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It is well known that if a regular matrix sums every subsequence of a sequence x, then x converges. It follows trivially from this result and row finiteness of the Cesáro summability matrix that if A is a regular matrix such that Ay is Cesáro summable for every subsequence y of x, then x is convergent (not merely Cesáro summable). The purpose of the present paper is to give some general results of this type involving matrix methods that are not necessarily row finite. For example, it is shown that if T is any regular matrix summability method and A is a regular matrix such that Ay is absolutely T-summable for every stretching y of x, then x is absolutely convergent. This is done without assuming that x is bounded, and consequently, without the benefit of associativity.

The well known result mentioned above is due to R. C. Buck [2], and the trivial consequence involving the Cesàro summability matrix $(C, 1)$ can be seen as follows. If A is regular and Ay is Cesàro summable for every subsequence y of x, then $(C, 1)A$ is a regular matrix which sums every subsequence of x, since row finiteness of $(C, 1)$ gives the associativity relation $(C, 1)(Ay) = [(C, 1)A]y$. Consequently by Buck's theorem, x is convergent.

When we say that a matrix A is semiregular, we will mean that A is regular over the set of all convergent sequences of 0's and 1's. Thus $A = (a_{pq})$ is semiregular iff A satisfies the first two of the following three conditions for regularity:

1) $a_{pq} \to 0$ as $p \to \infty$, $q = 1, 2, 3, \cdots$,
2) $\sum_{q=1}^{\infty} a_{pq} \to 1$ as $p \to \infty$,
3) $\sum_{q=1}^{\infty} |a_{pq}| < K_A$, $\quad p = 1, 2, 3, \cdots$.

If ε is a positive term null sequence and each of x and y is a complex sequence, then the statement that y contains an ε-copy of x means that y contains a subsequence $\{y_{n_p}\}$ such that $|y_{n_p} - x_p| < \varepsilon_p$, $p = 1, 2, 3, \cdots$.

THEOREM 1. *If $T = (t_{pq})$ is a matrix such that $\sum_{q=1}^{\infty} |t_{pq}| < L_p$, $p = 1, 2, 3, \cdots$, A is a regular matrix, and Ay is T-summable for every subsequence y of x, then either x converges or TA is a Schur matrix,* i.e., *TA sums every bounded sequence.*

Proof. Suppose x is unbounded. Clearly A is row finite since

Az is defined for every subsequence z of x. It is somewhat less clear, but nonetheless true, that T is row finite, and we gives a proof. Suppose the pth row of T contains infinitely many nonzero terms. Using only the semiregularity of A, we can construct a subsequence y of x such that $|t_{pj}\sum_{q=1}^{\infty} a_{jq}y_q| = |t_{pj}(Ay)_j| > 1$ for infinitely many values of j, thus ruling out convergence of $\sum_{q=1}^{\infty} t_{pq}(Ay)_q$, and contradicting the fact that $T(Ay)$ is defined. We see this as follows. Suppose a finite subsequence $y_1, y_2, \cdots, y_n$ of x has been determined. From the semiregularity of A, there exists a positive integer v such that if $i > v$, then $\sum_{q=1}^{n} |a_{iq}| < 1/2$ and $|\sum_{q=1}^{\infty} a_{iq} - 1| < 1/2$. Choose $j > v$ such that $t_{pj} \neq 0$. Let a_{jr} be the last nonzero term in the jth row of A. Then from the inequalities above, $r > n$. Determine $y_{n+1}, \cdots, y_r$ such that $y_1, \cdots, y_r$ is a finite subsequence of x and

$$|a_{jr}y_r| > \frac{1}{|t_{pj}|} + \left|\sum_{q=1}^{r-1} a_{jq}y_q\right| .$$

Then regardless of how the remaining terms of y are chosen,

$$|(Ay)_j| = \left|\sum_{q=1}^{\infty} a_{jq}y_q\right| = \left|\sum_{q=1}^{r} a_{jq}y_q\right| > \frac{1}{|t_{pj}|} ,$$

thus establishing our assertion above. Therefore T must be row finite. Hence the associativity $T(Az) = (TA)z$ holds for all z. Therefore TA sums every subsequence of the divergent sequence x. Thus by the theorem in [9], TA is a Schur matrix. This completes the proof for the case that x is unbounded.

