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# A REMARK ON INFINITELY NUCLEARLY DIFFERENTIABLE FUNCTIONS

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**There is an infinitely nuclearly differentiable function of bounded type from $E$ to $R$ which is not of bounded-compact type, when $E = l_1$, the Banach space of all summable sequences of real numbers.**

Let $E$ and $F$ be two real Banach spaces. A mapping $f: E \to F$ is said to be weakly uniformly continuous on bounded subsets of $E$ if for each bounded set $B \subset E$ and each $\varepsilon > 0$, there are $\phi_1, \phi_2, \cdots, \phi_k \in E'$ and $\delta > 0$ such that if $x, y \in B$, $|\phi_i(x) - \phi_i(y)| < \delta (i = 1, 2, \cdots, k)$, then $\|f(x) - f(y)\| < \varepsilon$. $C_w^m(E; F)$ is the space of $m$-times continuously differentiable mappings $f: E \to F$ satisfying the following conditions:

(1) $\hat{d}^j f(x) \in \mathscr{P}_w({}^jE; F)(x \in E, j \leqq m)$

(2) $\hat{d}^j f: E \to \mathscr{P}_w({}^jE; F)$ is weakly uniformly continuous on bounded subsets of $E$, where $\mathscr{P}_w({}^mE; F)(m \in N)$ is the Banach space of continuous $m$-homogeneous polynomials which are weakly uniformly continuous on bounded subsets of $E$, its norm being the one induced on it by the current norm of $\mathscr{P}({}^mE; F)$. Set

$$C_w^\infty(E; F) = \bigcap_{m=0}^{+\infty} C_w^m(E; F) .$$

$C_w^m(E; F)$ is endowed with the topology $\tau_b^m$ generated by the following system of semi-norms

$$f \in C_w^m(E; F) \sup \{\|\hat{d}^j f(x)\|; x \in B, j \leqq m\} ,$$

where $B$ runs through the bounded subsets of $E$.

For further details we refer to Aron-Prolla [1].

PROPOSITION 1 (Aron-Prolla [1]). *If $E'$ has the bounded approximation property, then $\mathscr{P}_f(E; F)$ is $\tau_b^m$-dense in $C_w^m(E; F)$, for all $m \geqq 1$.*

*Hence, since $\|P\| \leqq \|P\|_N$ for every $P \in \mathscr{P}_N({}^mE; F)(m \in N)$, then $\mathscr{E}_{Nbc}(E; F)$ is contained in $C_w^\infty(E; F)$.*

PROPOSITION 2 (Aron-Prolla [1]). *Let $f: E \to F$ be a weakly uniformly continuous mapping on bounded sets. If $B \subset E$ is a bounded set, then $f(B)$ is precompact.*

PROPOSITION 3. $\mathscr{E}_{Nbc}(l_1) \neq \mathscr{E}_{Nb}(l_1)$, *that is, there is an infinitely* *nuclearly differentiable function of bounded type from $l_1$ to $R$ which is not of bounded-compact type.*

*Proof.* Set

$$g\colon \boldsymbol{R} \longrightarrow \boldsymbol{R} \quad t \longmapsto g(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leqq 0 . \end{cases}$$

Let us define

$$f\colon l_1 \longrightarrow \boldsymbol{R} \quad (x_n)_n \longrightarrow f((x_n)_n) = \sum_{n=1}^{+\infty} g(x_n) .$$

Then $f$ is an infinitely nuclearly differentiable function of bounded type, but it is not of bounded-compact type. Indeed,

(a) $f \in \mathscr{E}_{Nb}(l_1)$. (i) $f$ is bounded on bounded subsets of $l_1$. More precisely, there is $\varepsilon > 0$ such that if $x \in l_1$, $\|x\|_1 \leqq R$, then $|f(x)| \leqq R(1 + 1/\varepsilon)$. Indeed, since $\lim_{t\to 0} 1/t \cdot g(t) = 0$, there is $\varepsilon > 0$ such that if $|t| < \varepsilon$, then $g(t) < |t|$. Now, if $\|x\|_1 \leqq R$, then we get that $\operatorname{card}(\{n; |x_n| \geqq \varepsilon\}) \leqq R/\varepsilon$. Therefore, if $\|x\|_1 \leqq R$, we have that

$$|f(x)| = \sum_{n=1}^{+\infty} g(x_n) = \sum_{x_n \geqq \varepsilon} e^{-1/x_n} + \sum_{|x_n| < \varepsilon} g(x_n) \leqq R/\varepsilon + \|x\|_1 \leqq R(1 + 1/\varepsilon) .$$

Hence $f$ is bounded of bounded sets.

