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# TYPE ANALYSIS OF THE REGULAR REPRESENTATION OF A NON-UNIMODULAR GROUP

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**This paper is concerned with finding necessary and sufficient conditions for the von Neumann algebra $\mathscr{M}(G)$ generated by the left regular representation $\lambda^G$ of a locally compact, separable, non-unimodular group $G$ to be type I, semifinite, or to have a central summand of type III. In the case where the modular function $\delta_G$ of $G$ has closed range, we are able to give a complete solution in terms of the orbit structure of the natural action of $G$ on the reduced quasi-dual $(\Gamma_H, \mu_H)$ of the maximal unimodular subgroup $H=$ kernel $\delta_G$. Thus $\mathscr{M}(G)$ is semifinite if and only if the action is smooth with isotropy subgroup $H$, and of type $\mathrm{III}_0$ if and only if the action is completely nonsmooth. Conditions of a similar type are given which are necessary and sufficient for $\mathscr{M}(G)$ to have a summand of type $\mathrm{III}_\lambda$, $\lambda \in (0, 1]$.**

In §2 we develop the necessary preliminary material for the later work, establishing the connection between semidirect products of groups and crossed products of the corresponding group algebras. Sections 3 and 4 give the proof of the above mentioned criterion of semi-finiteness; this proof relies heavily on the theory of modular automorphisms, and crossed products as developed in [3], [20], and [21]. In §5 we turn to examples; we exhibit groups $G_\lambda$, $\lambda \in [0, 1]$ with $\mathscr{M}(G_\lambda)$ a factor of type $\mathrm{III}_\lambda$, and also groups $G_{0,\lambda}$ with $\mathscr{M}(G_{0,\lambda})$ a factor of type $\mathrm{III}_0$ with $T(\mathscr{M}(G_{0,\lambda})) = 2\pi/\log \lambda \boldsymbol{Z}$. In fact, we construct two such families of groups; one is a variation on Godements example of a group with type III regular representation, and for these groups the associated von Neumann algebras are not hyperfinite; the other family is constructed using the semi- direct product of an abelian group by a solvable group so that the associated von Neumann algebras are hyperfinite. This second family of examples is due to A. Connes (private communication). In the final section we use the results of §3 to give a form of the Plancherel theorem for locally compact separable groups $G$ for which $\delta_G(G) = \boldsymbol{R}_+$ and $\mathscr{M}(G)$ is semifinite. For the most part this is an adaption of the more general formula in [18].

**2. Preliminaries.** Throughout, $G$ will denote a locally compact separable non-unimodular group, with modular function $\delta_G$.

PROPOSITION 2.1. *Suppose $\delta_G(G) = \boldsymbol{R}_+$. Then there is a continuous one parameter subgroup $L = \{g_t : t \in \boldsymbol{R}\}$ of $G$ with $\delta_G(g_t) = e^t$.*

*Proof.* Suppose first that $G$ is connected. Then by a theorem of Iwasawa [14] there are continuous one parameter subgroups $V_1, \cdots, V_r$ of $G$, and a maximal compact subgroup $K$ of $G$ such that $G = KV_1V_2, \cdots, V_r$. Clearly $\delta_G(K) = \{1\}$. Thus there is a one parameter subgroup, which we may take as $V_1$, such that $\delta_G(V_1) \neq \{1\}$. Since $V_1$ is connected, we will have $\delta_G(V_1) = \boldsymbol{R}_+$. Write $V_1 = \{h_t : t \in \boldsymbol{R}\}$; since the map $t \in \boldsymbol{R} \to \delta_G(h_t) \in \boldsymbol{R}_+$ is continuous and onto, we have $\delta_G(h_t) = e^{at}$ for some $a \in \boldsymbol{R}$. But then if $g_t = h_{a^{-1}t}$, $\delta_G(g_t) = e^t$ and $L = \{g_r : r \in \boldsymbol{R}\}$ is the desired subgroup.

If $G$ is not connected, let $G_0$ denote the connected component of the identity of $G$. Now either $\delta_G(G_0) = \{1\}$, or $\delta_G(G_0) = \boldsymbol{R}_+$. In the latter case the argument above shows we may find the desired subgroup within $G_0$. If on the other hand $\delta_G(G_0) = \{1\}$, then $\delta_G$ induces a continuous homomorphism of $G/G_0$ onto $\boldsymbol{R}_+$. Since $G/G_0$ has a basis consisting of compact open subgroups, this homomorphism is locally constant on $G/G_0$. But this contradicts the assumed separability of $G$. Thus in fact the desired subgroup may be found within $G_0$.

