Pacific Journal of Mathematics

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GHAN SHYAM PANDEY

Vol. 79, No. 1 May 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 1, 1978

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G. S. PANDEY

In the present paper we improve the conditions of all previously known theorems on the absolute $(C, 1)$ summability factors of Fourier series.

1. Let the formal expansion of a function $f(x)$, periodic with period 2π and integrable in the sense of Lebesgue over $[-\pi, \pi]$, in a Fourier-trigonometric series be given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) . \tag{1.1}$$

We write

$$\phi(u) = f(x+u) + f(x-u) - 2f(x)$$

and throughout this paper A will denote a positive constant, not necessarily the same at each occurrence.

Whittaker [5], in 1930, proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^{\alpha} , \qquad \alpha > 0 ,$$

is summable $|A|$ almost everywhere.

Later, Prasad [4] demonstrated that the series

$$\sum_{n=n_0}^{\infty} A_n(x)/\mu_n ,$$

where

$$\mu_n = \left(\prod_{\nu=1}^{k-1} \log^{\nu} n\right)(\log^k n)^{1+\varepsilon} , \qquad \log^k n_0 > 0 , \quad \varepsilon > 0 ,$$

and

$$\log^k n = \log(\log^{k-1} n), \cdots, \log^2 n = \log\log n ;$$

is summable $|A|$ almost everywhere.

Chow [2], on the other hand, has shown that the series $\sum \lambda_n A_n(x)$ is summable $|C, 1|$ almost everywhere, provided $\{\lambda_n\}$ is a convex sequence satisfying the condition $\sum n^{-1}\cdot\lambda_n < \infty$.

Cheng [1], in 1948, established the following:

THEOREM A. *If*

$$\Phi(t) \equiv \int_0^t |\phi(u)|\, du = O(t)$$

as $t \to 0$, *then the series*

$$\sum_{n=2}^{\infty} A_n(x)/(\log n)^{1+\delta}, \qquad \delta > 0,$$

is summable $|C, \alpha|$, $\alpha > 1$.

In a recent paper, Hsiang [3] has proved the following theorems:

THEOREM B. *If*

$$\Phi(t) = O(t) \qquad (t \longrightarrow +0), \tag{1.2}$$

then the series $\sum_{n=1}^{\infty} A_n(x)/n^{\alpha}$ *is summable* $|C, 1|$ *for every* $\alpha > 0$.

THEOREM C. *If*

$$\Phi(t) = O\left\{t/\prod_{\nu=1}^{k} \log^{\nu}(1/t)\right\} \tag{1.3}$$

as $t \to +0$, *then the series*

$$\sum_{n=0}^{\infty} A_n(x)\Big/\left(\prod_{\nu=1}^{k-1} \log^{\nu} n\right)(\log^{k} n)^{1+\varepsilon} \tag{1.4}$$

is summable $|C, 1|$ *for every* $\varepsilon > 0$.

In the present paper we prove the following theorem, which includes the theorem of Cheng and both the theorems of Hsiang:

THEOREM. *If*

$$\varphi(t) \equiv \int_t^{\delta} \frac{|\phi(u)|}{u} du = O\{(\log^{k}(1/t))^{\eta}\} \quad as \quad t \longrightarrow +0, \tag{1.5}$$

$0 < \delta \leqq \pi$, *then the series* (1.4) *is summable* $|C, 1|$ *for* $0 < \eta < \varepsilon$.

The conditions of our theorem are less stringent than those of Cheng and Hsiang.

2. The proof of the theorem is based on the following lemmas:

LEMMA 1. *Let* $S_n(x)$ *be the* nth *partial sum of the series* (1.1), *then under the condition* (1.5), *we have*

$$\sum_{\nu=0}^{n} |S_{\nu}(x) - f(x)| = O\{n(\log^{k} n)^{\eta}\}. \tag{2.1}$$

Proof. Let $\varepsilon_\nu = \operatorname{sign}[S_\nu(x) - f(x)]$, so that $\varepsilon_\nu = \pm 1$ and it depends only upon x and ν, and is independent of t. Also, we write

$$K_n(t) = \sum_{\nu=0}^{n} \varepsilon_\nu \sin \nu t \ .$$

Thus, we have

$$\begin{aligned}\sum_{\nu=0}^{n} |S_\nu(x) - f(x)| &= \frac{2}{\pi}\int_0^\pi \frac{\phi(t)}{t} K_n(t)dt + o(n) \\ &= \frac{2}{\pi}\left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_\delta^\pi\right] + o(n) \\ &= I_1 + I_2 + I_3 + o(n) \ ,\end{aligned}$$

say. Now,

$$\begin{aligned}I_1 &\leqq \int_0^{1/n} |\phi(t)| \cdot O(n^2)dt \\ &= O(n^2)\int_0^{1/n} - t\varphi'(t)dt \ , \qquad \varphi'(t) = \frac{d}{dt}\varphi(t) \ . \\ &= O(n^2)[-t\varphi(t)]_0^{1/n} + O(n^2)\int_0^{1/n} \varphi(t)dt \\ &= O\{n(\log^k n)^\eta\} \ .\end{aligned} \tag{2.3}$$

Also, for $nt \geqq 1$, we have

$$\begin{aligned}I_2 &\leqq \int_{1/n}^{\delta} \frac{|\phi(t)|}{t} \cdot ndt \\ &= O\{n(\log^k n)^\eta\} \ .\end{aligned} \tag{2.4}$$

Since, by Riemann-Lebesgue theorem,

$$\int_\delta^\pi \frac{\phi(t)}{t} \sin ntdt = o(1) \ ,$$

we have

$$I_3 = O(n) \ . \tag{2.5}$$

Combining (2.1), (2.2), $\cdots$, (2.5), the lemma follows.

