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CONTRACTION SEMIGROUPS IN LEBESGUE SPACE

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Let $(T_t: t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X, \Sigma, \mu)$, where (X, Σ, μ) is a σ -finite measure space. Without assuming the initial continuity of the semigroup it is shown that $(T_t: t > 0)$ is dominated by a strongly continuous semigroup $(S_t: t > 0)$ of positive linear contractions on $L_1(X, \Sigma, \mu)$, i.e., that $|T_t f| \leq S_t |f|$ holds a.e. on X for all $f \in L_1(X, \Sigma, \mu)$ and all $t > 0$. As an application, a representation of $(T_t: t > 0)$ in terms of $(S_t: t > 0)$ is obtained, and the question of the almost everywhere convergence of $1/b \int_0^b T_t f dt$ as $b \rightarrow +0$ is considered.

Introduction. Let (X, Σ, μ) be a σ -finite measure space and let $L_p(X) = L_p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, be the usual Banach spaces of real or complex functions on (X, Σ, μ) . For a set $A \in \Sigma$, $L_p(A)$ denotes the Banach space of all $L_p(X)$ -functions that vanish a.e. on $X - A$. If $f \in L_p(X)$, we define $\text{supp } f$ to be the set of all $x \in X$ at which $f(x) \neq 0$. Relations introduced below are assumed to hold modulo sets of measure zero. A linear operator T on $L_p(X)$ is called a *contraction* if $\|T\|_p \leq 1$, and *positive* if $f \geq 0$ implies $Tf \geq 0$.

Let $(T_t: t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X)$, i.e.,

- (i) each T_t is a linear contraction on $L_1(X)$,
- (ii) $T_t T_s = T_{t+s}$ for all $t, s > 0$,
- (iii) for every $f \in L_1(X)$ and every $s > 0$, $\lim_{t \rightarrow s} \|T_t f - T_s f\|_1 = 0$.

Under the additional hypothesis of $\text{strong-}\lim_{t \rightarrow +0} T_t = I$ (I denotes the identity operator), Kubokawa [6] proved that there exists a strongly continuous semigroup $(S_t: t > 0)$ of positive linear contractions on $L_1(X)$ such that $|T_t f| \leq S_t |f|$ a.e. on X for all $f \in L_1(X)$ and all $t > 0$. The main purpose of this paper is to prove the same result, without assuming any additional hypothesis. We then obtain a representation of $(T_t: t > 0)$ in terms of $(S_t: t > 0)$ which is a continuous extension of Akcoglu-Brunel's representation ([1], Theorem 3.1), and a decomposition of the space X for $(T_t: t > 0)$ which asserts the existence of a set $Y \in \Sigma$ such that $T_t f \in L_1(Y)$ for all $f \in L_1(X)$ and all $t > 0$ and also such that if $f \in L_1(Y)$ then $T_t f$ converges in the norm topology of $L_1(X)$ as $t \rightarrow +0$ and $1/b \int_0^b T_t f dt$ converges a.e. on X as $b \rightarrow +0$.

Existence theorem. Our main result is the following existence theorem.

THEOREM 1. *If $(T_t; t > 0)$ is a strongly continuous semigroup of linear contractions on $L_1(X)$, then there exists a strongly continuous semigroup $(S_t; t > 0)$ of positive linear contractions on $L_1(X)$, called the semigroup modulus of $(T_t; t > 0)$, such that*

$$(1) \quad |T_t f| \leq S_t |f| \quad (f \in L_1(X), t > 0).$$

If $0 \leq f \in L_1(X)$, S_t is given by

$$(2) \quad S_t f = \sup \left\{ \tau_{t_1} \cdots \tau_{t_n} f : \sum_{i=1}^n t_i = t, t_i > 0, n \geq 1 \right\}$$

where τ_t denotes the linear modulus of T_t in the sense of Chacon-Krengel ([3]).