Proposition 2.1 greatly simplifies many of our later computations, the main reason being that the cocycle naturally associated with the cross-section for $G/H$ may be assumed to be trivial. We note also that in the case $\delta_G(G)$ is singly generated, say $\{e^{nT} : n \in \boldsymbol{Z}\}$ there is trivially a subgroup $\{g_n : n \in \boldsymbol{Z}\}$ of $G$ with $\delta_G(g_n) = e^{nT}$, $(n \in \boldsymbol{Z})$.

Our next preparatory result is on the interconnection between crossed products for von Neumann algebras, and semidirect products for groups.

First, let $N$ be a locally compact group, with left Haar measure $d_N$, and let $\alpha$ be a continuous automorphism of $N$. It is well known that there is a constant $\delta(\alpha)$ such that

$$\int_N \xi(n)d_N(n) = \delta(\alpha)\int_N \xi(\alpha^{-1}(n))d_N(n)$$

for every function $\xi$ which is continuous and of compact support on $N$. Define a unitary $U(\alpha)$ on $L^2(N, d_N)$ by

$$(U(\alpha)\xi)(n) = \delta(\alpha)^{1/2}\xi(\alpha^{-1}(n)) , \qquad \xi \in L^2(N) .$$

Then we have

$$\begin{aligned}(U(\alpha)\lambda^N(n)U(\alpha)^*\xi)(m) &\\ &= \xi(\alpha(n^{-1}\alpha^{-1}(m))) \\ &= (\lambda(\alpha(n))\xi)(m)\end{aligned}$$

i.e.,
$$U(\alpha)\lambda^N(n)U(\alpha)^* = \lambda^N(\alpha(n)) \qquad n \in N .$$

Now suppose $K$ is a locally compact group and $\alpha : K \to \operatorname{Aut} N$ is an action of $K$ on $N$ by automorphisms, such that $(k, n) \rightarrow \alpha_k(n)$, $(k, n) \in K \times N$, is continuous. For $k \in K$, let $U(k) = U(\alpha_k)$ be as above, and define an action of $K$ on $\mathscr{M}(N)$ by $\bar{\alpha}_k(x) = U(k)xU(k)^*$. Since $U(k)U(h)$, we have $\bar{\alpha}_k\bar{\alpha}_h = \bar{\alpha}_{kh}$. Since $k \rightarrow U(k)$ is strong *-continuous, we have a continuous action of $K$ on $\mathscr{M}(N)$ in the terminology of [21].

We may also consider the semidirect product $N \times_\alpha K$ of $N$ by $K$; the group multiplication is given by $(n, k)(m, l) = (n\alpha_k(m), kl)$ on $N \times K$. The following result is certainly well known, but the author knows of no proof in the literature.

PROPOSITION 2.2. *The crossed product* $\mathscr{R}(\mathscr{M}(N); \bar{\alpha}, K)$ *is unitarily equivalent with* $\mathscr{M}(N \times_\alpha K)$.

*Proof*. Noting that the automorphism group $\{\bar{\alpha}_k: k \in K\}$ is implemented by the family of unitary operators $\{U(k): k \in K\}$, the indicated crossed product is generated by the operators on $L^2(N \times K, d_N \times d_K)$

$$\begin{cases} (\pi^{\bar{\alpha}}(\lambda^N(n))\xi)(m, k) = \xi(n^{-1}m, k) \\ (V(h)\xi)(m, k) = \delta(\alpha_h)^{1/2}\xi(\alpha_h^{-1}(m), h^{-}k) \end{cases} \tag{2.1}$$

for $n \in N$, $h \in K$.

On the other hand $\mathscr{M}(N \times_\alpha K)$ is generated on $L^2(N \times K, d_N \times d_K)$ by the operators

$$(\lambda(n, h)\xi)(m, k) = \delta(\alpha_h)^{1/2}\xi(\alpha_h^{-1}(n^{-1}m), h^{-1}k) \tag{2.2}$$

for $(n, h) \in N \times K$. But specializing the equations (2.2) to the group elements $(n, e)$, $n \in N$, and $(e, h)$, $h \in K$, we obtain the equations (2.1) as desired.

Now, with $N$ an arbitrary locally compact group, we let $\phi^N$ be the canonical weight on the algebra $\mathscr{M}(N)$ (see [20]). If $\alpha$ is a continuous automorphism of $N$, and $\bar{\alpha}$ the corresponding automorphism of $\mathscr{M}(N)$ we have

LEMMA 2.3. $\phi^N(\bar{\alpha}(x)) = \delta(\alpha)\phi^N(x)$; $x \in \mathscr{M}(N)_+$.

