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**PROJECTIVE MODULES OVER SUBRINGS OF $k[X, Y]$
GENERATED BY MONOMIALS**

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In this paper we study finitely generated projective modules over affine subrings A of $k[X, Y]$ generated by monomials. If A is normal, then all finitely generated projective A -modules are free. If A is not normal, we show that finitely generated projective A -modules stably have the form $\text{free} \oplus \text{rank one}$

1. **Introduction.** In this paper we study projective modules over subrings A of $k[X, Y]$ generated by monomials. We study conditions on A so that all finitely generated projective A -modules have the form $\text{free} \oplus \text{rank one}$. In §IV we use Seshadri's localization technique to show that all finitely generated projective A -modules are free when A is an affine normal subring of $k[X, Y]$ generated by monomials. If we drop the assumption that A is normal it need not be true that all finitely generated projective A -modules are free. However, in §V we show that finitely generated projective A -modules stably have the form $\text{free} \oplus \text{rank one}$. We also give sufficient conditions on k for finitely generated projective A -modules to have the form $\text{free} \oplus \text{rank one}$. These results do not generalize to arbitrary subrings of $k[X, Y]$.

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2. **Preliminaries.** All rings A will be commutative with 1. $\text{Spec}(A)$ is the set of all prime ideals of A and $\max(A)$ is the subset of $\text{spec}(A)$ consisting of maximal ideals. We give $\text{spec}(A)$ the Zariski topology. If X is a topological space, the combinatorial dimension of X will be denoted by $\dim X$. If A is a ring, the group of units of A is A^* . $\text{SL}(n, A)$ is the group of $n \times n$ matrices over A with determinant 1, and $E(n, A)$ is the subgroup of $\text{SL}(n, A)$ generated by elementary matrices. The Krull dimension of A will be denoted by $\dim A$. k will always be a field. Let P be a finitely generated projective A -module and $Q \in \text{spec}(A)$. We define $\text{rank}_Q P$ to be $\dim_{A_Q/QA_Q} P_Q/QP_Q$. If $\text{rank}_Q P$ is constant, we will denote it by $\text{rank } P$. Our K -theory notation will follow Bass [4].

$\tilde{K}_0(A)$ is the subgroup of $K_0(A)$ generated by $[A^{\text{rank } P}] - [P]$ for finitely generated projective A -modules P , and $\text{Pic}(A)$ is the group of isomorphism classes of finitely generated projective A -modules of rank

one. There is a natural determinant epimorphism $\det: \tilde{K}_0(A) \rightarrow \text{Pic}(A)$ defined by $\det([P]) = A^n(P)$ where $n = \text{rank } P$. We denote the kernel of this map by $SK_0(A)$. Clearly $SK_0(A) = 0$ iff every finitely generated projective A -module stably has the form free \oplus rank one. In this case P is stably isomorphic to $A^{n-1} \oplus A^n(P)$.

A commutative square of rings

$$\begin{array}{ccc} A & \xrightarrow{f_1} & A_1 \\ f_2 \downarrow & & \downarrow g_1 \\ A_2 & \xrightarrow{g_2} & B \end{array}$$

is cartesian if $g_1(x) = g_2(y)$ implies there is a unique $z \in A$ with $f_1(z) = x$ and $f_2(z) = y$.

THEOREM 2.1 (Milnor [10]). *Given a cartesian square of rings with g_1 surjective, the following ("Mayer-Vietoris") sequences are exact*

$$(1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A^* & \longrightarrow & A_1^* \oplus A_2^* & \longrightarrow & B^* \xrightarrow{\partial} \text{Pic}(A) \\ & & & & & & \longrightarrow \text{Pic}(A_1) \oplus \text{Pic}(A_2) \longrightarrow \text{Pic}(B) \end{array}$$

$$(2) \quad \begin{array}{ccccccc} K_1(A) & \longrightarrow & K_1(A_1) \oplus K_1(A_2) & \longrightarrow & K_1(B) & \xrightarrow{\partial} & \tilde{K}_0(A) \\ & & & & & & \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) \longrightarrow \tilde{K}_0(B) . \end{array}$$

Moreover, if $\text{GL}(n, A_1) \rightarrow \text{GL}(n, B)$ is surjective for all n and all finitely generated projective A_1 and A_2 -modules are free, then all finitely generated projective A -modules are free.