Proof. For $0 \leq f \in L_1(X)$ and $t > 0$, put

$$M(t, f) = \left\{ \tau_{t_1} \cdots \tau_{t_n} f : \sum_{i=1}^n t_i = t, t_i > 0, n \geq 1 \right\}.$$

Since $\|\tau_t\|_1 = \|T_t\|_1 \leq 1$ and $\tau_t \tau_s f \geq \tau_{t+s} f$ for all $t, s > 0$, we see that if g_1 and g_2 are in $M(t, f)$, then there exists a function h in $M(t, f)$ such that

$$\max(g_1, g_2) \leq h \text{ and } \|h\|_1 \leq \|f\|_1.$$

Thus it is possible to define a function $S_t f$ in $L_1(X)$ by the relation:

$$S_t f = \sup \{g : g \in M(t, f)\}.$$

It is clear that $\|S_t f\|_1 \leq \|f\|_1$ and $S_t f \geq 0$. It is easily seen that if c is a positive constant and f and g are nonnegative functions in $L_1(X)$, then

$$S_t(cf) = cS_t f \text{ and } S_t(f + g) = S_t f + S_t g.$$

Therefore S_t may be regarded as a positive linear contraction on $L_1(X)$. By the definition of S_t it follows that

$$S_t S_s = S_{t+s} \quad (t, s > 0).$$

It is now enough to prove the strong continuity of $(S_t; t > 0)$. To do this, we first show the following result:

$$(3) \quad \lim_{t \rightarrow s+0} \|\tau_t f - \tau_s f\|_1 = 0 \quad (f \in L_1(X), s > 0).$$

To see this, we may and do assume without loss of generality

that f is nonnegative. Let $\varepsilon > 0$ be given. By [3] there exist functions $g_i \in L_1(X)$, $1 \leq i \leq n$, such that

$$|g_i| \leq f \text{ and } \|\tau_s f - \max_{1 \leq i \leq n} |T_s g_i|\|_1 < \varepsilon .$$

Since $(T_t: t > 0)$ is strongly continuous on $(0, \infty)$, we can take a $\delta > 0$ so that $|s - t| < \delta$ implies $\|T_s g_i - T_t g_i\|_1 < \varepsilon/n$ for each $1 \leq i \leq n$. Fix a $t > 0$ so that $|s - t| < \delta$. We then have $\| |T_s g_s| - |T_t g_t| \|_1 \leq \|T_s g_t - T_t g_t\|_1 < \varepsilon/n$ for each $1 \leq i \leq n$, and so it follows that

$$\|\tau_s f - \max_{1 \leq i \leq n} |T_t g_i|\|_1 < 2\varepsilon .$$

By this and the fact that $\tau_t f \geq \max_{1 \leq i \leq n} |T_t g_i|$, we get

$$\|(\tau_t f - \tau_s f)^-\|_1 \leq \|(\max_{1 \leq i \leq n} |T_t g_i| - \tau_s f)^-\|_1 < 2\varepsilon .$$

Therefore

$$(4) \quad \lim_{t \rightarrow s} \|(\tau_t f - \tau_s f)^-\|_1 = 0 .$$

Next, let $t > s$ and write $t = s + a$. Since

$$\|(\tau_a \tau_s f - \tau_s f)^-\|_1 \leq \|(\tau_t f - \tau_s f)^-\|_1 ,$$

it follows that

$$(5) \quad \lim_{a \rightarrow +0} \|(\tau_a \tau_s f - \tau_s f)^-\|_1 = 0 .$$

On the other hand,

$$\tau_a \tau_s f = (\tau_a \tau_s f - \tau_s f)^+ - (\tau_a \tau_s f - \tau_s f)^- + \tau_s f .$$

Thus, by (5), we have that

$$\begin{aligned} \|(\tau_{a+s} f - \tau_s f)^+\|_1 &\leq \|(\tau_a \tau_s f - \tau_s f)^+\|_1 \\ &\leq \|(\tau_a \tau_s f - \tau_s f)^-\|_1 + \|\tau_a \tau_s f\|_1 - \|\tau_s f\|_1 \\ &\leq \|(\tau_a \tau_s f - \tau_s f)^-\|_1 \rightarrow 0 \end{aligned}$$

as $a \rightarrow +0$, because $\|\tau_a\|_1 \leq 1$. This and (4) establish (3).