*Proof*. Let $\mathscr{K}(N)$ denote the space of continuous compactly supported functions on $N$ with the usual structure of a left Hilbert algebra. For $\xi \in \mathscr{K}(N)$, we let $\pi_l(\xi)$ denote the operator of "left multiplication" by $\xi$ on $L^2(N)$. Recall that $\phi^N(\pi_l(\xi^\# * \xi)) = \|\xi\|_2^2$ and that it is sufficient to check the desired identity for operators $x$ of the form $\pi_l(\xi^\# * \xi)$ for $\xi \in \mathscr{K}(G)$. (The general case when $\xi$ is in the full left Hilbert algebra determined by $\mathscr{K}(G)$ is in fact identical.)

So let $\xi \in \mathscr{K}(G)$. Then

$$\begin{aligned}\bar{\alpha}(\pi_l(\xi)) &= \bar{\alpha}\Big(\int_N \xi(n)\lambda^N(n)d_N(n)\Big)\\ &= \int_N \xi(n)\lambda^N(\alpha(n))d_N(n)\\ &= \int_N \xi(\alpha^{-1}(n))\lambda^N(n)d_N(n)\cdot\delta(\alpha)\\ &= \delta(\alpha)^{1/2}\pi_l(U(\alpha)\xi)\ .\end{aligned}$$

Thus $\bar{\alpha}(\pi_l(\xi^\# * \xi)) = \delta(\alpha)\pi_l((U(\alpha)\xi)^\# * U(\alpha)\xi)$ and

$$\begin{aligned}\phi^N(\bar{\alpha}(\pi_l(\xi^\# * \xi))) &= \delta(\alpha)\phi^N(\pi_l((U(\alpha)\xi)^\# U(\alpha)\xi))\\ &= \delta(\alpha)\,||U(\alpha)\xi||_2^2\\ &= \delta(\alpha)\,||\xi||_2^2\\ &= \delta(\alpha)\phi^N(\pi_l(\xi^\# * \xi))\ .\end{aligned}$$

We recall that if $G$ is a locally compact group and $H = \ker\delta_G$, then $H$ is a closed normal unimodular subgroup. Furthermore, for $g \in G$, we may define an automorphism $\alpha_g$ of $H$ by $\alpha_g(h) = ghg^{-1}$; clearly $\alpha_g\alpha_{g'} = \alpha_{gg'}$, and the map $(g, h) \in G \times H \to \alpha_g(h) \in H$ is continuous. According to the results of [22], we have $\delta(\alpha_g) = \delta_G(g)$, so that also $\phi^H \circ \bar{\alpha}_g = \delta_G(g)\phi^H$. (We note that in this case $\phi^H$ is actually a trace since $H$ is unimodular; we will write $\tau^H$ or $\tau$ for $\phi^H$.)

As a matter of notation, if $\alpha$ is a continuous automorphism of a locally compact group $N$, then $\bar{\alpha}$ will denote the corresponding automorphism of $\mathscr{M}(N)$, and $\tilde{\alpha}$ will denote the restriction of $\bar{\alpha}$ to the centre $\mathscr{Z}(N)$ of $\mathscr{M}(N)$.

Finally, we recall the following basic facts from [18]. Suppose $G$ is separable and non-unimodular, and $H = \ker\delta_G$. Let $\{\sigma_t: t \in \boldsymbol{R}\}$ denote the canonical modular automorphism of $\mathscr{M}(G)$, so that $\sigma_t(\lambda^G(g)) = \delta_G(g)^{it}\lambda^G(g))$. Let $\lambda^G = \int_{\Gamma_G}^{\oplus} \lambda_\omega^G d\mu_G(\omega)$ and $\lambda^H = \int_{\Gamma_H}^{\oplus} \lambda_\gamma^H d\mu_H(\gamma)$ denote the central decompositions of $\lambda^G$ and $\lambda^H$. For any representation $\pi$ of $H$, and any element $g \in G$ define $(\hat{\alpha}_g\pi)(h) = \pi(g^{-1}hg)$. Since $\hat{\alpha}_g\lambda^H$ is unitarily equivalent with $\lambda^H$, we may consider $\hat{\alpha}_g$ as a transformation on the reduced quasi-dual $(\Gamma_H, \mu_H)$ of $H$. For the proof of the following facts, see [20].

THEOREM 2.4. (i) *The fixed point subalgebra* $\mathscr{M}(G)_0$ *of* $\mathscr{M}(G)$ *under* $\{\sigma_t: t \in \boldsymbol{R}\}$ *is* $\{\lambda^G(h): h \in H\}''$. *Thus* $\mathscr{Z}(G) \subset \{\lambda^G(h): h \in H\}''$.

