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## **A REMARK ON INFINITELY NUCLEARLY DIFFERENTIABLE FUNCTIONS**

TEÓFILO ABUABARA

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**There is an infinitely nuclearly differentiable function of bounded type from  $E$  to  $R$  which is not of bounded-compact type, when  $E = l_1$ , the Banach space of all summable sequences of real numbers.**

Let  $E$  and  $F$  be two real Banach spaces. A mapping  $f: E \rightarrow F$  is said to be weakly uniformly continuous on bounded subsets of  $E$  if for each bounded set  $B \subset E$  and each  $\varepsilon > 0$ , there are  $\phi_1, \phi_2, \dots, \phi_k \in E'$  and  $\delta > 0$  such that if  $x, y \in B$ ,  $|\phi_i(x) - \phi_i(y)| < \delta (i = 1, 2, \dots, k)$ , then  $\|f(x) - f(y)\| < \varepsilon$ .  $C_w^m(E; F)$  is the space of  $m$ -times continuously differentiable mappings  $f: E \rightarrow F$  satisfying the following conditions:

(1)  $\hat{d}^j f(x) \in \mathcal{P}_w^j(E; F) (x \in E, j \leq m)$

(2)  $\hat{d}^j f: E \rightarrow \mathcal{P}_w^j(E; F)$  is weakly uniformly continuous on bounded subsets of  $E$ , where  $\mathcal{P}_w^m(E; F) (m \in \mathbb{N})$  is the Banach space of continuous  $m$ -homogeneous polynomials which are weakly uniformly continuous on bounded subsets of  $E$ , its norm being the one induced on it by the current norm of  $\mathcal{P}^m(E; F)$ . Set

$$C_w^\infty(E; F) = \bigcap_{m=0}^{+\infty} C_w^m(E; F).$$

$C_w^m(E; F)$  is endowed with the topology  $\tau_w^m$  generated by the following system of semi-norms

$$f \in C_w^m(E; F) \sup \{ \|\hat{d}^j f(x)\|; x \in B, j \leq m \},$$

where  $B$  runs through the bounded subsets of  $E$ .

For further details we refer to Aron-Prolla [1].

**PROPOSITION 1** (Aron-Prolla [1]). *If  $E'$  has the bounded approximation property, then  $\mathcal{P}_f(E; F)$  is  $\tau_w^m$ -dense in  $C_w^m(E; F)$ , for all  $m \geq 1$ .*

*Hence, since  $\|P\| \leq \|P\|_N$  for every  $P \in \mathcal{P}_N^m(E; F) (m \in \mathbb{N})$ , then  $\mathcal{E}_{Nbc}(E; F)$  is contained in  $C_w^\infty(E; F)$ .*

**PROPOSITION 2** (Aron-Prolla [1]). *Let  $f: E \rightarrow F$  be a weakly uniformly continuous mapping on bounded sets. If  $B \subset E$  is a bounded set, then  $f(B)$  is precompact.*

**PROPOSITION 3.**  $\mathcal{E}_{Nbc}(l_1) \neq \mathcal{E}_{Nb}(l_1)$ , that is, there is an infinitely

nuclearly differentiable function of bounded type from  $l_1$  to  $R$  which is not of bounded-compact type.

*Proof.* Set

$$g: R \longrightarrow R \quad t \longmapsto g(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Let us define

$$f: l_1 \longrightarrow R \quad (x_n)_n \longrightarrow f((x_n)_n) = \sum_{n=1}^{+\infty} g(x_n).$$

Then  $f$  is an infinitely nuclearly differentiable function of bounded type, but it is not of bounded-compact type. Indeed,

(a)  $f \in \mathcal{E}_{N^b}(l_1)$ . (i)  $f$  is bounded on bounded subsets of  $l_1$ . More precisely, there is  $\varepsilon > 0$  such that if  $x \in l_1$ ,  $\|x\|_1 \leq R$ , then  $|f(x)| \leq R(1 + 1/\varepsilon)$ . Indeed, since  $\lim_{t \rightarrow 0} 1/t \cdot g(t) = 0$ , there is  $\varepsilon > 0$  such that if  $|t| < \varepsilon$ , then  $g(t) < |t|$ . Now, if  $\|x\|_1 \leq R$ , then we get that  $\text{card}(\{n; |x_n| \geq \varepsilon\}) \leq R/\varepsilon$ . Therefore, if  $\|x\|_1 \leq R$ , we have that

$$|f(x)| = \sum_{n=1}^{+\infty} g(x_n) = \sum_{x_n \geq \varepsilon} e^{-1/x_n} + \sum_{|x_n| < \varepsilon} g(x_n) \leq R/\varepsilon + \|x\|_1 \leq R(1 + 1/\varepsilon).$$

Hence  $f$  is bounded of bounded sets.