We next show that

$$(6) \quad \lim_{t \rightarrow s+0} \|S_t f - S_s f\|_1 = 0 \quad (f \in L_1(X), s > 0) .$$

To see this, we may and do assume without loss of generality that f is nonnegative. Let $\varepsilon > 0$ be given, and choose a function $g \in M(s, f)$ so that

$$\|S_s f - g\|_1 < \varepsilon,$$

where g is of the form

$$g = \tau_{t_1} \cdots \tau_{t_n} f, \quad \sum_{i=1}^n t_i = s, \quad \text{and } t_i > 0 \quad (1 \leq i \leq n).$$

Let $s_n > t_n$. Then

$$\begin{aligned} \|g - \tau_{t_1} \cdots \tau_{t_{n-1}} \tau_{s_n} f\|_1 &= \|\tau_{t_1} \cdots \tau_{t_{n-1}} (\tau_{t_n} f - \tau_{s_n} f)\|_1 \\ &\leq \|\tau_{t_n} f - \tau_{s_n} f\|_1, \end{aligned}$$

and hence, by (3),

$$(7) \quad \lim_{s_n \rightarrow t_n + 0} \|g - \tau_{t_1} \cdots \tau_{t_{n-1}} \tau_{s_n} f\|_1 = 0.$$

Let us write $t = t_1 + \cdots + t_{n-1} + s_n (> s)$. Since

$$S_t f - S_s f \geq (\tau_{t_1} \cdots \tau_{t_{n-1}} \tau_{s_n} f - g) + (g - S_s f),$$

it follows that

$$(S_t f - S_s f)^- \leq |\tau_{t_1} \cdots \tau_{t_{n-1}} \tau_{s_n} f - g| + |g - S_s f|.$$

This and (7) yield that

$$\limsup_{t \rightarrow s + 0} \|(S_t f - S_s f)^-\|_1 \leq \varepsilon.$$

Since ε is arbitrary,

$$\lim_{t \rightarrow s + 0} \|(S_t f - S_s f)^-\|_1 = 0.$$

Hence

$$\begin{aligned} \|(S_t f - S_s f)^+\|_1 &= \|(S_t f - S_s f)^-\|_1 + \|S_t f\|_1 - \|S_s f\|_1 \\ &\leq \|(S_t f - S_s f)^-\|_1 \longrightarrow 0 \end{aligned}$$

as $t \rightarrow s + 0$, because $\|S_t f\|_1 \leq \|S_s f\|_1$ for all $t > s$. This proves (6).

Using (6), it is now direct to show that the semigroup $(S_t; t > 0)$ is strongly continuous on $(0, \infty)$, and we omit the details.

THEOREM 2. *Let $(T_t; t > 0)$ and $(S_t; t > 0)$ be as in Theorem 1. Then T_t converges strongly as $t \rightarrow +0$ if and only if S_t converges strongly as $t \rightarrow +0$.*

Proof. If $T_0 = \text{strong-}\lim_{t \rightarrow +0} T_t$ exists, then $(T_t; t \geq 0)$ is a semigroup and strongly continuous on $[0, \infty)$. Hence we can apply the same arguments as in the proof of Theorem 1 to obtain that $\lim_{t \rightarrow +0} \|S_t f - \tau_0 f\|_1 = 0$ for all $f \in L_1(X)$, where τ_0 denotes the linear modulus of T_0 .

Conversely, if $S_0 = \text{strong-}\lim_{t \rightarrow +0} S_t$ exists, then, for all $f \in L_1(X)$, the set $\{T_t f: 0 < t < 1\}$ is weakly sequentially compact in $L_1(X)$, since $|T_t f| \leq S_t |f|$ and $\lim_{t \rightarrow +0} \|S_t |f| - S_0 |f|\|_1 = 0$ (cf. Theorem IV. 8.9 in [4]). Thus, by Lemma 1 of the author [8], T_t converges strongly as $t \rightarrow +0$.

The hypothesis of being a contraction semigroup can not be weakened in Theorem 1. To see this, we give the following example, motivated by S. Tsurumi.