(ii) *The map* $(g, \gamma) \in G \times \Gamma_H \to \hat{\alpha}_g(\gamma) \in \Gamma_H$ *is Borel. Under the identification of* $L^\infty(\Gamma_H, \mu_H)$ *with* $\mathscr{Z}(H)$, $\hat{\alpha}_g$ *is a point realization of* $\tilde{\alpha}_g$.

(iii) *There is an algebraic isomorphism* $\kappa$ *carrying* $\mathscr{M}(G)_0$ *to* $\mathscr{M}(H)$ *such that*

(a) $\kappa(\lambda^G(h)) = \lambda^H(h)$

(b) $\kappa(\mathscr{Z}(G)) \subset \mathscr{Z}(H)$, *and* $\kappa(\mathscr{Z}(G))$ *is the fixed point subalgebra of* $\mathscr{Z}(H)$ *under* $\{\tilde{\alpha}_g : g \in G\}$.

(iv) *Let* $\mu_H = \int_X^\oplus \mu_\zeta^H dm(\zeta)$ *be the ergodic decomposition of* $\mu_H$ (*with respect to* $\{\hat{\alpha}_g : g \in G\}$) *and* $\lambda_\zeta^H = \int_{\Gamma_H}^\oplus \lambda_\gamma^H d\mu_\zeta^H(\gamma)$. *Then* $\lambda^G = \int_X^\oplus \mathrm{Ind}_H^G \lambda_\zeta^H dm(\zeta)$ *is the central decomposition of* $\lambda^G$, *so that* $(X, m)$ *is measure isomorphic with* $(\Gamma_G, \mu_G)$.

**3. The structure theorems; the case $\delta_G(G) = \boldsymbol{R}_+$.** According to Theorem 2.4 (iv), in order to study the components in the central decomposition of $\lambda^G$, it is necessary to know the ergodic decomposition of the measure $\mu_H$ on $\Gamma_H$. Our first task then is to find an alternate description of the automorphism group $\{\tilde{\alpha}_g : g \in G\}$ on $\mathscr{Z}(H)$. We note the automorphisms $\tilde{\alpha}_h (h \in H)$ act trivially, so that we should regard the action as an action of $G/H$. In the case $\delta_G(G) = \boldsymbol{R}_+$, we identify $G/H$ with the subgroup $L$ of Proposition 2.1, and for $g_t \in L$, we let $\tilde{\alpha}_t = \tilde{\alpha}_{g_t}$ on $\mathscr{Z}(H)$, and $\bar{\alpha}_t = \bar{\alpha}_{g_t}$ on $\mathscr{M}(H)$.

Throughout the rest of this section $u(t)$, $v(t)$ will denote the operators on $L^2(\boldsymbol{R})$ defined by

$$(u(t)\xi)(s) = e^{-ist}\xi(s)$$
$$(v(t)\xi)(s) = \xi(s - t) .$$

THEOREM 3.1. *Suppose* $\delta_G(G) = \boldsymbol{R}_+$. *Then there is an algebraic isomorphism of the crossed product* $\mathscr{R}(\mathscr{M}(G); \sigma_t)$ *with* $\mathscr{M}(H) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$, *which carries the automorphism group* $\{\theta_t : t \in \boldsymbol{R}\}$ *dual to* $\{\sigma_t : t \in \boldsymbol{R}\}$ *to the automorphism group* $\{\bar{\alpha}_t \otimes \mathrm{Ad}\, v(t) : t \in \boldsymbol{R}\}$. *Thus the restriction* $\{\tilde{\theta}_t : t \in \boldsymbol{R}\}$ *of* $\{\theta_t : t \in \boldsymbol{R}\}$ *to the centre of* $\mathscr{R}(\mathscr{M}(G); \sigma_t)$ *is equivalent to the action* $\{\tilde{\alpha}_t : t \in \boldsymbol{R}\}$ *of* $\boldsymbol{R}$ *on* $\mathscr{Z}(H)$.

*Proof.* From the results of [21], the second crossed product $\mathscr{R}(\mathscr{R}(\mathscr{M}(G); \sigma_t); \theta_t)$ is isomorphic with $\mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$, and $\mathscr{R}(\mathscr{M}(G);\sigma_t)$ is isomorphic with the fixed point subalgebra of $\mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ under the automorphism group $\{\sigma_t \otimes \mathrm{Ad}\, v(t)^*\}$.

