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## DISTRIBUTION ESTIMATES OF BARRIER-CROSSING PROBABILITIES OF THE YEH-WIENER PROCESS

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## DISTRIBUTION ESTIMATES OF BARRIER-CROSSING PROBABILITIES OF THE YEH-WIENER PROCESS

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Let  $Q = [0, S] \times [0, T]$  be a rectangle and  $\{X(s, t): s, t \ge 0\}$ be the two-parameter Yeh-Wiener process. This paper finds probabilities of X(s, t) crossing barriers of the type ast + bs + ct + d on the boundary  $\partial Q$ . These probabilities give lower bounds for the yet unknown probabilities of X(s, t) crossing ast + bs + ct + d on Q. The paper also discusses sharper bounds for the latter probabilities.

1. Introduction. Let  $\{X(s, t): s, t \ge 0\}$  be the standard Yeh-Wiener process of two parameters such that it is a separable real Gaussian stochastic process satisfying:

(1.1) 
$$X(s, t) = 0$$
 a.s. if s or t is 0,

(1.2) the expected value  $E\{X(s, t)\} = 0$  at every  $s, t \ge 0$ ,

(1.3) 
$$E\{X(s, t)X(s', t')\} = \min(s, s') \cdot \min(t, t').$$

Further properties of the process are found in Yeh's [8] and [9].

For the square  $D = [0, 1] \times [0, 1]$  and its boundary  $\partial D$ , Paranjape and Park [6] showed that the probability

$$(1.4) \qquad P\left\{\sup_{\partial D} X(s, t) \ge \lambda\right\} = 3N(-\lambda) - e^{4\lambda^2}N(-3\lambda) , \quad \lambda \ge 0 ,$$

where  $N(\cdot)$  stands for the standard normal distribution function. This probability is a lower bound of the yet unknown probability,  $P\{\sup_D X(s, t) \ge \lambda\}$ . It is known (see [4] or [7]) that

(1.5) 
$$P\left\{\sup_{D} X(s, t) \geq \lambda\right\} \leq 4P\{X(1, 1) \geq \lambda\} = 4N(-\lambda).$$

Recently Chan [1] showed that, for every  $\varepsilon > 0$ ,

(1.6) 
$$P\left\{\sup_{D} X(s, t) \ge \lambda\right\} \le N(\varepsilon)^{-1} P\left\{\sup_{D} X(s, t) \ge \lambda - \varepsilon\right\}.$$

By the same technique as he used in his paper, the upper bound can easily be improved to  $N(\varepsilon)^{-1}P\{\sup X(1, t) \ge \lambda - \varepsilon: 0 \le t \le 1\} = 2N(-\lambda + \varepsilon)/N(\varepsilon)$ . However it turns out to be that even this improved upper bound is not as good as  $4N(-\lambda)$  for any  $\varepsilon > 0$ . In fact

$$4N(-\lambda) < N(arepsilon)^{-1}Pigg\{ \sup_{0 \leq t \leq 1} X(1,\,t) \geq \lambda - arepsilon igg\} \,, \ \ arepsilon > 0 \,,$$

and

$$\lim_{\epsilon o 0^+} N(\epsilon)^{-1} P \left\{ \sup_{0 \le t \le 1} X(1, t) \ge \lambda - \epsilon 
ight\} = 4 N(-\lambda) \; .$$

More recently Goodman [3] showed that for  $\lambda \ge 0$ ,

(1.7) 
$$2\left\{N(-\lambda)+\lambda\int_{\lambda}^{\infty}N(-s)ds\right\} \leq P\left\{\sup_{D}X(s,t)\geq\lambda\right\}.$$

Obviously the left-hand side of (1.7) is a much better lower bound of  $P\{\sup_D X(s, t) \ge \lambda\}$  than (1.4). He subsequently proves that

(1.8) 
$$\lim_{\lambda \to \infty} \frac{2\{N(-\lambda) + \lambda \int_{\lambda}^{\infty} N(-s) ds}{4N(-\lambda)} = 1,$$

thus showing that both  $2\left\{N(-\lambda) + \lambda \int_{\lambda}^{\infty} N(-s)ds\right\}$  and  $4N(-\lambda)$  are very good approximations of  $P\{\sup_{D} X(s, t) \ge \lambda\}$  for all sufficiently large  $\lambda$ .