Using the natural determinant maps, sequences (1) and (2) may be connected to obtain the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ SK_1(A) & \longrightarrow & SK_1(A_1) \oplus SK_1(A_2) & \longrightarrow & SK_1(B) & \xrightarrow{\partial} & SK_0(A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_1(A) & \longrightarrow & K_1(A_1) \oplus K_1(A_2) & \longrightarrow & K_1(B) & \xrightarrow{\partial} & \tilde{K}_0(A) \\ \downarrow & & \downarrow h & & \downarrow & & \downarrow \\ 0 \longrightarrow & A^* & \longrightarrow & A_1^* \oplus A_2^* & \longrightarrow & B^* & \xrightarrow{\partial} \text{Pic}(A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 & & 0 \end{array}$$

$$\begin{array}{ccccc}
& 0 & & 0 & \\
& \downarrow & & \downarrow & \\
\longrightarrow & SK_0(A_1) \oplus SK_0(A_2) & \longrightarrow & SK_0(B) & \\
& \downarrow & & \downarrow & \\
\longrightarrow & \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) & \longrightarrow & \tilde{K}_0(B) & \\
& \downarrow & & \downarrow & \\
\longrightarrow & \text{Pic}(A_1) \oplus \text{Pic}(A_2) & \longrightarrow & \text{Pic}(B) & \\
& \downarrow & & \downarrow & \\
& 0 & & 0 &
\end{array}$$

The following lemma is obvious.

LEMMA 2.2. Suppose that $SK_0(A_1) = SK_0(A_2) = 0$, then

- (1) $SK_0(A) = \partial(SK_1(B))$.
- (2) $SK_1(B) = 0$ implies $SK_0(A) = 0$.
- (3) If h is an isomorphism, then $SK_1(B) \approx SK_0(A)$.

We review a localization technique due to Seshadri which will be used in §IV. For details one may consult [4]. A set S of ideals of A is multiplicative if $I, J \in S$ implies $IJ \in S$. A prime ideal P is special if P is invertible and A/P is a PID for which $E(n, A/P) = \text{SL}(n, A/P)$ for all n . A multiplicative set of ideals is special if it is generated by special prime ideals. If S is any multiplicative set of invertible ideals, we define $S^{-1}A = \bigcup_{I \in S} I^{-1}$. For M an A -module, $S^{-1}M = S^{-1}A \otimes_A M$.

THEOREM 2.3 (Seshadri). Let A be a commutative noetherian ring and S a special multiplicative set of invertible ideals. Let P be a finitely generated projective A -module and suppose that $S^{-1}P \approx L'_1 \oplus \cdots \oplus L'_n$ where each L'_i is a finitely generated projective $S^{-1}A$ -module of rank one. Then

- (1) There are finitely generated projective A -modules L_i of rank one with $L'_i \approx S^{-1}A \otimes_A L_i$ for $i = 1, \dots, n$.
- (2) For each choice of L_i in (1) there is an I in the group of invertible ideals generated by S such that $P \approx IL_1 \oplus L_2 \oplus \cdots \oplus L_n$.

COROLLARY 2.4. Let A, S , and P be as above. If $S^{-1}P$ is the direct sum of a free $S^{-1}A$ -module and a projective $S^{-1}A$ -module of rank one, then P is also the direct sum of a free A -module and a projective A -module of rank one.

The next two lemmas will also be used in §IV. I do not know a reference for Lemma 2.6, however compare [16, p. 7].

LEMMA 2.5 ([11]). *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded affine normal domain with A_0 a field, then $\text{Pic}(A) = 0$.*

LEMMA 2.6. *Let A be a commutative ring with $\max(A)$ noetherian and $V(I) = F \subset \max(A)$ closed with $\dim(\max(A) \setminus F) \leq 1$. Let P be a finitely generated projective A -module with $\text{rank } P = n \geq 2$, and assume that P/IP is a free A/I -module. Then $P \approx A^{n-1} \oplus A^n(P)$.*

Proof. It is sufficient to show that if P is a finitely generated projective A -module with $\text{rank } P \geq 2$ and P/IP a free A/I -module, then $P \approx A \oplus P'$ with P'/IP' A/I -free.

Let $\max(A) \setminus F = U_1 \cup \cdots \cup U_t$ be a decomposition into closed irreducible components and pick $M_i \in U_i$. For $s \in P$ and $N \in \max(A)$, let $s(N)$ be the image of s in P_N/NP_N . P/IP is free, so by the Chinese Remainder Theorem there is a $s \in P$ with $s(M_i) \neq 0$, $1 \leq i \leq t$, and \bar{s} a basis element for P/IP . Clearly $s(M) \neq 0$ for $M \supset I$.