We first show that $\mathscr{R}(\mathscr{M}(G); \sigma_t)$ is isomorphic with $\mathscr{M}(N) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. Consider the automorphism group $\{\sigma_t \otimes \mathrm{Ad}\, v(t)^*\}$ on $\mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. For $\xi \in L^2(G \times \boldsymbol{R})$, define $(\mathscr{F}\xi)(g, p) = \int_{\boldsymbol{R}} e^{-isp}\xi(g, s)ds$. Noting that $\mathscr{F}(\mathscr{M}(G)) \otimes \mathscr{B}(L^2(\boldsymbol{R}))\mathscr{F}^* = \mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ and that $\mathscr{F}(1 \otimes v(t)^*)\mathscr{F}^* = 1 \otimes u(t)^*$, we are invited to consider the action $\{\sigma_t \otimes \mathrm{Ad}\, u(t)^*\}$ of $\boldsymbol{R}$ on $\mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$.

Define a unitary operator $W$ on $L^2(G \times \boldsymbol{R})$ by $(W\xi)(g, p) =$

$\xi(g_p^{-1}g, p)$ ($L = \{g_p \colon p \in \boldsymbol{R}\}$). Note that if $\rho^G$ denotes the right regular representation of $G$ on $L^2(G)$ then $W(\rho^G(g) \otimes 1)W^* = \rho^G(g) \otimes 1$, so that $W \in \mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ and $W(\mathscr{M}(G)) \otimes \mathscr{B}(L^2(\boldsymbol{R}))W^* = \mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. Furthermore, if $\Delta$ is the canonical modular operator of $\mathscr{M}(G)$, then

$$\begin{aligned}(W(\Delta^{it} \otimes u(t)^*)&W^*\xi)(g, p)\\ &= \delta_G(g_p^{-1}g)^{it}e^{itp}\xi(g, p)\\ &= ((\Delta^{it} \otimes 1)\xi)(g, p) .\end{aligned}$$

Thus $W(\Delta^{it} \otimes u(t)^*)W^* = \Delta^{it} \otimes 1$. Consequently for $x \in \mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$, $x$ is fixed under $\sigma_t \otimes \operatorname{Ad} u(t)^*$ if and only if $WxW^*$ is fixed under $\sigma_t \otimes i$ (where $i$ denotes the identity automorphism). But, by Theorem 2.4 (i) and (iii), the fixed point subalgebra of $\{\sigma_t \otimes i \colon t \in \boldsymbol{R}\}$ on $\mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ is isomorphic with $\mathscr{M}(H) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ as required.

It remains to identify the automorphism $\{\theta_t \colon t \in \boldsymbol{R}\}$ on $\mathscr{M}(H) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. This is possible mainly because Takesaki's Duality theorem has a particularly explicit formulation in this context. Following [21], the generators of $\mathscr{R}(\mathscr{M}(G); \sigma_t)$ on $L^2(L^2(G); \boldsymbol{R})$ are

$$\tag{3.1} \begin{cases}(\lambda_1^G(g)\xi)(s) = \delta_G(g)^{-is}\lambda^G(g)\xi(s) ; & g \in G\\ (v_1(t)\xi)(s) = \xi(s - t) & s \in \boldsymbol{R} .\end{cases}$$

The generators of the second crossed product, acting on $L^2(L^2(G); \boldsymbol{R} \times \hat{\boldsymbol{R}})$ are then

$$\tag{3.2} \begin{cases}(\lambda_2^G(g)\xi)(s, p) = \delta_G(g)^{-is}\lambda^G(g)\xi(s, p) ; & g \in G\\ (v_2(t)\xi)(s, p) = e^{-itp}\xi(s - t, p) ; & t \in \boldsymbol{R}\\ (u_2(q)\xi)(s, q) = \xi(s, p - q) ; & q \in \boldsymbol{R}\end{cases}$$

where the operators $\lambda_2^G(g)$, $v_2(t)$ are the images of the operators described in (3.1). Performing the Fourier transform in the second variable as in [21], we obtain generators

$$\tag{3.3} \begin{cases}(\lambda_3^G(g)\xi)(s, p) = \delta_G(g)^{-is}\lambda^G(g)\xi(s, p)\\ (v_3(t)\xi)(s, p) = \xi(s - t, p - t)\\ (u_3(q)\xi)(s, p) = e^{-iqp}\xi(s, p) .\end{cases}$$

Letting $\mathscr{N}$ be the algebra generated by the operators described in (3.3) and $\mathscr{P}$ the subalgebra generated by $\{v_3(t) \colon t \in \boldsymbol{R}\}$ and $\{u_3(q) \colon q \in \boldsymbol{R}\}$, then $\mathscr{N}$ is generated by $\mathscr{P}$ and an isomorphic copy $\mathscr{M} = \pi(\mathscr{M}(G))$ of $\mathscr{M}(G)$ in such a way that $\mathscr{N} = \mathscr{P} \otimes \mathscr{M}$; the normal isomorphism $\pi \colon \mathscr{M}(G) \to \mathscr{N} \cap \mathscr{P}'$ is defined by