The main purpose of this paper is to generalize the above results for more general barriers, namely, to find a formula for

$$P\left\{\sup_{s \in D} X(s, t) - (ast + bs + ct + d) \ge 0
ight\}$$
 , a, b, c,  $d \ge 0$  ,

and then find a lower bound for  $P\{\sup_D X(s, t) - (ast + bs + ct + d) \ge 0\}$  for which (1.7) is a special case. It is apparent that for all  $a, b, c, d \ge 0$ 

(1.9) 
$$P\left\{\sup_{D} X(s, t) - (ast + bs + ct + d) \ge 0\right\} \le 4N(-d)$$

In addition we obtain a formula for

$$P\{\sup_{\mathfrak{d} D} | X(s, t)| - (ast + bs + ct + d) \geqq 0\}$$
 ,  $a, b, c, d \geqq 0$  .

Some results on two-parameter Brownian bridge are also included.

2. Some lemmas. To avoid unnecessary repetitions in the proofs of the theorems, the following lemmas are given. Throughout this paper W(t) and X(s, t) will denote the standard Wiener process and the Yeh-Wiener process, respectively.

LEMMA 1. (Doob [2: p. 398]). If 
$$a \ge 0, b > 0, \alpha \ge 0, \beta > 0$$
, then  

$$P\left\{\sup_{0 \le t < \infty} [W(t) - (at + b)] \ge 0 \quad or \quad \inf_{0 \le t < \infty} [W(t) + \alpha t + \beta] \le 0\right\}$$

$$= \sum_{m=1}^{\infty} \exp\left\{-2[m^2ab + (m - 1)^2\alpha\beta + m(m - 1)(a\beta + \alpha b)]\right\}$$

 $+ \exp \{-2[(m-1)^2ab + m^2\alpha\beta + m(m-1)(a\beta + \alpha b)]\}$ 

$$-\exp\left\{-2[m^2(ab+lphaeta)+m(m-1)lphaeta+m(m+1)lpha b]
ight\}
ight. \ -\exp\left\{-2[m^2(ab+lphaeta)+m(m-1)aeta+m(m-1)lpha b]
ight\}.$$

LEMMA 2. Let f(t) be a Borel measurable function. Then for each Borel set E of real numbers,

(2.1) 
$$P\{W(t) - f(t) \in E, \ 0 < t \leq 1 | W(1) = u\} = P\{W(t) + u - (t+1)f(\frac{1}{t+1}) \in \frac{1}{t}E, \ 0 < t < \infty\}.$$

*Proof.* The basic technique used here is the same as the one used by Malmquist in [5]. Observe that W(t) and tW(1/t) are equivalent processes for t > 0. Thus, the left-hand side of (2.1) reduces to

$$\begin{split} P\Big\{W\Big(\frac{1}{t}\Big) &- \frac{1}{t}f(t) \in \frac{1}{t}E, \ 0 < t \leq 1 \ \middle| \ W(1) = u \Big\} \\ &= P\Big\{W\Big(\frac{1}{t}\Big) - W(1) - \left[\frac{1}{t}f(t) - u\right] \\ &\in \frac{1}{t}E, \ 0 < t \leq 1 \ \middle| \ W(1) = u \Big\} \;. \end{split}$$

Upon using the fact that W(1/t - 1) and W(1/t) - W(1) are equivalent processes for t > 0, and W(1/t) - W(1) and W(1) are independent for  $1 \ge t > 0$ , we have the result by the transformation  $1/t - 1 \rightarrow t$ .

LEMMA 2.a. If f(t) is a Borel measurable function on [0, 1], then

$$egin{aligned} &Pigg\{\sup_{0\leq t\leq 1}|X(1,\,t)|-f(t)\geq 0\,|\,X(1,\,1)=uigg\}\ &=Pigg\{\sup_{0\leq t<\infty}|X(1,\,t)+u\,|-(t\,+\,1)figg(rac{1}{t\,+\,1}igg)\geq 0igg\}\ , \end{aligned}$$

and the same holds for X(t, 1).