Let $Z(s) = \{J \in \max(A) \mid s(J) = 0\}$; then $Z(s) \subset \max(A) \setminus F$ and $Z(s)$ is closed [16, p. 6]. Each $M_i \notin Z(s)$, so $Z(s)$ is 0-dimensional and hence finite, say $Z(s) = \{I_1, \dots, I_l\}$. $\text{Rank } P \geq 2$, so as above we may choose $t \in P$ such that (1) $t(I_i) \neq 0$ for $1 \leq i \leq l$, (2) \bar{t} and \bar{s} form part of a basis for P/IP , and (3) $s(M_i)$ and $t(M_i)$ are linearly independent for $1 \leq i \leq t$.

As above $Z(s, t) = \{M \in \max(A) \mid s(M) \text{ and } t(M) \text{ are linearly dependent}\}$ is finite. Let $Y = Z(s, t) \setminus Z(s) = \{J_1, \dots, J_m\}$, then pick $0 \neq a \in (J_1 \cap \cdots \cap J_m) \setminus (I_1 \cup \cdots \cup I_l)$, or let $a = 1$ if $Y = \emptyset$. Let $u = s + at$, then $u(M) \neq 0$ for all $M \in \max(A)$, so Au is a direct summand of P [16, p. 6]. Note that $\bar{u} = \bar{s} + \bar{a}\bar{t}$ is part of a basis for P/IP .

$(a_0, \dots, a_n) \in A^{n+1}$ is unimodular if $Aa_0 + \cdots + Aa_n = A$. $U_{n+1}(A)$ is the set of all unimodular elements in A^{n+1} . The stable range of A , denoted by $\text{sr}(A)$, is $\leq d$ if given any unimodular row (a_0, \dots, a_d) , there exist $c_0, \dots, c_{d-1} \in A$ so that $(a_0 + c_0a_d, \dots, a_{d-1} + c_{d-1}a_d) \in A^d$ is unimodular. It is well-known that $\text{sr}(A) \leq 1 + \dim A$ and that $A^{n+1} \approx A \oplus P$ implies $A^n \approx P$ whenever $n \geq \text{sr}(A)$ ([4, p. 239]).

3. Subrings of $k[X, Y]$ generated by monomials. Subrings A of $B = k[X, Y]$ generated by monomials arise naturally as either the ring of invariants of an automorphism of B of finite order or as the kernel of a k -derivation of B when $\text{char } k = p \neq 0$. Clearly if A is as above, then $A \subset B$ is integral, so not all affine normal subrings of B generated by monomials are one of these two types. However,

over an algebraically closed field of $\text{char} = 0$, any affine normal subring of B generated by monomials is isomorphic to B^G where G is the cyclic group generated by an automorphism of the form $\phi: X \mapsto aX, Y \mapsto bY, a, b \in k$. We state the following three propositions without proof; for details see [1] or [2].

PROPOSITION 3.1. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Then $A \approx A'$ where $A' = k[X, Y]$ or $A' = k[X^n, XY^j, X^2Y^{2j}, \dots, X^{n-1}Y^{(n-1)j}, Y^n]$ where $0 < j < n$, $\gcd(j, n) = 1$, and “ \approx ” denotes mod n .*

PROPOSITION 3.2. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials. If $\dim A = 1$, then $A \approx A'$ where A' is an affine subring of $k[X]$ generated by monomials. If $\dim A = 2$, then $A \approx A''$ where A'' is an affine subring of B generated by monomials with $A'' \subset k[X, Y]$ integral.*

PROPOSITION 3.3. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Then \bar{A} , the integral closure of A , is also an affine subring of B generated by monomials. The conductor of \bar{A}/A contains a nonzero monomial.*

We recall that $\text{sing}(A) = \{P \in \text{spec}(A) \mid A_P \text{ is not regular}\}$. If A is an affine normal domain of $\dim 2$, then $\text{sing}(A)$ is a closed subset of $\text{spec}(A)$ of $\dim 0$, and hence finite [9, p. 245]. If in addition $A \subset k[X, Y]$ is generated by monomials, we can explicitly describe $\text{sing}(A)$.

PROPOSITION 3.4. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral and A not regular. Then the origin is the only singularity of A , that is, $\text{sing}(A) = \{M = (X, Y)B \cap A\}$.*

Proof. The proof of Proposition 3.1 [2, Thm. 2.5] shows that the isomorphism is just a change of variables which does not change the origin. Thus we may assume that $A = k[X^n, XY^j, X^2Y^{2j}, \dots, X^{n-1}Y^{(n-1)j}, Y^n]$ where $0 < j < n$ and $\gcd(j, n) = 1$. It is sufficient to show that for each of the generators $f_1 = X^n, f_2 = XY^j, \dots, f_{n+1} = Y^n$ of $M, A[1/f_i]$ is regular. For if N is any other maximal ideal, then some $f_i \notin N$, and thus A_N is regular since it is a localization of the regular ring $A[1/f_i]$.