Next suppose $|x_p| < M$, $p = 1, 2, 3, \cdots$. We note that if y is any subsequence of x, then $|\sum_{q=1}^{\infty} t_{pq}\sum_{s=1}^{\infty} a_{qs}y_s| < MK_A\sum_{q=1}^{\infty}|t_{pq}| < MK_AL_p$. Thus we can interchange the order of summation and obtain

$$(*) \qquad \sum_{q=1}^{\infty} t_{pq}\left(\sum_{s=1}^{\infty} a_{qs}y_s\right) = \sum_{s=1}^{\infty}\left(\sum_{q=1}^{\infty} t_{pq}a_{qs}\right)y_s .$$

The left side of (*) is the pth term of the sequence $T(Ay)$ and the right side of (*) is the pth term of the sequence $(TA)y$. Thus again we have the associativity $T(Ay) = (TA)y$. Hence the matrix TA sums every subspace y of x. Therefore if x is not convergent, then TA is a Schur matrix by the theorem in [9]. This completes the proof.

THEOREM 2. *Suppose T is any regular matrix summability method. If A is a regular matrix such that Ay is T-summable for every subsequence y of x, then x is convergent.*

Proof. Since the hypothesis of Theorem 1 is satisfied, then either x converges or TA is a Schur matrix. But TA is regular since it is the product of regular matrices, and no regular matrix is a Schur matrix. This completes the proof.

For stretchings, we obtain the following theorem which is analogous to (but more comprehensive than) Theorem 2.

THEOREM 3. *Suppose T is any regular matrix summability method. If A is a regular matrix such that Ay is T-summable (absolutely T-summable) for every stretching y of x, then x is convergent (absolutely convergent).*

We note that Theorem 3 is an immediate consequence of the following result which we shall call the "Copy Theorem."

THEOREM 4. *If each of T and A is a regular matrix, x is any complex sequence (bounded or not), and ε is any positive term null sequence, then there exists a stretching y of x such that $T(Ay)$ exists and contains an ε-copy of x.*

Proof. Let $K = K_A + K_T + \max \varepsilon_p + 1$, $M_p = 1 + \sum_{q=1}^{p} |x_q|$, $\delta_p = \min\{\varepsilon_1, \cdots, \varepsilon_p\}$, and $Q_p = KM_p + 1$. There exists a positive integer n_1 such that if $p \geqq n_1$, then

$$\left| \sum_{q=1}^{\infty} a_{pq} - 1 \right| < \frac{\delta_1}{16Q_1}.$$

There exists r_1 such that

$$\sum_{q=1}^{n_1} |t_{r_1 q}| < \frac{\delta_1}{8Q_1} \quad \text{and} \quad \left| \sum_{q=n_1}^{\infty} t_{r_1 q} - 1 \right| < \frac{\delta_1}{8Q_1}.$$

There exists $m_1 > n_1$ such that if $1 \leqq p \leqq r_1$ then

$$\sum_{q=m_1}^{\infty} |t_{pq}| < \frac{\delta_1}{8Q_2}.$$

There exists an integer $s_1 > 1$ such that if $1 \leqq p \leqq m_1$, then

$$\sum_{q=s_1}^{\infty} |a_{pq}| < \frac{\delta_1}{16Q_2}.$$

Suppose the finite increasing sequences $\{n_p\}_{p=1}^{\alpha-1}$, $\{r_p\}_{p=1}^{\alpha-1}$, $\{m_p\}_{p=1}^{\alpha-1}$, and $\{s_p\}_{p=1}^{\alpha-1}$ of positive integers have been determined. Choose $n_\alpha > m_{\alpha-1}$ such that if $p \geqq n_\alpha$, then

$$\left| \sum_{q=s_{\alpha-1}}^{\infty} a_{pq} - 1 \right| < \frac{\delta_\alpha}{16Q_\alpha} \tag{1}$$

and

$$(2)\qquad \sum_{q=1}^{s_{\alpha-1}} |a_{pq}| < \frac{\delta_\alpha}{8Q_{\alpha-1}}, \qquad \text{where } s_0 = 1 \text{ and } M_0 = 1.$$

Choose $r_\alpha > r_{\alpha-1}$ such that

$$(3)\qquad \sum_{q=1}^{n_\alpha} |t_{r_\alpha q}| < \frac{\delta_\alpha}{8Q_\alpha} \quad \text{and} \quad \left| \sum_{q=n_\alpha}^{\infty} t_{r_\alpha q} - 1 \right| < \frac{\delta_\alpha}{8Q_\alpha}.$$

Choose $m_\alpha > n_\alpha$ such that if $1 \leqq p \leqq r_\alpha$, then

$$(4)\qquad \sum_{q=m_\alpha}^{\infty} |t_{pq}| < \frac{\delta_\alpha}{2^{2+\alpha}Q_{\alpha+1}}.$$

Choose $s_\alpha > s_{\alpha-1}$ such that if $1 \leqq p \leqq m_\alpha$, then

$$(5)\qquad \sum_{q=s_\alpha}^{\infty} |a_{pq}| < \frac{\delta_\alpha}{2^{3+\alpha}Q_{\alpha+1}}.$$