LEMMA 3. Let f(s, t) be a Borel measurable function on D. Then for each Borel set E of real numbers,

$$P\{X(s, t) - f(s, t) \in E, (s, t) \in (0, 1]^2 | X(1, 1) = u\}$$
  
=  $P\{X(s + 1, t + 1) - X(1, 1)$   
 $-\left[(s + 1)(t + 1)f\left(\frac{1}{s + 1}, \frac{1}{t + 1}\right) - u\right] \in \frac{E}{st}, (s, t) \in (0, \infty)^2\}.$ 

*Proof.* This lemma is a two-parameter analogue of Lemma 2, and it can be proved similarly by observing that X(s, t) and stX(1/s, 1/t) are equivalent processes for s, t > 0.

LEMMA 4. Let f(t) and g(t) be any Borel measurable functions on [0, 1]. Then for any Borel sets  $E_1$  and  $E_2$  of real numbers,

$$P\{X(s, 1) - f(s) \in E_1, X(1, t) - g(t) \in E_2, (s, t) \in D \mid X(1, 1) = u\}$$

$$(2.2) = P\{X(s, 1) - f(s) \in E_1, 0 \leq s \leq 1 \mid X(1, 1) = u\}$$

$$\cdot P\{X(1, t) - g(t) \in E_2, 0 \leq t \leq 1 \mid X(1, 1) = u\}.$$

*Proof.* Observe first that X(s, 1) and sX(1/s, 1) are equivalent standard Wiener processes for s > 0, and so are X(1, t) and tX(1, 1/t) for t > 0. Now s[X(1/s, 1) - X(1, 1) + u] and t[X(1, 1/t) - X(1, 1) + u] are independent processes for  $1 \ge s, t > 0$ , and they are also independent of  $\{X(s, t): (s, t) \in D\}$ . Hence (2.2) gives:

$$P\{X(s, 1) - f(s) \in E_1, X(1, t) - g(t) \in E_2, (s, t) \in D \mid X(1, 1) = u\}$$

$$(2.3) = P\{s[X(1/s, 1) - X(1, 1) + u] - f(s) \in E_1, 0 < s \leq 1\}$$

$$\cdot P\{t[X(1, 1/t) - X(1, 1) + u] - g(t) \in E_2, 0 < t \leq 1\}.$$

But the two probabilities on the right-hand side of (2.3) are equal to  $P\{X(s, 1) - f(s) \in E_1, 0 \leq s \leq 1 \mid X(1, 1) = u\}$  and  $P\{X(1, t) - g(t) \in E_2, 0 \leq t \leq 1 \mid X(1, 1) = u\}$  respectively, and hence the proof is complete.

3. Main results and proofs. In what follows  $\{X(s, t): s, t \ge 0\}$  will be used exclusively for the Yeh-Wiener process.

THEOREM 1. If a, b, c,  $d \ge 0$ , then with  $\overline{a} = a + b + c + d$ ,

$$P\left\{\sup_{\partial D} X(s, t) - (ast + bs + ct + d) \ge 0
ight\}$$
  
=  $N(-\overline{a}) + e^{-2(a+b)(c+d)}N(a + b - c - d)$   
+  $e^{-2(a+c)(b+d)}N(a - b + c - d)$   
-  $e^{2(d-a)(b+c+2d)}N(a - b - c - 3d)$ .

Proof. First observe that

$$P_{1} \equiv P\left\{\sup_{s \neq D} X(s, t) - (ast + bs + ct + d) \ge 0\right\}$$
  
=  $P\left\{\sup_{0 \le s \le 1} X(s, 1) - [(a + b)s + (c + d)] \ge 0\right\}$   
(3.1)  $+ P\left\{\sup_{0 \le t \le 1} X(1, t) - [(a + c)t + (b + d)] \ge 0\right\}$ 

$$-P\left\{\sup_{0\leq s\leq 1}X(s, 1) - [(a + b)s + (c + d)] \geq 0, \sup_{0\leq t\leq 1}X(1, t) - [(a + c)t + (b + d)] \geq 0\right\}.$$

Since X(s, 1) and X(1, t) are equivalent to the standard Wiener process W(t), the first two probabilities on the right of (3.1) can be evaluated explicitly.