(ii) $f \in C^\infty(l_1)$. Indeed, for every fixed $x = (x_n)_n \in l_1$, let $K = \overline{\{x_n\}_n} \subset \boldsymbol{R}$ and let

$$L_k(x) = \sum_{n=1}^{+\infty} g^{(k)}(x_n) \overbrace{e_n \times e_n \times \cdots \times e_n}^{k\text{-times}} ,$$

for $k = 1, 2, \cdots$, where $e_n = (0, 0, \cdots, 0, \underset{\substack{\uparrow \\ n\text{th}}}{1}, 0, \cdots)$. Notice that $L_k(x) \in \mathscr{L}({}^k l_1)$, since if $M = \sup_n |g^{(k)}(x_n)|$, then $\|L_k(x)(h_1, h_2, \cdots, h_k)\| \leqq M\|h_1\|_1\|h_2\|_1 \cdots \|h_k\|_1$. Let us show that $d^k f(x)$ exists and $d^k f(x) = L_k(x)$ for $k = 1, 2, \cdots$, using induction on $k$. Indeed, for $k = 1$, since $g$ is uniformly differentiable on compact sets, given $\varepsilon > 0$ there is $\delta > 0$ such that

$$|v| < \delta \Longrightarrow |g(t + v) - g(t) - g'(t)v| < \varepsilon|v| ,$$

for every $t \in K$. Therefore,

$$h \in l_1,\ \|h\|_1 < \delta \Longrightarrow |f(x + h) - f(x) - L_1(x)h| < \varepsilon\|h\|_1 .$$

It follows that $df(x) = L_1(x)$. Let us assume that $d^k f(x) = L_k(x)$. Then,

$$\begin{aligned}
&\|d^k f(x+h) - d^k f(x) - L_{k+1}(x)h\| \\
&= \left\|\sum_{n=1}^{+\infty} (g^{(k)}(x_n+h_n) - g^{(k)}(x_n) - g^{(k+1)}(x_n)h_n)\cdot e_n \times e_n \times \cdots \times e_n\right\| \\
&= \sum_{n=1}^{+\infty} \left| g^{(k)}(x_n+h_n) - g^{(k)}(x_n) - g^{(k+1)}(x_n)h_n \right| .
\end{aligned}$$

Now, since $g^{(k)}$ is uniformly differentiable on compact sets, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$|v| < \delta \Longrightarrow |g^{(k)}(t+v) - g^{(k)}(t) - g^{(k+1)}(t)v| < \varepsilon|v| ,$$

for every $t \in K$. Thus,

$$h \in l_1,\ \|h\|_1 < \delta \Longrightarrow \|d^k f(x+h)\text{-}d^k f(x) - L_{k+1}(x)h\| < \varepsilon\|h\|_1 .$$

Hence, $d^{k+1}f(x) = L_{k+1}(x)$. It follows that $f \in C^\infty(l_1)$.

(iii) $\hat{d}^k f(x) = \sum_{n=1}^{+\infty} g^{(k)}(x_n)\cdot e_n^k \in \mathscr{P}_N({}^k l_1)$.

Moreover, $\hat{d}^k f\colon l_1 \to \mathscr{P}_N({}^k l_1)$ is bounded on bounded sets. Indeed, since $\lim_{t\to 0} 1/t \cdot g^{(k)}(t) = 0$, there is $\varepsilon > 0$ such that if $|t| < \varepsilon$, then $|g^{(k)}(t)| < |t|$. Now, if $x \in l_1$, $\|x\|_1 \leqq R$, then card $(\{n;\ |x_n| \geqq \varepsilon\}) \leqq R/\varepsilon$. Therefore, if $\|x\|_1 \leqq R$, we have that

$$\begin{aligned}
\|\hat{d}^k f(x)\|_N &\leqq \sum_{n=1}^{+\infty} |g^{(k)}(x_n)| \\
&= \sum_{x_n \geqq \varepsilon} |P(1/x_n)| e^{-1/x_n} + \sum_{|x_n| < \varepsilon} g^{(k)}(x_n) \\
&\leqq |P|(1/\varepsilon)\cdot R/\varepsilon + \|x\|_1 \\
&\leqq R(1 + |P|(1/\varepsilon)/\varepsilon) ,
\end{aligned}$$

where if $P = \Sigma a_n z^n$, then $|P| = \Sigma |a_n| z^n$. Hence the assertion follows.

