of R. If $US = eQ$, then $UAA^*A = eQA = eQ$. However $UAA^*A \subset U$. Thus in this case U is divisible. Hence US is finitely generated as S-module [8, Lemma (3.2)]. This gives U_R is finitely generated, since as proved above S_R is finitely generated.

Thus every uniform torsion free right R-module is injective or projective. Consider any right R-module M and let T be its torsion submodule. T admits a basic submodule B by Theorem 1. Then B is a pure submodule of M and T/B is divisible; further T/B is the torsion submodule of M/B. So we can write

$$M/B = L/B \oplus T/B \oplus K/B$$

where L/B is torsion free, divisible R-module and K/B is a torsion free reduced R-module. If $K/B = 0$, we get B itself is a basic submodule. So let $K/B \neq 0$. We can find a maximal uniform submodule U/B of K/B. By what has been proved above U/B is finitely generated and hence projective. So by Lemma 1, U is a pure submodule of K, and $U = U_1 \oplus B$, where U_1 is a finitely generated uniform torsion submodule. By Zorns lemma, we can find a maximal direct sum $E = B \oplus \sum \oplus U_i$, in K such that E is a pure submodule of K, U_i are finitely generated uniform, torsion free R-modules. By Lemma 1, K/E is torsion free. If K/E is not divisible, then as before we get a nonzero finitely generated uniform submodule V/E of K/E such that V/E is pure in K/E. Then $V = V_1 \oplus E$ and V is a pure submodule of K. This contradicts the maximality of E. Hence K/E is divisible. E is clearly decomposable and is a basic submodule of M. This completes the proof.

We remark that any two basic submodules of a module over the ring of the above theorem, can be shown to be isomorphic.

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