Clearly $A[1/Y^n] = k[XY^j, Y^n][1/Y^n]$ which is regular. Similarly $A[1/X^n] = k[X^n, YX^j][1/X^n]$ where $YX^j \in A$. If $a, b \neq 0$, then $A[1/X^a Y^b]$

contains $1/X^n$ and $1/Y^n$ and thus is a localization of $A[1/Y^n]$. So $A[1/X^n Y^n]$ is also regular.

We note that a subring of $B = k[X, Y]$ generated by monomials is a graded ring with the natural grading it inherits from B .

4. Projective modules over affine normal subrings of $k[X, Y]$ generated by monomials.

THEOREM 4.1. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials, then all projective A -modules are free.*

Proof. Let P be a projective A -module. If P is not finitely generated, then P is free by a result of Bass [5] or Hinohara [8]. So we may assume that P is finitely generated.

If $\dim A = 1$, then by Serre's theorem [4, p. 173], P has the form free \oplus rank one. But $\text{Pic}(A) = 0$ by Lemma 2.5, so P is free. Thus, we may assume that $\dim A = 2$. By Propositions 3.1 and 3.2 we may assume that $A = k[X^n, XY^j, X^2Y^{2j}, \dots, X^{n-1}Y^{(n-1)j}, Y^n]$ where $0 < j < n$ and $\gcd(j, n) = 1$.

Let $\bar{B} = \bar{k}[X, Y]$ where \bar{k} is the algebraic closure of k . The maximal ideals of \bar{B} are of the form $M_{a,b} = (X - a, Y - b)$, and thus the maximal ideals of A are of the form $A \cap M_{a,b}$ because $A \subset \bar{B}$ is integral.

For each $0 \neq b \in \bar{k}$ let $Q_b = (Y - b)\bar{B} \cap A$. Clearly

$$A/Q_b \approx k[T^n, b^j T, \bar{b}^{2j} T^2, \dots, b^{(n-1)j} T^{n-1}, b^n].$$

$b \in \bar{k}$ is algebraic over k , so $k' = k[b^n]$ is a field. Each $b^{\bar{j}} T^i = (b^n)^l (b^j T)^i$ for some integer l . Thus $A/Q_b \approx k'[b^j T] = k'[S]$ is a euclidean ring because $S = b^j T$ is transcendental over k' .

Next we show that $Q_b (b \neq 0)$ is invertible. It is sufficient to show that $(Q_b)_N$ is principal for each maximal ideal N of A . If $Q_b \not\subset N$, then $(Q_b)_N = A_N$. If $Q_b \subset N$, then clearly $N \neq (X, Y)\bar{B} \cap A$. So by Proposition 3.4, A_N is a regular local ring and hence factorial. Thus $(Q_b)_N$ is a ht one prime ideal in a factorial ring and hence principal. So Q_b is locally principal and thus invertible. Q_b is actually principal because $\text{Pic}(A) = 0$ by Lemma 2.5. In fact $Q_b = (Y - b)\bar{B} \cap A = fA$ where $f \in k[Y^n]$ is the polynomial of least degree satisfying $f(b) = 0$.

For each $0 \neq b \in \bar{k}$ Q_b is an invertible ideal such that A/Q_b is a euclidean ring, and thus $E(n, A/Q_b) = \text{SL}(n, A/Q_b)$ for all n . Thus each Q_b is a special prime ideal. Let S be the multiplicative set generated by the Q_b for $0 \neq b \in \bar{k}$. We show that all finitely generated projective $S^{-1}A$ -modules have the form free \oplus rank one.

Let $I = (Y\bar{B} \cap A)S^{-1}A$ and $Z = \max(S^{-1}A \setminus V(I))$. S kills all the maximal ideals $(X - a, Y - b) \cap A$ when $b \neq 0$; so $\dim Z \leq 1$.

$$A/(Y\bar{B} \cap A) \approx k[T] \quad \text{and} \quad S^{-1}A/I \approx S^{-1}(A/Y\bar{B} \cap A),$$

so $\text{Pic}(S^{-1}A/I) = 0$ also. Thus all finitely generated projective $S^{-1}A/I$ -modules are free. By Lemma 2.6 all finitely generated projective $S^{-1}A$ -modules have the form free \oplus rank one. Thus all finitely generated projective A -modules have the form free \oplus rank one by Corollary 2.4. But $\text{Pic}(A) = 0$, so all finitely generated projective A -modules are free.