Now,

$$\begin{split} P_2 &\equiv P \Big\{ \sup_{0 \leq s \leq 1} X(s, 1) - [(a + b)s + (c + d)] \geqq 0, \sup_{0 \leq t \leq 1} X(1, t) \\ &- [(a + c)t + (b + d)] \geqq 0 \Big\} \\ &= P \{ X(1, 1) \geqq \bar{a} \} \\ &+ \int_{-\infty}^{a+b+c+d} P \Big\{ \sup_{0 \leq s \leq 1} X(s, 1) - [(a + b)s + (c + d)] \geqq 0 \ , \\ &\sup_{0 \leq t \leq 1} X(1, t) - [(a + c)t + (b + d)] \geqq 0 | X(1, 1) = u \Big\} dN(u) \ . \end{split}$$

Due to the fact that

$$P\left\{\sup_{0 \le s \le 1} X(s, 1) - [(a + b)s + (c + d)] \ge 0, \sup_{0 \le t \le 1} X(1, t) \\ -[(a + c)t + (b + d)] \ge 0 | X(1, 1) = u\right\}$$

$$(3.2)$$

$$P\left\{\sup_{0 \le s \le 1} X(s, 1) - [(a + b)s + (c + d)] \ge 0 | X(1, 1) = u\right\}$$

$$\cdot P\left\{\sup_{0 \le t \le 1} X(1, t) - [(a + c)t + (b + d)] \ge 0 | X(1, 1) = u\right\},$$

we may use Lemma 2 to get

$$\begin{split} P_{2} &= N(-\bar{a}) \\ &+ \int_{-\infty}^{a+b+c+d} P\Big\{\sup_{s \ge 0} X(s,1) - [(c+d)s + (\bar{a}-u)] \ge 0 \Big\} \\ &\cdot P\Big\{\sup_{t \ge 0} X(1,t) - [(b+d)t + (\bar{a}-u)] \ge 0 \Big\} dN(u) \\ &= N(-\bar{a}) \\ &+ \int_{-\infty}^{a+b+c+d} e^{-2(c+d)(\bar{a}-u)} e^{-2(b+d)(\bar{a}-u)} dN(u) \\ &= N(-\bar{a}) + e^{2(d-a)(b+c+2d)} N(a-b-c-3d). \end{split}$$

The result now readily follows.

COROLLARY. If  $d \geq 0$ , then

$$P\left\{\sup_{\delta D} X(s, t) \ge d\right\} = 3N(-d) - e^{4d^2}N(-3d)$$
.

This corollary agrees with the result in [6: p. 877].

THEOREM 2. If  $\{Y(s, t): (s, t) \in D\}$  is the two-parameter Brownian bridge, i.e.,  $\{Y(s, t): (s, t) \in D\} = \{X(s, t): (s, t) \in D \mid X(1, 1) = 0\}$  and a, b, c,  $d \ge 0$ , then

$$P\{\sup_{s_{D}}Y(s, t) - (ast + bs + ct + d) \ge 0\}$$
  
=  $e^{-2(b+d)\overline{a}} + e^{-2(b+d)\overline{a}} - e^{-2(b+c+2d)\overline{a}}$ .

*Proof.* This follows from (3.2) by setting u = 0.

THEOREM 3. If a, b,  $c \ge 0$  and d > 0, then with  $\bar{a} = a + b + c + d$ and  $\bar{c} = c + d$ ,

$$Pig\{ \sup_{\mathfrak{d}^D} rac{|X(s,t)|}{ast+bs+ct+d} \geq 1 ig\} = 2f(a, b, c, d)$$
 ,

where

$$egin{aligned} f(a, b, c, d) &= N(-ar{a}) + \sum_{K=1}^{\infty} (-1)^{k+1} \Big[ e^{-2(a+b)ar{c}k^2} \int_{-ar{a}-2ar{c}k}^{ar{a}-2ar{c}k} dN(u) \ &+ e^{-2(a+c)(b+d)k^2} \int_{-ar{a}-2(b+d)k}^{ar{a}-2(b+d)k} dN(u) \Big] \ &- \sum_{j,k=1}^{\infty} (-1)^{j+k} e^{-2ar{a}[ar{c}j^2+(b+d)k^2]} \Big\{ e^{2[ar{c}j+(b+d)k]^2} \ & imes \int_{ar{a}-2[ar{c}j+(b+d)k]}^{ar{a}-2ar{c}[ar{c}j+(b+d)k]} dN(u) \ &+ e^{2ar{c}i-2ar{c}(ar{c}j-(b+d)k]^2} \int_{-ar{a}-2ar{c}ar{c}(b+d)k]}^{ar{a}-2ar{c}[ar{c}j+(b+d)k]} dN(u) \Big\} \ . \end{aligned}$$