LEMMA 2. *Let*

$$t_n(x) = \frac{1}{(n+1)} \sum_{\nu=1}^{n} \nu A_\nu(x) \ .$$

Then

$$T_n(x) \equiv \sum_{\nu=1}^{n} |t_\nu(x)| = O\{n(\log^k n)^\eta\}$$

and

$$\sum_{n=n_0}^{\infty} (\mu_n)^{-1} \cdot n^{-1} |t_\nu(x)| < \infty .$$

Proof. Let

$$\sigma_n(x) = \frac{1}{(n+1)} \sum_{\nu=0}^{n} S_\nu(x) .$$

Thus, we have

$$\sigma_n(x) - f(x) = \frac{1}{(n+1)} \sum_{\nu=0}^{n} \{S_\nu(x) - f(x)\}$$

$$(2.6) \quad \begin{aligned} \Longrightarrow |\sigma_n(x) - f(x)| &\leqq \frac{1}{(n+1)} \sum_{\nu=0}^{n} |S_\nu(x) - f(x)| \\ &= O\{(\log^k n)^\eta\} \end{aligned}$$

by Lemma 1.

Therefore, we find that

$$(2.7) \quad \begin{aligned} T_n(x) &= \sum_{\nu=1}^{n} |t_\nu(x)| \\ &= \sum_{\nu=1}^{n} |S_\nu(x) - \sigma_\nu(x)| \\ &\leqq \sum_{\nu=1}^{n} |S_\nu(x) - f(x)| + \sum_{\nu=1}^{n} |\sigma_\nu(x) - f(x)| \\ &= O[n(\log^k n)^\eta] \end{aligned}$$

by (2.6) and Lemma 1.

Finally, by Abel's transformation, we have

$$(2.8) \quad \begin{aligned} \sum_{n=m}^{M} (\mu_n)^{-1} \cdot n^{-1} |t_n(x)| &= \sum_{n=m}^{M-1} T_n(x) \Delta\{(\mu_n)^{-1} \cdot n^{-1}\} \\ &\quad - (\mu_{m-1})^{-1}(m-1)^{-1} T_{m-1}(n) + \mu_M^{-1} \cdot M^{-1} T_M(x) \\ &= \sum_{n=m}^{M-1} \Delta\{(\mu_n)^{-1}\} \cdot n^{-1} T_n(x) \\ &\quad + \sum_{m=m}^{M-1} (\mu_{n+1})^{-1} \cdot n^{-1}(n+1)^{-1} T_n(x) + O(1) \\ &= \sum_{n=m}^{M-1} \Delta\{(\mu_n)^{-1}\} \cdot (\log^k n)^\eta \\ &\quad + \sum_{n=m}^{M-1} (\mu_{n+1})^{-1}(n+1)^{-1}(\log^k n)^\eta + O(1) \\ &\leqq \sum_{n=m}^{M-1} \frac{A \cdot (\log^k n)^\eta}{n\left(\prod_{\mu=1}^{k-1} \log^\mu n\right)(\log^k n)^{1+\varepsilon}} + O(1) \\ &= O(1) , \end{aligned}$$

for $m \to \infty$ and $M \to \infty$.

In view of (2.7) and (2.8) the lemma is proved.

3. Proof of the theorem. Let $\tau_n(x)$ denotes the nth Cesàro mean of the sequence $\{n(\mu_n^{-1}) \cdot A_n(x)\}$.

By Abel's transformation, we have

$$
\begin{aligned}
\tau_n(x) &= \frac{1}{(n+1)} \sum_{\nu=n_0}^{n} \nu(\mu_\nu)^{-1} \cdot A_\nu(x) \\
&= \frac{1}{(n+1)} \sum_{\nu=n_0}^{n-1} \Delta(\mu_\nu)^{-1} \cdot (\nu+1)t_\nu(x) + (\mu_n)^{-1}t_n(x) \qquad (3.1)\\
&= J_1^{(n)}(x) + J_2^{(n)}(x) ,
\end{aligned}
$$

say. Now, by Lemma 2, we find that

$$
\begin{aligned}
\sum_{n=m_0}^{m} J_1^{(n)}(x)/n &\leqq \sum_{n=m_0}^{m} n^{-1}(n+1)^{-1} \sum_{\nu=n_0}^{n-1} \Delta(\mu_\nu)^{-1}(\nu+1)|t_\nu(x)| , \quad \log^k m_0 > 0 \\
&\leqq A \sum_{\nu=m_0}^{m} \Delta(\mu_\nu)^{-1}(\nu+1)|t_\nu(x)| \sum_{n=\nu+1}^{m} n^{-1}(n+1)^{-1} \\
&\leqq A \sum_{\nu=m_0}^{m} \Delta(\mu_\nu)^{-1}|t_\nu(x)| \qquad (3.2)\\
&= A \sum_{\nu=m_0}^{m-1} \Delta^2[(\mu_\nu)^{-1}] \cdot T_\nu(x) + \Delta(\mu_m^{-1})T_m(x) + O(1) \\
&= O(1) .
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\sum_{n=m_0}^{m} J_2^{(n)}(x)/n &\leqq \sum_{n=m_0}^{m} (\mu_n)^{-1} \cdot n^{-1} t_n(x)| \qquad (3.3)\\
&= O(1) .
\end{aligned}
$$

From (3.1), (3.2), and (3.3), we have

$$
\sum_{n=m_0}^{m} \frac{|\tau_n(x)|}{n} = O(1) .
$$

This completes the proof of the theorem.

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Received January 16, 1975 and in revised form April 3, 1978.

MOSUL UNIVERSITY
MOSUL, IRAQ

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 79, No. 1 May, 1978