Seshadri [17] first showed that all finitely generated projective $k[X, Y]$ -modules are free. Murthy and Pedrini [12] showed that all finitely generated projective A -modules are free if $A = k[X^n, XY, Y^n]$ or $A = k[X^n, XY^{n-1}, \dots, X^{n-1}Y, Y^n]$. Our result generalizes these. Quillen [15] and Suslin have recently, and independently, proved Serre's problem. That is, all finitely generated projective $k[X_1, \dots, X_n]$ -modules are free. The following conjecture thus seems reasonable.

Conjecture. Let A be an affine normal subring of $k[X_1, \dots, X_n]$ generated by monomials, then all finitely generated projective A -modules are free.

We can however prove a weaker version of this conjecture. First a result which follows from Quillen's work [15]. The author learned of this result in a course given by R. G. Swan.

PROPOSITION 4.2. *Let A be a commutative ring and $f \in A[X]$ a monic polynomial. Let P and Q be finitely generated projective $A[X]$ -modules with*

(1) *Q is extended from A .*

(2) *$fQ \subset P \subset Q$.*

Then P and Q are isomorphic.

Proof (sketch). The proof is similar to that of [15, Thm. 3]. Let $A(X)$ denote the localization of $A[X]$ with respect to the multiplicative system of monic polynomials. Let $Q \approx Q_0 \otimes_A A[X]$. Since $f \in A[X]$ is monic, by (2), $P \otimes_{A[X]} A(X) \approx Q \otimes_{A[X]} A(X) \approx Q_0 \otimes_A A(X)$. Then as in [15, Thm. 3], P is extended from A , say $P \approx P_0 \otimes_A A[X]$. Thus $P_0 \approx Q_0$, and so $P \approx Q$.

THEOREM 4.3. *Let A be an affine normal subring of $k[X, Y]$ generated by monomials, then all finitely generated projective $A[X_1, \dots, X_n]$ -modules are free.*

Proof. By induction on n , the case $n = 0$ is just Theorem 4.1. Suppose $A = k[f_1, \dots, f_r]$ with $f_i \in k[X, Y]$ and let $B = A[X_1, \dots, X_n]$. Let S be the set of monic polynomials in $k[X_{n+1}]$. Then $B[X_{n+1}]_S = k(X_{n+1})[f_1, \dots, f_r][X_1, \dots, X_n]$. Clearly $k(X_{n+1})[f_1, \dots, f_r]$ is still an affine normal subring of $k(X_{n+1})[X, Y]$ generated by monomials. So by induction all finitely generated projective $B[X_{n+1}]_S$ -modules are free. Let P be a finitely generated projective $B[X_{n+1}]$ -module; then P_S is free, say $P_S \approx F_S$ where F is a finitely generated free $B[X_{n+1}]$ -module. Thus there exists a $g \in S$ so that $P_g \approx F_g$ and hence $g^m F \subset P \subset F$ for some m . By Proposition 4.2 $P \approx F$, and hence is free.

Thus affine normal subrings A of $k[X, Y]$ generated by monomials are nontrivial examples of nonregular rings for which $NK_0(A) = 0$, where $NK_0(A) = \ker(K_0(A[T]) \rightarrow K_0(A))$, induced by $T \mapsto 0$. In fact, it is an open question if $NK_0(A) = 0$ for all normal domains.

5. Projective modules over subrings of $k[X, Y]$ generated by monomials. If we drop the assumption that A is normal, all finitely generated projective A -modules need not be free.

EXAMPLE 5.1 ([7]). Let $A = k[X^2, X^3, Y]$, then not all finitely generated projective A -modules are free. $P = (1 + XY, 1 + XY + X^2Y^2)$ is a rank one projective A -module (invertible ideal in $k(X, Y)$) which is not free. In fact, $K_0(A) \approx \mathbb{Z} \oplus \text{Pic}(A) \approx \mathbb{Z} \oplus k[Y]$. That $K_0(A) \approx \mathbb{Z} \oplus \text{Pic}(A)$ is just Theorem 5.5. We show that $\text{Pic}(A) \approx k[Y]$. We have the following cartesian square.

$$\begin{array}{ccc} A = k[X^2, X^3, Y] & \hookrightarrow & B = k[X, Y] \\ \downarrow & & \downarrow \\ A/I = k[Y] & \hookrightarrow & B/I = k[\varepsilon][Y] \end{array}$$

Here $I = (X^2, X^3)B$ is contained in the conductor ideal and $\varepsilon^2 = 0$. By (1) of Theorem 2.1, $\text{Pic}(A) \approx k[\varepsilon][Y]^*/k^*$. But, as abelian groups, $k[\varepsilon][Y]^*/k^* \approx k[Y]$.