*Proof.* Observe that

$$egin{aligned} P_{\mathfrak{z}}(u) &\equiv P\Big\{ \sup_{\mathfrak{d} D} rac{|X(s,t)|}{ast+bs+ct+d} &\geqq 1 \, \Big| \, X(1,\,1) = u \Big\} \ &= P\Big\{ \sup_{\mathfrak{d} \leq s \leq 1} rac{|X(s,1)|}{(a+b)s+(c+d)} &\geqq 1 \quad ext{or} \ && \sup_{\mathfrak{d} \leq t \leq 1} rac{|X(1,\,t)|}{(a+c)t+(b+d)} &\geqq 1 \, \Big| \, X(1,\,1) = u \Big\} \;. \end{aligned}$$

Upon applying Lemma 4, we obtain

$$egin{aligned} P_3(u) &= Pigg\{ \sup_{0 \leq s \leq 1} rac{|X(s, 1)|}{(a + b)s + (c + d)} \geq 1 \, \Big| \, X(1, 1) = u igg\} \ &+ Pigg\{ \sup_{0 \leq t \leq 1} rac{|X(1, t)|}{(a + c)t + (b + d)} \geq 1 \, \Big| \, X(1, 1) = u igg\} \end{aligned}$$

$$- P\left\{ \sup_{0 \le s \le 1} \frac{|X(s, 1)|}{(a + b)s + (c + d)} \ge 1 \, \Big| \, X(1, 1) = u \right\} \\ \cdot P\left\{ \sup_{0 \le t \le 1} \frac{|X(1, t)|}{(a + c)t + (b + d)} \ge 1 \, \Big| \, X(1, 1) = u \right\} \, .$$

Due to Lemma 2.a., it follows

$$\begin{split} P\Big\{ \sup_{0 \leq s \leq 1} \frac{|X(s, 1)|}{(a+b)s+(c+d)} &\geq 1 \, \Big| \, X(1, 1) = u \Big\} \\ &= P\Big\{ \sup_{0 \leq s < \infty} \frac{|X(s, 1) + u|}{(c+d)s + \bar{a}} &\geq 1 \Big\} \\ &= P\Big\{ \sup_{0 \leq s < \infty} X(s, 1) - [(c+d)s + (\bar{a} - u)] &\geq 0 \\ \text{or } \inf_{0 \leq s < \infty} X(s, 1) + [(c+d)s + (\bar{a} + u)] &\leq 0 \Big\} \,. \end{split}$$

Lemma 1 applied to the last expression gives:

$$P\left\{\sup_{0 \le s < \infty} X(s, 1) - [(c + d)s(\bar{a} - u)] \ge 0$$
  
or  $\inf_{0 \le s < \infty} X(s, 1) + [(c + d)s + (\bar{a} + u)] \le 0\right\}$   
$$= \sum_{m=1}^{\infty} \left\{ e^{-2\bar{a}(c+d)(2m-1)^2} [e^{2(c+d)(2m-1)u} + e^{-2(c+d)(2m-1)u}] - e^{-2\bar{a}(c+d)(2m)^2} [e^{2(c+d)(2m)u} + e^{-2(c+d)(2m)u}] \right\}.$$

Therefore

$$P\left\{\sup_{0\leq s\leq 1}rac{|X(s, 1)|}{(a+b)s+(c+d)}\geq 1\,\Big|\,X(1, 1)=u
ight\}\ =\sum_{j=1}^{\infty}(-1)^{j+1}e^{-2\overline{a}(c+d)j^2}[e^{2(c+d)ju}+e^{-2(c+d)ju}]$$

and

$$P\left\{\sup_{0\leq t\leq 1}\frac{|X(1,t)|}{(a+c)t+(b+d)}\geq 1 \,\middle|\, X(1,1)=u\right\}$$
$$=\sum_{k=1}^{\infty}(-1)^{k+1}e^{-2\overline{a}(b+d)k^2}[e^{2(b+d)ku}+e^{-2(b+d)ku}].$$

Since X(1, 1) is the standard normal random variable, the result now follows by:

$$egin{aligned} &Pig\{\sup_{\mathfrak{d}\mathcal{D}}rac{|X(s,t)|}{ast+bs+ct+d} \geqq 1ig\}\ &= P\{|X(1,1)|\geqq ar{a}\}+\int_{-ar{a}}^{ar{a}}P_{\mathfrak{z}}(u)dN(u) \end{aligned}$$

$$=2N(-ar{a})+\int_{-ar{a}}^{ar{a}}P_{3}(u)dN(u)$$
 .