$$(\pi(x)\xi)(s, p) = \sigma_{s-p}^{-1}(x)\xi(s, p) .$$

In our situation,

$$\begin{aligned}(\pi(\lambda^C(g))\xi)(s, p) &= \delta_G(g)^{-i(s-p)}\lambda^G(g)\xi(s, p) \\ &= (\lambda_3^G(g)u_3(\chi(g))\xi)(s, p) ,\end{aligned}$$

where $\chi(g) = \log \delta_G(g)$. Thus, in $\mathcal{N}$, the first crossed product is generated by the operators

$$\tag{3.4} \begin{cases}\pi(\lambda_3^G(g))u_3(\chi(g)) , & g \in G \\ v_3(t) , & t \in \boldsymbol{R} .\end{cases}$$

Thus under the identification of $\mathcal{N}$ with $\mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ the first crossed product is generated by the operators

$$\tag{3.5} \begin{cases}\lambda^G(g) \otimes u(\chi(g)) , & g \in G \\ 1 \otimes v(t) , & t \in \boldsymbol{R} .\end{cases}$$

Since the dual automorphism $\{\theta_t : t \in \boldsymbol{R}\}$ satisfies $\theta_t(\lambda_1^G(g)) = \lambda_1^G(g)$, and $\theta_t(v_1(s)) = e^{ist}v_1(t)$, examination of the generators (3.4) shows that the dual automorphism on the first crossed product, viewed as a subalgebra of $\mathscr{M}(G) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$, is given by $\theta_t = \operatorname{Ad}(1 \otimes u(t))$. Performing the Fourier transform as in the first part of the proof, our generators become $\lambda^G(g) \otimes v(\chi(g))$ $(g \in G)$ and $1 \otimes u(t)$ $(t \in \boldsymbol{R})$ and the automorphism group in question $\{\operatorname{Ad}(1 \otimes v(t)) : t \in \boldsymbol{R}\}$. Thus, we compute the image of $1 \otimes v(t)$ under the unitary operator $W$ defined in the first part of the proof.

$$\begin{aligned}(W(1 \otimes v(t))W^*\xi)(g, p) &= \xi(g_{p-t}g_p^{-1}g, p - t) \\ &= \xi(g_t^{-1}g, p - t) \\ &= (\lambda^G(g_t) \otimes v(t)\xi)(g, p) .\end{aligned}$$

So $W(1 \otimes v(t))W^* = \lambda^G(g_t) \otimes v(t) = w(t)$. Thus the dual automorphism group on $\mathscr{M}(G)_0 \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ is given by $\operatorname{Ad} w(t)$. But when we identify $\mathscr{M}(G)_0$ with $\mathscr{M}(H)$ as in Theorem 2.4 (iii), we obtain precisely the automorphism group $\{\bar{\alpha}_t \otimes \operatorname{Ad} v(t) : t \in \boldsymbol{R}\}$ as required. The final conclusion of the theorem follows trivially.

REMARK. (i) Suppose that the subgroup $L$ of Proposition 2.1 may be chosen to be closed in $G$ (examples indicate that such a choice is always possible, although author knows of no proof to support this contention). Then $G = H \times_s L$ and the theorem just proven is in effect the Duality theorem of Takesaki applied to the convariant system $(\mathscr{M}(H); \bar{\alpha}_{-t})$. Even without the assumption that $L$ is closed, however, a straight-forward computation shows that $\mathscr{R}(\mathscr{M}(H); \bar{\alpha}_{-t} : t \in \boldsymbol{R}) \simeq \mathscr{M}(G)$.

(ii) A similar theorem is possible (and will be proven) in case

$\delta_G(G)$ is closed in $\boldsymbol{R}_+$. However, in case $\delta_G(G)$ is not closed, the analogue of Theorem 3.1 remains unknown. It is not known whether or not in this case we even have $\mathscr{R}(\mathscr{M}(G); \sigma_t) \simeq \mathscr{M}(H) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. To some extent this difficulty may be removed by considering $G \times S$ in place of $G$, where $S$ is a group with $\delta_S(S) = \boldsymbol{R}_+$, and for simplicity $\mathscr{M}(S)$ a type I factor $\left(\text{e.g., } S = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0,\, a, b \in \boldsymbol{R}\right\}\right)$. But then further analysis is difficult since the relationship between $\ker \delta_G$ and $\ker \delta_{G\times S}$ appears to be very complicated.

Our next objective is to combine Theorem 3.1 with the direct integral theory, and, at the same time, to clarify the relationship between the crossed product description of $\mathscr{M}(G)$ as $\mathscr{R}(\mathscr{M}(H); \bar{\alpha}_t)$ and the description of the central decomposition of $\lambda^G$ given in Theorem 2.4 (iv).