EXAMPLE. Let X be the positive integers, Σ all possible subsets of X , and μ the counting measure. Let $\varepsilon > 0$ be given. By an elementary computation, there exists a real constant r , with $1/e < r < 1$, such that

$$(8) \quad 1 < \sup \{r^t(|\cos t| + |\sin t|): t \geq 0\} < 1 + \varepsilon .$$

For $f \in L_1(X)$ and $t > 0$, define

$$T_t f(2n - 1) = r^{nt} [f(2n - 1) \cos nt - f(2n) \sin nt] \quad (n \geq 1)$$

and

$$T_t f(2n) = r^{nt} [f(2n - 1) \sin nt + f(2n) \cos nt] \quad (n \geq 1) .$$

It is easily seen that $(T_t: t > 0)$ is a strongly continuous semigroup of linear operators on $L_1(X)$ satisfying $\|T_t\|_1 \leq 1 + \varepsilon$ for all $t > 0$. Furthermore

$$(9) \quad \lim_{m \rightarrow \infty} \|(\tau_{1/m})^m\|_1 = \infty .$$

To see this, let 1_n denote the indicator function of $\{n\}$. Then

$$\begin{aligned} \|(\tau_{1/m})^m\|_1 &\geq \|(\tau_{1/m})^m(1_{2n-1} + 1_{2n})\|_1 / \|1_{2n-1} + 1_{2n}\|_1 \\ &= \left[r^{n/m} \left(\left| \cos \frac{n}{m} \right| + \left| \sin \frac{n}{m} \right| \right) \right]^m \quad (n \geq 1) , \end{aligned}$$

as has been pointed out by S. Koshi. Hence (8) implies (9).

By (9) it is now immediate to see that $(T_t: t > 0)$ can not be dominated by a semigroup of positive linear operators on $L_1(X)$.

Representation theorem. Let $(T_t: t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X)$. It is well known that given an $f \in L_1(X)$ there exists a scalar function $g(t, x)$ on $(0, \infty) \times X$, measurable with respect to the product of Lebesgue measure and μ , such that for each $t > 0$, $g(t, x)$, as a function of x , belongs to the equivalence class of $T_t f$. In the sequel $g(t, x)$ will

be denoted by $T_t f(x)$. Using Fubini's theorem, we see that there exists a set $E(f) \in \Sigma$, with $\mu(E(f)) = 0$, such that if $x \notin E(f)$ then the scalar function $t \mapsto T_t f(x)$ is Lebesgue integrable on every finite interval (a, b) and the integral $\int_a^b T_t f(x) dt$, as a function of x , belongs to the equivalence class of $\int_a^b T_t f dt$, where $\int_a^b T_t f dt$ denotes the Bochner integral of the vector valued function $t \mapsto T_t f$ with respect to Lebesgue measure on (a, b) .

If $(S_t: t > 0)$ denotes the semigroup modulus of $(T_t: t > 0)$, then the ratio ergodic theorem holds for $(S_t: t > 0)$, i.e., for any f and g in $L_1(X)$, with $g \geq 0$, the ratio ergodic limit

$$\lim_{b \rightarrow \infty} \left(\int_0^b S_t f(x) dt \right) / \left(\int_0^b S_t g(x) dt \right)$$

exists and is finite a.e. on the set $\{x: \int_0^\infty S_t g(x) dt > 0\}$ (cf. [5]). Thus Hopf's decomposition holds, i.e., X decomposes into two measurable sets C and D , called respectively the conservative and dissipative parts of X , such that if $0 \leq g \in L_1(X)$ then $\int_0^\infty S_t g(x) dt = \infty$ or 0 a.e. on C and $\int_0^\infty S_t g(x) dt < \infty$ a.e. on D . A set $A \in \Sigma$ is called *invariant* (under $(S_t: t > 0)$), if $S_t L_1(A) \subset L_1(A)$ for all $t > 0$. It is immediate that A is invariant under $(S_t: t > 0)$ if and only if it is invariant under $(T_t: t > 0)$. It is known (cf. [7]) that C is invariant and the class Σ_i of all invariant subsets of C forms a σ -field in the class of all measurable subsets of C .

We are now in a position to state our representation theorem.