THEOREM 4. If a, b, c,  $d \ge 0$  and  $\bar{a} = a + b + c + d$ ,  $\bar{b} = b + c + d$ ,  $\bar{c} = c + d$ , then for  $u < \bar{a}$ ,

$$egin{aligned} P_4 &\equiv Pigg\{ \sup_D X(s,\,t) - (ast+bs+ct+d) &\geq 0 \, \Big| \, X(1,\,1) = u igg\} \ &\geq igg\{ &rac{ar b}{b} [e^{2ar c(u-ar a)} - e^{-2ar b(u-ar a)}] + e^{2ar b(u-ar a)} \,, \quad b > 0 \ &e^{2ar c(u-ar a)} [1 + 2ar c(ar a - u)] \,, \quad b = 0 \,. \end{aligned}$$

Proof. Upon applying Lemma 3, we obtain

$$P_{4} = P\left\{\sup_{s,t\geq 0} X(s+1,t+1) - X(1,t+1) + X(1,t+1) - X(1,1) \\ - \left[d(s+1)(t+1) + c(s+1) + b(t+1) + a - u\right] \geq 0\right\}.$$

Consider the fact that X(s + 1, t + 1) - X(1, t + 1) and X(1, t + 1) - X(1, 1) are independent processes equivalent to X(s, t + 1) and X(1, t), respectively. The latter X(1, t) will be denoted by  $X^*(1, t)$  to signify that it is independent of X(s, t + 1). Due to the fact that  $c(s + 1) \leq c(s + 1)(t + 1)$  for all  $c, s, t \geq 0$ , it follows from (3.3)

$$P_{4} \geq P\left\{\sup_{s,t\geq 0} X(s, t+1) + X^{*}(1, t) - [\bar{c}(t+1)s + \bar{b}t + \bar{a} - u] \geq 0\right\}$$

$$(3.4) \geq \int_{u-\bar{a}}^{\infty} P\left\{\sup_{s\geq 0} X(s, t+1) - [\bar{c}(t+1)s - r] \geq 0 \middle| \sup_{t\geq 0} X^{*}(1, t) - (\bar{b}t + \bar{a} - u) = r\right\} p(r, u) dr,$$

where p(r, u) is the probability density of

$$egin{aligned} &Pigg\{\sup_{t \geq 0} X^*(1,\,t) - (ar{b}t + ar{a} - u) \leq rigg\} \ &= igg\{ 1 - e^{-2ar{b}(ar{a} + r - u)} \ , \ \ u - ar{a} \leq r \ 0 \ \ \ , \ \ ext{otherwise} \ . \end{aligned}$$

Thus

(3.5) 
$$p(r, u) = \begin{cases} 2\overline{b}e^{-2\overline{b}(\overline{a}+r-u)}, & u-\overline{a} \leq r \\ 0, & \text{otherwise} \end{cases}$$

Observe that the probability in the integrand of (3.4) becomes

(3.6)  
$$P\left\{\sup_{s\geq 0} X(s, t+1) - [\bar{c}(t+1)s - r] \geq 0\right\} = \left\{ \begin{aligned} e^{s\bar{c}r} &, r \leq 0\\ 1 &, r > 0 \end{aligned} \right.$$

Therefore, (3.5) and (3.6) together with (3.4) give

$$P_4 \geqq \int_{u-\overline{a}}^{_0} e^{\overline{c} \overline{c} r} 2\overline{b} \overline{e}^{-2\overline{b}(\overline{a}+r-u)} dr + \int_{_0}^{_\infty} 2\overline{b} \overline{e}^{-2\overline{b}(\overline{a}+r-u)} dr$$
 ,

from which the result readily follows.

The following is a special case (u = 0) of Theorem 4, which has broad application in Kolmogorov-Smirnov statistics.