Of course all finitely generated projective A -modules may be free even though A is not normal.

EXAMPLE 5.2. Let $A = k[X^2, XY, Y]$, then all finitely generated projective A -modules are free. We have the following cartesian square.

$$\begin{array}{ccc} A = k[X^2, XY, Y] & \longrightarrow & B = k[X, Y] \\ \downarrow & & \downarrow \\ A/I = k[X^2] & \longrightarrow & B/I = k[X] \end{array}$$

Here $I = YB$ is the conductor ideal. All finitely generated projective B and A/I -modules are free and all $\text{GL}(n, B) \rightarrow \text{GL}(n, B/I)$ are surjective. Thus by Theorem 2.1, all finitely generated projective A -modules are free.

We show that if A is an affine subring of $B = k[X, Y]$ generated by monomials, then $SK_0(A) = 0$, that is, the natural map $\det: \tilde{K}_0(A) \rightarrow \text{Pic}(A)$ is an isomorphism. This just means that stably any finitely generated projective A -module has the form $\text{free} \oplus \text{rank one}$.

Let A be an affine subring of $B = k[X, Y]$ generated by monomials, \bar{A} the integral closure of A , and I the conductor ideal. $I \neq 0$, so let J be any nonzero ideal contained in I . Thus $\dim A/J, \dim \bar{A}/J \leq 1$, so $SK_0(A/J) = SK_0(\bar{A}/J) = 0$ by Serre's theorem [4, p. 173]. By Theorems 3.3 and 4.1 $SK_0(\bar{A}) = 0$, so by Lemma 2.2 it is sufficient to show that $SK_1(\bar{A}/J) = 0$. Again we may assume that $\dim A = 2$, and thus by Propositions 3.1, 3.2, and 3.3 we may assume that $\bar{A} = k[X^n, XY^j, X^2Y^{2j}, \dots, X^{n-1}Y^{(n-1)j}, Y^n]$ where $0 < j < n$ and $\gcd(j, n) = 1$. Also I contains a nonzero monomial $f = X^a Y^b$ with $a, b \neq 0$; let $J = f\bar{A}$.

LEMMA 5.3. *Let A and B be commutative rings with $A \subset B$ integral. If $I \subset A$ is an ideal of A , then $\sqrt[A]{I} = \sqrt[B]{IB} \cap A$.*

Proof. Clearly $\sqrt[A]{I} \subset \sqrt[B]{IB} \cap A$. Let $P \in \text{spec}(A)$ with $I \subset P$. $A \subset B$ is integral, so there is a $\bar{P} \in \text{spec}(B)$ with $P = \bar{P} \cap A$. $I \subset P \subset \bar{P}$, so $IB \subset \bar{P}$. Thus $\sqrt[B]{IB} \subset \bar{P}$ and $\sqrt[B]{IB} \cap A \subset \bar{P} \cap A = P$. So $\sqrt[A]{I} = \sqrt[B]{IB} \cap A$.

LEMMA 5.4. $SK_1(\bar{A}/f\bar{A}) = 0$.

Proof. By above $\bar{A}/f\bar{A} = k[X^n, XY^j, \dots, X^{n-1}Y^{(n-1)j}, Y^n]/X^a Y^b \bar{A}$. Let $B = k[X, Y]$, then $\sqrt[B]{fB} = XYB$. By Lemma 5.3 $\sqrt[A]{f\bar{A}} = XYB \cap \bar{A}$. By [4, p. 469], $SK_1(\bar{A}/f\bar{A}) \approx SK_1((\bar{A}/f\bar{A})/(\sqrt[A]{f\bar{A}}/f\bar{A})) \approx SK_1(\bar{A}/\sqrt[A]{f\bar{A}})$. Clearly $\bar{A}/\sqrt[A]{f\bar{A}} \approx k[X, Y]/(XY)$. But $SK_1(k[X, Y]/(XY)) = 0$, so $SK_1(\bar{A}/f\bar{A}) = 0$.

THEOREM 5.5. *Let A be an affine subring of $k[X, Y]$ generated by monomials, then $SK_0(A) = 0$. Thus all finitely generated projective A -modules stably have the form $\text{free} \oplus \text{rank one}$.*

COROLLARY 5.6. *Let A be a subring of $k[X, Y]$ generated by monomials, then all finitely generated projective A -modules stably have the form $\text{free} \oplus \text{rank one}$.*

Proof. This follows from the following well-known result. Let M be a finitely presented A -module, then there is a noetherian subring R of A and a finitely presented R -module M' with $M \approx M' \otimes_R A$. If M is projective, M' may also be chosen to be projective.