(ii)  $f \in C^\infty(l_1)$ . Indeed, for every fixed  $x = (x_n)_n \in l_1$ , let  $K = \overline{\{x_n\}_n} \subset R$  and let

$$L_k(x) = \sum_{n=1}^{+\infty} \overbrace{g^{(k)}(x_n) e_n \times e_n \times \cdots \times e_n}^{k\text{-times}},$$

$\uparrow$   
 $n\text{th}$

for  $k = 1, 2, \dots$ , where  $e_n = (0, 0, \dots, 0, 1, 0, \dots)$ . Notice that  $L_k(x) \in \mathcal{L}^{(k)}(l_1)$ , since if  $M = \sup_n |g^{(k)}(x_n)|$ , then  $\|L_k(x)(h_1, h_2, \dots, h_k)\| \leq M \|h_1\|_1 \|h_2\|_1 \cdots \|h_k\|_1$ . Let us show that  $d^k f(x)$  exists and  $d^k f(x) = L_k(x)$  for  $k = 1, 2, \dots$ , using induction on  $k$ . Indeed, for  $k = 1$ , since  $g$  is uniformly differentiable on compact sets, given  $\varepsilon > 0$  there is  $\delta > 0$  such that

$$|v| < \delta \implies |g(t+v) - g(t) - g'(t)v| < \varepsilon |v|,$$

for every  $t \in K$ . Therefore,

$$h \in l_1, \|h\|_1 < \delta \implies |f(x+h) - f(x) - L_1(x)h| < \varepsilon \|h\|_1.$$

It follows that  $df(x) = L_1(x)$ . Let us assume that  $d^k f(x) = L_k(x)$ . Then,

$$\begin{aligned} & \|d^k f(x+h) - d^k f(x) - L_{k+1}(x)h\| \\ &= \left\| \sum_{n=1}^{+\infty} (g^{(k)}(x_n+h_n) - g^{(k)}(x_n) - g^{(k+1)}(x_n)h_n) \cdot e_n \times e_n \times \dots \times e_n \right\| \\ &= \sum_{n=1}^{+\infty} \left| g^{(k)}(x_n+h_n) - g^{(k)}(x_n) - g^{(k+1)}(x_n)h_n \right|. \end{aligned}$$

Now, since  $g^{(k)}$  is uniformly differentiable on compact sets, given  $\varepsilon > 0$ , there is  $\delta > 0$  such that

$$|v| < \delta \implies |g^{(k)}(t+v) - g^{(k)}(t) - g^{(k+1)}(t)v| < \varepsilon |v|,$$

for every  $t \in K$ . Thus,

$$h \in l_1, \|h\|_1 < \delta \implies \|d^k f(x+h) - d^k f(x) - L_{k+1}(x)h\| < \varepsilon \|h\|_1.$$

Hence,  $d^{k+1}f(x) = L_{k+1}(x)$ . It follows that  $f \in C^\infty(l_1)$ .

$$(iii) \quad \hat{d}^k f(x) = \sum_{n=1}^{+\infty} g^{(k)}(x_n) \cdot e_n^k \in \mathcal{S}_N(kl_1).$$

Moreover,  $\hat{d}^k f: l_1 \rightarrow \mathcal{S}_N(kl_1)$  is bounded on bounded sets. Indeed, since  $\lim_{t \rightarrow 0} 1/t \cdot g^{(k)}(t) = 0$ , there is  $\varepsilon > 0$  such that if  $|t| < \varepsilon$ , then  $|g^{(k)}(t)| < |t|$ . Now, if  $x \in l_1, \|x\|_1 \leq R$ , then  $\text{card}(\{n; |x_n| \geq \varepsilon\}) \leq R/\varepsilon$ . Therefore, if  $\|x\|_1 \leq R$ , we have that

$$\begin{aligned} \|\hat{d}^k f(x)\|_N &\leq \sum_{n=1}^{+\infty} |g^{(k)}(x_n)| \\ &= \sum_{x_n \geq \varepsilon} |P(1/x_n)| e^{-1/x_n} + \sum_{|x_n| < \varepsilon} |g^{(k)}(x_n)| \\ &\leq |P|(1/\varepsilon) \cdot R/\varepsilon + \|x\|_1 \\ &\leq R(1 + |P|(1/\varepsilon)/\varepsilon), \end{aligned}$$

where if  $P = \Sigma a_n z^n$ , then  $|P| = \Sigma |a_n| z^n$ . Hence the assertion follows.