THEOREM 4.a. If  $\{Y(s, t): (s, t) \in D\}$  is the two-parameter Brownian bridge and if a, b, c,  $d \ge 0$ , then

$$egin{aligned} &P\{\sup_D Y(s,\,t)\,-\,(ast\,+\,bs\,+\,ct\,+\,d)\geqq 0\}\ &\ge& \left\{ egin{aligned} &ar{b}\ b\ c&=\,\overline{cac}\ -\,e^{-2\overline{a}\overline{b}}\ )\,+\,e^{-2\overline{a}\overline{b}}\ ,\ b\,>\,0\ &(1\,+\,2\overline{a}\overline{c})e^{-\overline{2}\overline{a}\overline{b}}\ ,\ b\,=\,0\ . \end{aligned}
ight. \end{aligned}$$

THEOREM 5. If a, b, c,  $d \ge 0$ , then

$$P\{\sup_{D} X(s, t) - (ast + bs + ct + d) \ge 0\}$$
  
 $\ge \begin{cases} N(-\bar{a}) + rac{ar{b}}{b} [N(ar{a} - 2ar{c})e^{-2ar{c}(a+b)} - N(ar{a} - 2ar{b})e^{-2aar{b}}] + N(ar{a} - 2ar{b})e^{-2aar{b}}, & b > 0 \end{cases}$ 
 $N(-ar{a}) + rac{2ar{c}}{\sqrt{2\pi}}e^{-ar{a}^2/2} + N(ar{a} - 2ar{c})(1 + 2ar{a}ar{c} - 4ar{c}^2)e^{-2aar{c}}, & b = 0 \end{cases}$ 

In particular,

$$P\{\sup_D X(s, t) - \lambda \ge 0\} \ge 2 \Big[ (1 - \lambda^2) N(-\lambda) + rac{\lambda}{\sqrt{2\pi}} e^{-\lambda^2/2} \Big] \ = 2 \Big[ N(-\lambda) + \lambda \int_{\lambda}^{\infty} N(-s) ds \Big], \quad \lambda \ge 0 \;.$$

*Proof.* The theorem now can be established by integrating lower estimates of the conditional probability  $P_4$  in Theorem 4 with respect to  $dP\{X(1, 1) \leq u\} = dN(u) = (2\pi)^{-1/2} \exp(-u^2/2) du$ . The special case when a = b = c = 0 and  $d = \lambda$  agrees with Goodman's result (Theorem 3 in [3]).

In order to find sharper upper bounds for the barrier-crossing

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probabilities we introduce the following: Let f(s, t) be a continuous function on D. If  $\sup_D X(s, t) - f(s, t) \ge 0$ , then define  $\tau_f = (s_0, t_0)$  where

$$egin{aligned} s_0 &= \inf \left\{ s \in [0,\,1] \, | \, X(s,\,t) = f(s,\,t) \, \, ext{for some } t \in [0,\,1] 
ight\}, \ t_0 &= \inf \left\{ t \in [0,\,1] \, | \, X(s_0,\,t) = f(s_0,\,t) 
ight\}, \end{aligned}$$

while if  $\sup_D X(s, t) - f(s, t) < 0$ , then set  $\tau_f = (\infty, \infty)$ . Thus with the convention that  $(s_1, t_1) \leq (s_2, t_2)$  if and only if  $s_1 \leq s_2$  and  $t_1 \leq t_2$ , we have that

$$P\left\{\sup_{D}X(s, t) - f(s, t) \ge 0\right\} = P\left\{ au_{f} \le (1, 1)\right\}.$$

THEOREM 6. If  $c, d \ge 0$ , then

$$egin{aligned} &Pigg\{ \sup_{D}X(s,\,t)-(ct\,+\,d)\geqq\mathbf{0}igg\}\ &\leqq 2Pigg\{ \sup_{0\le t \le 1}X(1,\,t)-(ct\,+\,d)\geqq\mathbf{0}igg\}\ &=2[1\,-\,N(c\,+\,d)\,+\,\exp{(-2cd)N(a\,-\,b)}] \ . \end{aligned}$$