Theorem 5.5 is rather unsatisfying because it does not say that any finitely generated projective A -module has the form $\text{free} \oplus \text{rank one}$, but only that this is stably true. Since $\dim A \leq 2$, by Bass' Cancellation Theorem [4, p. 184], if $\text{rank } P = n \geq 3$, then actually $P \approx A^{n-1} \oplus A^n(P)$. If $\text{rank } P = 2$, we only have $P \oplus A \approx A^2 \oplus A^2(P)$. If k is algebraically closed, by a cancellation theorem of Murthy and Swan [13], $P \approx A^{n-1} \oplus A^n(P)$. I know of no examples where $P \not\approx A^{n-1} \oplus A^n(P)$.

If $\text{Pic}(A) = 0$, then all finitely generated projective A -modules are stably free. If $\text{sr}(A) \leq 2$, then $E(3, A)$ acts transitively on $U_3(A) = \{(a_0, a_1, a_2) \in A^3 \mid (a_0, a_1, a_2) \text{ unimodular}\}$, so all finitely generated projective A -modules are free. This happens when k is algebraic over a finite field [18, p. 45]. We next show that this also happens whenever $1/2 \in k$.

We recall a few definitions. $KSp_0(A)$ is the Grothendieck group with generators $[P]$ for each symplectic A -module P and relations $[P] = [Q]$ if $P \approx Q$ and $[P \perp Q] = [P] + [Q]$. $W(A)$ is the kernel of the natural map $KSp_0(A) \rightarrow K_0(A)$ given by $[P] \mapsto [P]$ which forgets the symplectic structure. W is a functor from rings to abelian groups. For more details one is referred to [6] or [18].

LEMMA 5.7 (*C. Weibel and R. G. Swan*). *Let $A = A_0 \oplus A_1 \oplus \dots$ be a graded ring and F a functor on rings. If the natural map induces an isomorphism $F(A) \rightarrow F(A[T])$, then the natural map $A_0 \rightarrow A$ also induces an isomorphism $F(A_0) \rightarrow F(A)$.*

Proof. Define $f: A \rightarrow A[T]$ by $f: \sum a_i \rightarrow \sum a_i T^i$. By hypothesis the two maps $F(A[T]) \rightarrow F(A)$ induced by $T \mapsto 0$ and $T \mapsto 1$ are both isomorphisms. Consider the composition $A \xrightarrow{f} A[T] \rightarrow A$. If $T \mapsto 1$ we obtain the identity, while $T \mapsto 0$ gives the natural augmentation $A \rightarrow A_0$. Thus the natural map $A \rightarrow A[T] \rightarrow A_0$ induces a monomorphism $F(A) \rightarrow F(A_0)$. But this map is always surjective, so $F(A_0) \approx F(A)$.

PROPOSITION 5.8. *Let A be an affine subring of $k[X, Y]$ generated by monomials. If $\text{Pic}(A) = 0$ and $1/2 \in k$, then all finitely generated projective A -modules are free.*

Proof. By a theorem of Karoubi [6, p. 8], when R is a commutative ring with $1/2 \in R$, $R \rightarrow R[T]$ induces an isomorphism $W(R) \rightarrow$

$W(R[T])$. A is a graded ring with $A_0 = k$, so $W(A) \approx W(k)$ by Lemma 5.7. It is well-known that $W(k) = 0$ [6, p. 8], so also $W(A) = 0$.

By a result of Vaserstein [6, p. 7], there is a natural map

$$\phi: \mathrm{SL}(3, R) \backslash U_3(R) \longrightarrow W(R)$$

which is bijective if $E(r, R)$ acts transitively on $U_r(R)$ for all $r \geq 4$.

In our case $W(A) \approx W(k) = 0$ and $E(r, A)$ acts transitively on $U_r(A)$ for all $r \geq 4$ since $\mathrm{sr}(A) \leq 3$. Thus $\mathrm{SL}(3, A) \backslash U_3(A) = 1$; that is, $\mathrm{SL}(3, A)$ acts transitively on $U_3(A)$. So all finitely generated projective A -modules which are stably free are actually free. But $\mathrm{Pic}(A) = 0$, so all finitely generated projective A -modules are free by Theorem 5.5.

6. Subrings A of $k[X, Y]$ with $\mathrm{Pic}(A) = 0$. It is not hard to determine precisely which subrings A of $k[X, Y]$ generated by monomials have $\mathrm{Pic}(A) = 0$. If $\dim A = 1$, clearly $A \approx k[X]$ iff $\mathrm{Pic}(A) = 0$. If $\dim A = 2$, by Proposition 3.2 we may assume that $A \subset k[X, Y]$ is integral.