THEOREM 3. *Let $(T_t: t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X)$ and $(S_t: t > 0)$ denote the semigroup modulus of $(T_t: t > 0)$. Let C denote the conservative part of X with respect to $(S_t: t > 0)$ and let Σ_i be the σ -field of invariant subsets of C . Then there exists a (unique) set $\Gamma \in \Sigma_i$ and a function $u \in L_\infty(\Gamma)$ such that*

(i) $|u| = 1$ a.e. on Γ and $T_t f = (1/u) S_t(uf)$ for all $f \in L_1(\Gamma)$ and all $t > 0$,

(ii) if $\Delta = C - \Gamma$, then the closed linear hull of $\{f - T_t f: f \in L_1(\Delta), t > 0\}$ is $L_1(\Delta)$,

(iii) a function $v \in L_\infty(\Gamma)$, with $|v| > 0$ a.e. on Γ , satisfies $T_t f = (1/v) S_t(vf)$ for all $f \in L_1(\Gamma)$ and all $t > 0$ if and only if there exists a function $r \in L_\infty(\Gamma)$ such that $|r| > 0$ a.e. on Γ , $S_t^* r = r$ a.e. on Γ for all $t > 0$, and $v = ru$.

Proof. Let $h \in L_\infty(C)$ be such that $T_t^* h = h$ a.e. on C for all $t > 0$. Since $|h| = |T_t^* h| \leq \tau_t^* |h| \leq S_t^* |h|$ and the conservative part

of X with respect to each single operator S_i is exactly C (cf. [7]), it follows that $|h| = S_i^*|h|$ a.e. on C for all $t > 0$, and hence $\text{supp } h \in \Sigma_i$. By this, we can find a function $h \in L_\infty(C)$ such that $T_i^*h = h$ a.e. on C for all $t > 0$ and also such that if $f \in L_\infty(C)$ satisfies $T_i^*f = f$ a.e. on C for all $t > 0$, then $\text{supp } f \subset \text{supp } h$. Put $\Gamma = \text{supp } h$ and define $u \in L_\infty(\Gamma)$ by $u = h/|h|$ a.e. on Γ . If $0 \leq f \in L_1(\Gamma)$ and $t > 0$, then, as in [1],

$$\begin{aligned} \int (S_i f)|h| d\mu &= \int f S_i^*|h| d\mu = \int f|h| d\mu = \int (f/u)h d\mu \\ &= \int (f/u)T_i^*h d\mu = \int T_i(f/u)u|h| d\mu. \end{aligned}$$

Hence $S_i f = T_i(f/u)u$, since $S_i f \geq |T_i(f/u)| = |T_i(f/u)u|$, and (i) is established.

To prove (ii), let $h \in L_\infty(\Delta)$ be such that $\int (f - T_i f)h d\mu = 0$ for all $f \in L_1(\Delta)$ and all $t > 0$. Then $T_i^*h = h$ a.e. on Δ (and hence on C) for all $t > 0$. Therefore, by the definition of Γ , $h = 0$ a.e. on Δ , and (ii) follows from the Hahn-Banach theorem.

To prove (iii), let $v \in L_\infty(\Gamma)$ and $|v| > 0$ a.e. on Γ . Put $r = v/u$. Then $T_i f = (1/v)S_i(vf)$ for all $f \in L_1(\Gamma)$ and all $t > 0$ if and only if $(1/rv)S_i(rvf) = (1/u)S_i(uf)$ for all $f \in L_1(\Gamma)$ and all $t > 0$, or equivalently, $S_i(rf) = rS_i f$ for all $f \in L_1(\Gamma)$ and all $t > 0$, since $\{uf : f \in L_1(\Gamma)\} = L_1(\Gamma)$. And this is equivalent to the fact that $S_i^*r = r$ a.e. on Γ for all $t > 0$, by Lemma 2.4 in [1].

The proof is complete.

Decomposition theorem. It is shown that, after eliminating an uninteresting subset of X , a strongly continuous semigroup $(T_t : t > 0)$ of linear contractions on $L_1(X)$ can be made strongly continuous at the origin and the local ergodic theorem holds.

THEOREM 4. *Let $(T_t : t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X)$. Then X can be written as the union of two disjoint measurable sets Y and Z with the following properties:*

- (i) *For every $f \in L_1(X)$ and every $t > 0$, $T_t f \in L_1(Y)$.*
- (ii) *For every $f \in L_1(Y)$, $T_t f$ converges in the norm topology of $L_1(X)$ as $t \rightarrow +0$ and*

$$\lim_{b \rightarrow +0} \frac{1}{b} \int_0^b T_t f(x) dt$$

exists a.e. on X .