From (3) and (4) we can obtain

$$(6)\qquad \left| \sum_{q=n_\alpha}^{m_\alpha} t_{r_\alpha q} - 1 \right| < \frac{\delta_\alpha}{4Q_\alpha}.$$

From (1) and (5) we obtain

$$(7)\qquad \left| \sum_{q=s_{\alpha-1}}^{s_\alpha - 1} a_{pq} - 1 \right| < \frac{\delta_\alpha}{8Q_\alpha} \qquad \text{for} \quad n_\alpha \leqq p \leqq m_\alpha.$$

Thus we have defined the increasing sequences $\{n_p\}_{p=1}^{\infty}$, $\{r_p\}_{p=0}^{\infty}$, $\{m_p\}_{p=0}^{\infty}$, and $\{s_p\}_{p=0}^{\infty}$ of integers, where $r_0 = 0$ and $m_0 = 0$.

Let $\{y_p\}_{p=1}^{\infty}$ be the stretching of x induced by $\{s_p\}_{p=0}^{\infty}$ [3, p. 455]. If $\alpha > 1$ and $n_\alpha \leqq p \leqq m_\alpha$, then from (2), (5), and (7) we obtain

$$\begin{aligned}
\left| \sum_{q=1}^{\infty} a_{pq}y_q - x_\alpha \right| &\leqq \left| \sum_{q=1}^{s_{\alpha-1}-1} a_{pq}y_q \right| + \left| \sum_{q=s_{\alpha-1}}^{s_\alpha - 1} a_{pq}y_q - x_\alpha \right| \\
&\quad + \sum_{v=\alpha+1}^{\infty} \left| \sum_{q=s_{v-1}}^{s_v - 1} a_{pq}y_q \right| \\
&\leqq \left(\sum_{q=1}^{s_{\alpha-1}-1} |a_{pq}| \right) \max\{|x_1|, \cdots, |x_{\alpha-1}|\} \\
&\quad + |x_\alpha| \left| \sum_{q=s_{\alpha-1}}^{s_\alpha - 1} a_{pq} - 1 \right| + \sum_{v=\alpha+1}^{\infty} |x_v| \sum_{q=s_{v-1}}^{s_v - 1} |a_{pq}| \\
&< \delta_\alpha/8K + \delta_\alpha/8K + \sum_{v=\alpha+1}^{\infty} |x_v| \frac{\delta_{v-1}}{2^{2+v}Q_v} \\
&< 3\delta_\alpha/8K.
\end{aligned} \tag{8}$$

Also we can prove this inequality for $n_1 \leqq p \leqq m_1$. Thus we have for $\alpha \geqq 1$ and $n_\alpha \leqq p \leqq m_\alpha$,

$$(9) \qquad \sum_{q=1}^{\infty} a_{pq}y_q = x_\alpha + \mu_\alpha , \quad \text{where} \quad |\mu_\alpha| < 3\delta_\alpha/8K .$$

If $m_{\alpha-1} < p < n_\alpha$, then from (5) we obtain

$$\begin{aligned}\left|\sum_{q=1}^{\infty} a_{pq}y_q\right| &\leqq \left|\sum_{q=1}^{s_\alpha-1} a_{pq}y_q\right| + \sum_{v=\alpha}^{\infty}\left|\sum_{q=s_v}^{s_{v+1}-1} a_{pq}y_q\right| \\ &\leqq \left(\sum_{q=1}^{s_\alpha-1}|a_{pq}|\right)\max\{|x_1|, \cdots, |x_\alpha|\} + \sum_{q=\alpha}^{\infty}|x_{v+1}|\sum_{q=s_v}^{s_{v+1}-1}|a_{pq}| \\ &< KM_\alpha + \sum_{v=\alpha}^{\infty}|x_{v+1}|\frac{\delta_v}{2^{3+v}Q_{v+1}} \\ &\leqq KM_\alpha + (\delta_\alpha/K)\sum_{v=\alpha}^{\infty}1/2^{3+v} \\ &< KM_\alpha + 1 = Q_\alpha .\end{aligned}$$

From this inequality and (8) we can show that if $m_{\alpha-1} < p \leqq m_\alpha$, then

$$(10) \qquad \left|\sum_{q=1}^{\infty} a_{pq}y_q\right| \leqq \left|\sum_{q=1}^{\infty} a_{pq} - x_\alpha\right| + |x_\alpha| < KM_\alpha + 1 = Q_\alpha ,$$

since $|x_\alpha| < KM_\alpha$ and $3\delta_\alpha/8K < 1$.