PROPOSITION 6.1. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral and let \bar{A} be the integral closure of A . Then $\mathrm{Pic}(A) = 0$ iff*

(1) *Let X^m and Y^n be the lowest powers of X and Y in A , then $X^i, Y^j \in A$ imply $m|i$ and $n|j$.*

(2) *$XYB \cap \bar{A}$ is contained in the conductor of \bar{A}/A .*

Proof. We prove the notationally easier case with $\bar{A} = B = k[X, Y]$. Otherwise we may assume $\bar{A} = k[X^n, XY^j, \dots, X^{n-1}Y^{(n-1)j}, Y^n]$ and the proof is similar.

(\Leftarrow) Let $I = XYB$, then $A/I \approx k[X, Y]/(XY)$. Clearly $A^* = B^* = (A/I)^* = (B/I)^* = k^*$. Also $\mathrm{Pic}(A/I) = 0$. This follows from Theorem 2.1 applied to following cartesian square.

$$\begin{array}{ccc} k[X, Y]/(XY) & \longrightarrow & k[X, Y]/(X) \\ \downarrow & & \downarrow \\ k[X, Y]/(Y) & \longrightarrow & k[X, Y]/(X, Y) \end{array}$$

Thus also $\mathrm{Pic}(A) = 0$ by Theorem 2.1.

(\Rightarrow) Conversely assume that $\mathrm{Pic}(A) = 0$. Suppose that (1) fails. Say that not all powers of X are multiples of m . There is a retract of rings $R \rightarrow A \xrightarrow{\theta} R$ where R is the image in $k[X]$ of the map

$\theta: X \mapsto X, Y \mapsto 0$. Thus $\text{Pic}(R) \subset \text{Pic}(A)$. By Theorem 2.1 it is easy to see that $\text{Pic}(R) \neq 0$, and hence also $\text{Pic}(A) \neq 0$. So we may assume that (1) holds.

Pick $f = X^i Y^j$ in the conductor of B/A with $i > m, n$; this is possible by Proposition 3.3. Since $\text{Pic}(A) = 0$, also $(A/fB)^* = (B/fB)^*$ by Theorem 2.1. For each $g = X^a Y^b$ with $1 \leq a \leq m$ and $1 \leq b \leq n$, $1 + g + fB$ is a unit in B/fB , and hence also in A/fB . But thus $X^a Y^b \in A$, so XYB is contained in the conductor, and the proposition is proved.

For example, the affine subrings A of $B = k[X, Y]$ generated by monomials with integral closure B for which $\text{Pic}(A) = 0$ are precisely those of the form

$$A = k[X^m, \{X^i Y^j \mid 1 \leq i \leq m, 1 \leq j \leq n\}, Y^n].$$

For these rings $K_0(A) \approx \mathbb{Z}$. Since this does not depend on the field k , by an argument similar to that of Theorem 4.3 we see that all finitely generated projective $A[X_1, \dots, X_n]$ -modules are stably free. So these rings provide many examples of nonnormal rings for which $NK_0(A) = 0$.

We note that even though these rings are not normal, they are "power closed" in the sense that if f is in the quotient field of A and $f^n \in A$ for all large n , then actually $f \in A$. This condition is in fact necessary, for if A is not "power closed", then $\text{Pic}(A) \rightarrow \text{Pic}(A[T])$ is not an isomorphism (see Example 5.1).

One can also see that $NK_0(A) = 0$ for the rings of Proposition 6.1 by using the Mayer-Vietoris K -theory sequence for NK_1 and NK_0 ([14]). So the rings of Proposition 6.1 are precisely the nonnormal affine subrings of $k[X, Y]$ generated by monomials for which $NK_0(A) = 0$. Thus if A is an affine subring of $k[X, Y]$ generated by monomials, $\text{Pic}(A) = 0$ iff $NK_0(A) = 0$.

7. The general case. One can ask if these results generalize to more general subrings of $k[X, Y]$. This is studied in more detail in [1] or [3]. The analogue of Theorem 5.5 fails in general because there exist $f \in k[X, Y]$ with $SK_1(k[X, Y]/(f)) \neq 0$. This also depends on the field k . We close with one example.

EXAMPLE 7.1. Let $A = k[X, Y(X^2 - Y), Y^2(X^2 - Y)]$, then

(1) If k is algebraic over a finite field all finitely generated projective A -modules are free.

(2) If k is an algebraically closed field of char 0, then $SK_0(A) \approx \Omega_{k/\mathbb{Z}}^1 \neq 0$ and $\text{Pic}(A) = 0$. Thus there exist indecomposable finitely generated projective A -modules of rank 2.

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