- (iii) *For every $f \in L_1(Y)$ with $f > 0$ a.e. on Y ,*

$$Y = \bigcup_{n=1}^{\infty} \{x: \tau_{1/n} f(x) > 0\}.$$

Proof. Let $(S_t: t > 0)$ be the semigroup modulus of $(T_t: t > 0)$. Fix an $h \in L_1(X)$ with $h > 0$ a.e. on X , and put

$$Y = \bigcup_{n=1}^{\infty} \{x: S_{1/n} h(x) > 0\}$$

and $Z = X - Y$. It is easily seen that $S_t f \in L_1(Y)$ and hence $T_t f \in L_1(Y)$ for all $f \in L_1(X)$ and all $t > 0$. If we write $h_0 = \int_0^1 S_t h dt$, then $h_0 \in L_1(Y)$, $h_0 > 0$ a.e. on Y , and $\lim_{t \rightarrow +0} \|S_t h_0 - h_0\|_1 = 0$. Therefore, by approximation, the set $\{S_t f: 0 < t < 1\}$ is weakly sequentially compact in $L_1(X)$ for all $0 \leq f \in L_1(Y)$, from which we observe that the set $\{T_t f: 0 < t < 1\}$ is also weakly sequentially compact in $L_1(X)$ for all $f \in L_1(Y)$, since $|T_t f| \leq S_t |f|$ for all $t > 0$. Hence Lemma 1 of the author [8] implies that $T_t f$ converges in the norm topology of $L_1(X)$ as $t \rightarrow +0$ for all $f \in L_1(Y)$.

To prove the second part of (ii), we may and do assume without loss of generality that $X = Y$. Put $T_0 = \text{strong-}\lim_{t \rightarrow +0} T_t$, and let $f \in L_1(X)$. Then f can be written as $f = g + h$, where $g = T_0 f$ and $T_t h = 0$ for all $t \geq 0$, because $T_t T_0 = T_0 T_t = T_t$ for all $t \geq 0$. It follows that

$$\lim_{a \rightarrow +0} \left\| (f - h) - \frac{1}{a} \int_0^a T_t g dt \right\|_1 = 0.$$

If we write $f_a = h + 1/a \int_0^a T_t g dt$, then it is easily seen that

$$\lim_{b \rightarrow +0} \frac{1}{b} \int_0^b T_t f_a(x) dt = f_a(x) - h(x) \quad \text{a.e.}$$

on X . On the other hand, by Akcoglu-Chacon's local ergodic theorem ([2]),

$$\sup_{0 < b < 1} \left| \frac{1}{b} \int_0^b T_t f(x) dt \right| \leq \sup_{0 < b < 1} \frac{1}{b} \int_0^b S_t |f|(x) dt < \infty \quad \text{a.e.}$$

on X . Thus, the second part of (ii) follows from Banach's convergence theorem (cf. Theorem IV. 11. 3 in [4]).

For the proof of (iii), let $f \in L_1(Y)$, $f > 0$ a.e. on Y . Put

$$P = \bigcup_{n=1}^{\infty} \{x: \tau_{1/n} f(x) > 0\}.$$

Clearly, $P \subset Y$, and by the definition of Y and (i),

$$Y = \bigcup_{n=1}^{\infty} \{x: S_{1/n} f(x) > 0\}.$$

Let $1/n < t$. Then $\tau_t f \leq \tau_{1/n} \tau_{t-(1/n)} f$, and so $\text{supp } \tau_t f \subset \text{supp } \tau_{1/n} f$. Thus it follows that

$$\text{supp } S_t f \subset P \quad (t > 0).$$

Therefore $Y \subset P$, and (iii) is established.

The proof is complete.

In conclusion, the author would like to remark that the question of whether the almost everywhere convergence of $1/b \int_0^b T_t f(x) dt$ as $b \rightarrow +\infty$ holds for all $f \in L_1(Z)$ remains an open problem.

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