(iv) The mapping  $\hat{d}^k f: l_1 \rightarrow \mathcal{S}_N(kl_1)$  is differentiable of first order when  $\mathcal{S}_N(kl_1)$  is endowed with its nuclear norm. Indeed, set

$$T_k(x) = \sum_{n=1}^{+\infty} (g^{(k+1)}(x_n) \cdot e_n) \cdot e_n^k \in \mathcal{L}(l_1; \mathcal{S}_N(kl_1)),$$

for  $k = 0, 1, 2, \dots$ . Then

$$\begin{aligned} & \|\hat{d}^k f(x+h) - \hat{d}^k f(x) - T_k(x)h\|_N \\ &= \left\| \sum_{n=1}^{+\infty} (g^{(k)}(x_n+h_n) - g^{(k)}(x_n) - g^{(k+1)}(x_n)h_n) \cdot e_n^k \right\|_N \\ &\leq \sum_{n=1}^{+\infty} |g^{(k)}(x_n+h_n) - g^{(k)}(x_n) - g^{(k+1)}(x_n)h_n|. \end{aligned}$$

As in (iii), given  $\varepsilon > 0$ , there is  $\delta > 0$  such that

$$h \in l_1, \|h\|_1 < \delta \implies \|\hat{d}^k f(x+h) - \hat{d}^k f(x) - T_k(x)h\|_N < \varepsilon \|h\|_1.$$

Hence,  $d(\hat{d}^k f)(x) = T_k(x)$ , when  $\mathcal{S}_N(kl_1)$  is endowed with the nuclear norm. Moreover, the mapping  $T_k: l_1 \rightarrow \mathcal{L}(l_1; \mathcal{S}_N(kl_1))$  is continuous,

for  $k = 0, 1, 2, \dots$ . Indeed,

$$T_k(x + h) - T_k(x) = \sum_{n=1}^{+\infty} [(g^{(k+1)}(x_n + h_n) - g^{(k+1)}(x_n)) \cdot e_n] \cdot e_n^k.$$

Therefore,

$$\begin{aligned} \|T_k(x + h) - T_k(x)\| &= \sup_{\|w\|_1 \leq 1} \left\| \sum_{n=1}^{+\infty} (g^{(k+1)}(x_n + h_n) - g^{(k+1)}(x_n)) w_n \cdot e_n^k \right\|_N \\ &\leq \sup_{\|w\|_1 \leq 1} \sum_{n=1}^{+\infty} |g^{(k+1)}(x_n + h_n) - g^{(k+1)}(x_n)| |w_n| \\ &= \sum_{n=1}^{+\infty} |g^{(k+1)}(x_n + h_n) - g^{(k+1)}(x_n)| \\ &= \sum_{n=1}^{+\infty} |g^{(k+2)}(\theta_n)| |h_n|, \end{aligned}$$

where  $\theta_n \in (x_n, x_n + h_n)$ . Set  $\alpha = \sup_{y \in [0, \max_n |x_n| + 1]} |g^{(k+2)}(y)|$ . Given  $\varepsilon > 0$ , set  $\delta = \min \{\varepsilon/\alpha, 1\}$ . Then,

$$h \in l_1, \|h\|_1 < \delta \implies \|T_k(x + h) - T_k(x)\| < \varepsilon.$$

It follows that  $T_k$  is continuous. Thus,  $T_k$  is differentiable of first order.

(i)-(iv) imply  $f \in \mathcal{E}_{N^b}(l_1)$ .

(b)  $f \notin \mathcal{E}_{N^b}(l_1)$ . Indeed,  $df(e_n) = e^{-1} \cdot e_n$ . Therefore,  $df(B_1)$  is not a precompact subset of  $l'_1$ , where  $B_1$  is the unit ball of  $l_1$ . Hence the assertion follows of Propositions 1 and 2 above.

Hence Proposition 3 follows.

I thank Richard Aron for valuable conversations.

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