If $r_{i-1} < p \leqq r_i$, then from (4) and (10) we obtain

$$(11) \qquad \begin{aligned}\left|\sum_{q=m_i+1}^{\infty} t_{pq}(Ay)_q\right| &\leqq \sum_{\alpha=i}^{\infty}\sum_{j=m_\alpha+1}^{m_{\alpha+1}}|t_{pj}||(Ay)_j| \\ &\leqq \sum_{\alpha=i}^{\infty} Q_{\alpha+1}\sum_{j=m_\alpha+1}^{m_{\alpha+1}}|t_{pj}| \\ &\leqq \sum_{\alpha=i}^{\infty} Q_{\alpha+1}\frac{\delta_\alpha}{2^{2+\alpha}Q_{\alpha+1}} < \delta_i/4 .\end{aligned}$$

Thus we see that $T(Ay)$ is defined.

From (3), (6), (9), (10), and (11), we obtain

$$\begin{aligned}\left|\sum_{j=1}^{\infty} t_{r_ij}(Ay)_j - x_i\right| &\leqq \left|\sum_{j=1}^{n_i-1} t_{r_ij}(Ay)_j\right| + \left|\sum_{j=n_i}^{m_i} t_{r_ij}(Ay)_j - x_i\right| \\ &\quad + \left|\sum_{j=m_i+1}^{\infty} t_{r_ij}(Ay)_j\right| \\ &\leqq Q_i\sum_{j=1}^{n_i-1}|t_{r_ij}| + \left|\sum_{j=n_i}^{m_i} t_{r_ij}(x_i+\mu_i) - x_i\right| \\ &\quad + \sum_{\alpha=i}^{\infty}\left|\sum_{j=m_\alpha+1}^{m_{\alpha+1}} t_{r_ij}(Ay)_j\right| \\ &< \delta_i/8 + |x_i|\left|\sum_{j=n_i}^{m_i} t_{r_ij} - 1\right| + \sum_{j=n_i}^{m_i}|t_{r_ij}\mu_i| \\ &\quad + \sum_{\alpha=i}^{\infty} Q_{\alpha+1}\sum_{j=m_\alpha+1}^{m_{\alpha+1}}|t_{r_ij}| \\ &< \delta_i/8 + \delta_i/4 + 3\delta_i/8 + \sum_{\alpha=i}^{\infty} Q_{\alpha+1}\frac{\delta_\alpha}{2^{2+\alpha}Q_{\alpha+1}}\end{aligned}$$

$$< \delta_i/8 + \delta_i/4 + 3\delta_i/8 + \delta_i/4$$
$$= \delta_i$$
$$\leqq \varepsilon_i .$$

This completes the proof.

We can use Theorem 4 to prove the following extension of a theorem of Agnew [1].

THEOREM 5. *Suppose T is any regular matrix summability method. If A is a regular matrix and x is a sequence having a finite limit point, then there exists a subsequence y of x such that every finite limit point of x is a T-limit point of Ay.*

Proof. Using the separability of the complex plane, we write the finite limit points of x in a sequence denoted by u. Let v denote the sequence $u_1; u_1, u_2; u_1, u_2, u_3; \cdots$, and let ε be a positive term null sequence. By the "Copy Theorem," there exists a stretching z of v such that $T(Az)$ is defined and contains an ε-copy of v. Let y be a subsequence of x such that $z - y$ is a null suquence. Since $T(Ay) = T(A[y - z]) + T(Az)$, we see that $T(Ay)$ is the sum of a null sequence and a sequence which contains an ε-copy of v. Therefore every finite limit point of x is a limit point of $T(Ay)$. This completes the proof.

In [5] we proved theorems analogous to the results of this paper, except that T was the identity matrix (ordinary convergence) and A was a semiregular matrix. The following theorems are trivial consequences of associativity, the results in [5], and the fact that if T is a row finite regular matrix and A is a semiregular matrix, then TA is a semiregular matrix.

THEOREM 6. *Suppose T is any row finite regular matrix summability method. If A is a semiregular matrix such that Ay is T-summable for every subsequence y of x, then x is convergent.*

THEOREM 7. *Suppose T is any row finite regular matrix summability method. If A is a semiregular matrix such that Ay is T-summable (absolutely T-summable) for every stretching y of x, then x is convergent (absolutely convergent).*

REMARK. We give an example to show the necessity of "row finite" in the statement "$\cdots$ the fact that if T is a row finite regular matrix and A is a semiregular matrix, then TA is a semiregular matrix," which precedes Theorem 6. Let B and A be matrices defined as follows: $b_{pq} = 2^{p-q-1}$ if p is even and $q \geqq p$,

$b_{pq} = 0$ if p is even and $q < p$, $b_{pq} = 1$ if p is odd and $q = p$, $b_{pq} = 0$ if p is odd and $q \neq p$, $a_{pq} = 0$ if $q < 2p - 1$ or $q > 2p$, $a_{pq} = 2^{p-1} + 1$ if $q = 2p - 1$, $a_{pq} = -2^{p-1}$ if $q = 2p$. Simple calculations show that B is regular, A is semiregular, but BA is not semiregular.

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