*Proof.* Let  $\tau$  stand for  $\tau_f$  when f(s, t) = ct + d. Define

$$F(s, t) \equiv P\{\tau \leq (s, t)\}$$

Then

$$\begin{split} F(\mathbf{1},\,\mathbf{1}) &= P\Big\{\sup_{D} X(s,\,t) - (ct\,+\,d) \geqq 0\Big\} \\ &= P\Big\{\sup_{0 \le t \le 1} X(\mathbf{1},\,t) - (ct\,+\,d) \geqq 0\Big\} \\ &+ P\Big\{\sup_{0 \le t \le 1} X(\mathbf{1},\,t) - (ct\,+\,d) < \mathbf{0},\,\sup_{D} X(s,\,t) - (ct\,+\,d) \geqq 0\Big\} \\ (3.7) &= P\Big\{\sup_{0 \le t \le 1} X(\mathbf{1},\,t) - (ct\,+\,d) \geqq 0\Big\} \\ &+ \int_{\mathbf{0}}^{1} P\Big\{\sup_{0 \le t' \le 1} X(\mathbf{1},\,t') - (ct'\,+\,d) < \mathbf{0} \,\Big| \,\tau = (s,\,t)\Big\} dF(s,\,t) \\ & \le P\Big\{\sup_{0 \le t \le 1} X(\mathbf{1},\,t) - (ct\,+\,d) \geqq 0\Big\} \\ &+ \int_{\mathbf{0}}^{1} P\Big\{X(\mathbf{1},\,t) - (ct\,+\,d) < \mathbf{0} \,\Big| \,\tau = (s,\,t)\Big\} dF(s,\,t) \,. \end{split}$$

On account of the fact that  $\tau = (s, t)$  implies X(s, t) = ct + d and X(1, t) - X(s, t) is independent of the conditioning  $\tau = (s, t)$ , it follows that

(3.8) 
$$\int_{0}^{1} P \Big\{ X(1, t) - (ct + d) < 0 \Big| \tau = (s, t) \Big\} dF(st) \\= \int_{0}^{1} P \Big\{ X(1, t) - X(s, t) < 0 \Big\} dF(s, t) = \frac{1}{2} F(1, 1) .$$

The theorem now follows readily from (3.7) and (3.8).

COROLLARY 6.1. If b, c,  $d \ge 0$ , then

(3.9)  
$$P\left\{\sup_{D} X(s, t) - (bs + ct + d) \ge 0\right\} \\ \le 2P\left\{\sup_{0 \le t \le 1} X(1, t) - (b^*t + d) \ge 0\right\}, \quad b^* = \max\{b, c\}.$$

Proof. The result follows immediately by observing that

$$egin{aligned} &P\left\{\sup_{D}X(s,\,t)-(bs\,+\,ct\,+\,d)\geqq\mathbf{0}
ight\}\ &\le\min\left\{\mathit{P}\left[\sup_{D}X(s,\,t)-(bs\,+\,d)\geqq\mathbf{0}
ight],\ &P\left[\sup_{D}X(s,\,t)-(ct\,+\,d)\geqq\mathbf{0}
ight]
ight\}. \end{aligned}$$

The right-hand side of (3.9) can also serve as an upper bound of  $P\{\sup_D X(s, t) - (ast + bs + ct + d) \ge 0\}$ , and it is certainly a substantial improvement over (1.9). We state this fact formally as a corollary.

COROLLARY 6.2. If a, b, c,  $d \ge 0$ , then  $P\left\{\sup_{D} X(s, t) - (ast + bs + ct + d) \ge 0\right\}$ (3.10)  $\le 2P\left\{\sup_{0\le t\le 1} X(1, t) - (b^*t + d) \ge 0\right\}$  $\le 2P\left\{\sup_{0\le t\le 1} X(1, t) - d \ge 0\right\} = 4N(-d)$ ,

where  $b^* = \max{\{b, c\}}$ .

4. Supremum over rectangular regions. Some adjustments are needed to apply the results for the more general rectangular region  $Q = [0, S] \times [0, T]$ . The conversion formulas are given by:

(4.1)  
$$P\left\{\sup_{\partial Q} X(s, t) - (ast + bs + ct + d) \ge 0\right\}$$
$$= P\left\{\sup_{\partial D} X(s, t) - (a'st + b's + c't + d') \ge 0\right\},$$

where  $a' = a\sqrt{ST}$ ,  $b' = b\sqrt{S/T}$ ,  $c' = c\sqrt{T/S}$ , and  $d' = d/\sqrt{ST}$ .

(4.2)  
$$P\left\{\sup_{Q} X(s, t) - (ast + bs + ct + d) \ge 0 | X(S, T) = u\right\}$$
$$= P\left\{\sup_{D} X(s, t) - (a'st + b's + c't + d') \ge 0 | X(1, 1) = u'\right\},$$

where a', b', c', d' are as in (4.1) and  $u' = u/\sqrt{\overline{ST}}$ . In (4.1), if  $\partial Q$  is replaced by Q, then D replaces  $\partial D$ .

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