(iv) The mapping $\hat{d}^k f\colon l_1 \to \mathscr{P}_N({}^k l_1)$ is differentiable of first order when $\mathscr{P}_N({}^k l_1)$ is endowed with its nuclear norm. Indeed, set

$$T_k(x) = \sum_{n=1}^{+\infty} (g^{(k+1)}(x_n)\cdot e_n)\cdot e_n^k \in \mathscr{L}(l_1;\ \mathscr{P}_N({}^k l_1)) ,$$

for $k = 0, 1, 2, \cdots$. Then

$$\begin{aligned}
&\|\hat{d}^k f(x+h) - \hat{d}^k f(x) - T_k(x)h\|_N \\
&= \left\|\sum_{n=1}^{+\infty} (g^{(k)}(x_n+h_n) - g^{(k)}(x_n) - g^{(k+1)}(x_n)h_n)\cdot e_n^k\right\|_N \\
&\leqq \sum_{n=1}^{+\infty} |g^{(k)}(x_n+h_n) - g^{(k)}(x_n) - g^{(k+1)}(x_n)h_n| .
\end{aligned}$$

As in (iii), given $\varepsilon > 0$, there is $\delta > 0$ such that

$$h \in l_1,\ \|h\|_1 < \delta \Longrightarrow \|\hat{d}^k f(x+h) - \hat{d}^k f(x) - T_k(x)h\|_N < \varepsilon\|h\|_1 .$$

Hence, $d(d^k f)(x) = T_k(x)$, when $\mathscr{P}_N({}^k l_1)$ is endowed with the nuclear norm. Moreover, the mapping $T_k\colon l_1 \to \mathscr{L}(l_1;\ \mathscr{P}_N({}^k l_1))$ is continuous, for $k = 0, 1, 2, \cdots$. Indeed,

$$T_k(x + h) - T_k(x) = \sum_{n=1}^{+\infty} [(g^{(k+1)}(x_n + h_n) - g^{(k+1)}(x_n))\cdot e_n]\cdot e_n^k .$$

Therefore,

$$\begin{aligned} \| T_k(x + h) - T_k(x)\| &= \sup_{\|w\|_1 \leqq 1} \left\| \sum_{n=1}^{+\infty} (g^{(k+1)}(x_n + h_n) - g^{(k+1)}(x_n))w_n \cdot e_n^k \right\|_N \\ &\leqq \sup_{\|w\|_1 \leqq 1} \sum_{n=1}^{+\infty} | g^{(k+1)}(x_n + h_n) - g^{(k+1)}(x_n) | | w_n | \\ &= \sum_{n=1}^{+\infty} | g^{(k+1)}(x_n + h_n) - g^{(k+1)}(x_n) | \\ &= \sum_{n=1}^{+\infty} | g^{(k+2)}(\theta_n) | | h_n | , \end{aligned}$$

where $\theta_n \in (x_n, x_n + h_n)$. Set $\alpha = \sup_{y \in [0, \max_n |x_n| + 1]} | g^{(k+2)}(y) |$. Given $\varepsilon > 0$, set $\delta = \min \{\varepsilon/\alpha, 1\}$. Then,

$$h \in l_1, \|h\|_1 < \delta \Longrightarrow \| T_k(x + h) - T_k(x)\| < \varepsilon .$$

It follows that $T_k$ is continuous. Thus, $T_k$ is differentiable of first order.

(i)-(iv) imply $f \in \mathscr{E}_{Nb}(l_1)$.

(b) $f \notin \mathscr{E}_{Nbc}(l_1)$. Indeed, $df(e_n) = e^{-1}\cdot e_n$. Therefore, $df(B_1)$ is not a precompact subset of $l_1'$, where $B_1$ is the unit ball of $l_1$. Hence the assertion follows of Propositions 1 and 2 above.

Hence Proposition 3 follows.

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