Let $\mathscr{M}$ be an arbitrary von Neumann algebra with separable predual, and $\{\sigma_t : t \in \boldsymbol{R}\}$ an arbitrary modular automorphism group on $\mathscr{M}$. It is known from the results of [18] and [19] that if $\mathscr{M} = \int_\Gamma^\oplus \mathscr{M}(\omega)d\mu(\omega)$ is the central decomposition of $\mathscr{M}$ then

(a) the modular automorphism group $\{\sigma_t : t \in \boldsymbol{R}\}$ decomposes, $\sigma_t = \int_\Gamma^\oplus \sigma_{t,\omega}d\mu(\omega)$ so that $\{\sigma_{t,\omega} : t \in \boldsymbol{R}\}$ is a modular automorphism group on $\mathscr{M}(\omega)$.

(b) there is a canonical isomorphism of $\mathscr{R}(\mathscr{M}; \sigma_t)$ with $\int_\Gamma^\oplus \mathscr{R}(\mathscr{M}(\omega); \sigma_{t,\omega})d\mu(\omega)$.

(c) in the decomposition (b), the dual automorphism group $\{\theta_t : t \in \boldsymbol{R}\}$ decomposes, $\theta_t = \int_\Gamma^\oplus \theta_{t,\omega}d\mu(\omega)$ and $\{\theta_{t,\omega} : t \in \boldsymbol{R}\}$ is dual to $\{\sigma_{t,\omega} : t \in \boldsymbol{R}\}$ on $\mathscr{R}(\mathscr{M}(\omega); \sigma_{t,\omega})$.

(d) the diagonal subalgebra of the decomposition (b) is the fixed point subalgebra of the centre of $\mathscr{R}(\mathscr{M}; \sigma_t)$ under $\{\theta_t : t \in \boldsymbol{R}\}$. (This is not actually proven in [19] but may be seen easily be observing that $\{\theta_{t,\omega} : t \in \boldsymbol{R}\}$ is ergodic on the centre of $\mathscr{R}(\mathscr{M}(\omega); \sigma_{t,\omega})$ so that (c) gives the ergodic decomposition $\{\theta_t : t \in \boldsymbol{R}\}$ on the centre of $\mathscr{R}(\mathscr{M}; \sigma_t)$.)

In our situation, we consider the central decomposition

$$\{\mathscr{M}(G), \lambda^G\} = \int_{\Gamma_G}^\oplus \{\mathscr{M}(G)(\omega), \lambda^G_\omega\}d\mu_G(\omega)$$

of $\mathscr{M}(G)$ and $\lambda^G$. Thus (b) furnishes us with a decomposition of $\mathscr{M}(H) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ and hence of $\lambda^H$, with diagonal subalgebra $\kappa(\boldsymbol{Z}(G)) \subset \boldsymbol{Z}(H)$. By Theorem 3.1, the dual automorphism group on $\mathscr{M}(H) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ is given by $\bar{\alpha}_t \otimes \operatorname{Ad} v(t)$. Let $\lambda^H = \int_{\Gamma_G}^\oplus \lambda^H_\omega d\mu_G(\omega)$ be the decomposition of $\lambda^H$ over $\kappa(\mathscr{Z}(G))$ and let $\bar{\alpha}_t = \int_{\Gamma_G}^\oplus \bar{\alpha}_{t,\omega}d\mu_G(\omega)$

be the corresponding decomposition of the dual automorphism. Our next result is intended to clarify the description of the central decomposition of $\lambda^G$ given in Theorem 2.4 (iv) by relating it to the crossed product description.

PROPOSITION 3.2. *Let* $\lambda^G = \int_{\Gamma_G}^{\oplus} \lambda_\omega^G d\mu_G(\omega)$ *be the central decomposition of* $\lambda^G$, *and* $\lambda^H = \int_{\Gamma_G}^{\oplus} \lambda_\omega^H d\mu_G(\omega)$ *be the decomposition of* $\lambda^H$ *arising from the isomorphism of* $\mathscr{R}(\mathscr{M}(G); \sigma_t)$ *with* $\mathscr{M}(H) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. *Then the representations* $\lambda_\omega^G$ *and* $\mathrm{Ind}_H^G \lambda_\omega^H$ *are quasi-equivalent.*

*Proof.* Let $\pi_\omega^G = \mathrm{Ind}_H^G \lambda_\omega^H$; we first write down generators for the von Neumann algebra $\{\pi_\omega^G(G)\}''$. Recall that $\pi_\omega^G$ acts on the Hilbert space $\mathfrak{H}(\omega)$ of Borel functions $\eta: G \to \mathscr{H}(\omega)$ (the space of the representation $\lambda_\omega^H$) and which satisfy the properties:

(a) $\eta(gh) = \lambda_\omega^H(h)^{-1}\eta(g)$; $h \in H$

(b) $\int_{G/H} \|\eta(g)\|^2 d\dot{g} < \infty$

where $d\dot{g}$ is an invariant measure on $G/H$, which, in this case may be taken as Lebesque measure on $\boldsymbol{R}$. The action of $\pi_\omega^G$ on $\mathfrak{H}(\omega)$ is given by

$$(\pi_\omega^G(g)\eta)(g_1) = \eta(g^{-1}g_1) ; \qquad \eta \in \mathfrak{H}(\omega) , \qquad g \in G .$$

However, $\mathfrak{H}(\omega)$ may be identified with $L^2(\mathscr{H}(\omega); \boldsymbol{R})$ by means of the unitary operator $U: \mathfrak{H}(\omega) \to L^2(\mathscr{H}(\omega); \boldsymbol{R})$ defined by $(U\eta)(t) = \eta(g_t)$ so that $(U^*\xi)(g) = \lambda^H(g^{-1}g_{\chi(g)})\xi(\chi(g))$. We then compute

$$\begin{aligned}
&(U\pi_\omega^G(g)U^*\xi)(t) \\
&\quad = (\pi_\omega^G(g)U^*\xi)(g_t) \\
&\quad = (U^*\xi)(g^{-1}g_t) \\
&\quad = \lambda_\omega^H(g_t^{-1}gg_{t-\chi(g)})\xi(t - \chi(g)) .
\end{aligned}$$

Specializing to the case $g = h \in H$, $g = g_s \in L$ we obtain generators for $\{\pi_\omega^G(G)\}''$ on $L^2(\mathscr{H}(\omega); \boldsymbol{R})$ of the form

$$\begin{cases} (\bar{\pi}_\omega^G(h)\eta)(t) = \lambda_\omega^H(g_t^{-1}hg_t)\eta(t) ; & h \in H \\ (\bar{\pi}_\omega^G(g_s)\eta)(t) = \eta(t - s); & s \in \boldsymbol{R} . \end{cases} \tag{3.6}$$

On the other hand, the action of the automorphism group $\{\bar{\alpha}_{t,\omega}; t \in \boldsymbol{R}\}$ on $\mathscr{M}(H)(\omega) = \{\lambda_\omega^H(H)\}''$ is determined by $\bar{\alpha}_{t,\omega}\lambda_\omega^H(h) = \lambda_\omega^H(g_thg_t^{-1})$, so that the crossed product $\mathscr{R}(\mathscr{M}(H)(\omega); \bar{\alpha}_{t,\omega})$ has generators on $L^2(\mathscr{H}(\omega); \boldsymbol{R})$ given by

$$\begin{cases} (\bar{\lambda}(h)\xi)(t) = \lambda_\omega^H(g_t^{-1}hg_t)\xi(t) ; & h \in H \\ (\lambda(s)\xi)(t) = \xi(t - s) ; & s \in \boldsymbol{R} . \end{cases} \tag{3.7}$$

Comparison of (3.6) and (3.7) shows that $\{\bar{\pi}_\omega^G(G)\}''$ and $\mathscr{R}(\mathscr{M}(H)(\omega); \bar{\alpha}_{t,\omega})$ have identical generators.

We consider now the dual automorphism $\{\theta_{t,\omega}: t \in \boldsymbol{R}\}$ of $\mathscr{M}(H)(\omega)\otimes \mathscr{B}(L^2(\boldsymbol{R}))$; since $\theta_{t,\omega}(x) = (1 \otimes v(t))((\bar{\alpha}_{t,\omega} \otimes i)(x))(1 \otimes v(t)^*)$, and $\mathrm{Ad}\,(1 \otimes v(t))$ commutes with $\bar{\alpha}_{t,\omega}$, the automorphism groups $\{\theta_{t,\omega}: t \in \boldsymbol{R}\}$ and $\{\bar{\alpha}_{t,\omega} \otimes i: t \in \boldsymbol{R}\}$ are equivalent in the sense of [21]. We let $\pi_\omega^\theta$, $\pi_\omega^{\bar{\alpha}}$ denote the canonical inclusions of $\mathscr{M}(H)(\omega) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ in the respective crossed products; the above observation combined with the results of [21] shows there is a normal isomorphism $\psi$ of $\{\pi_\omega^G(G)\}''$ into $\mathscr{R}(\mathscr{M}(H)(\omega) \otimes \mathscr{B}(L^2(\boldsymbol{R})); \bar{\alpha}_{t,\omega} \otimes i)$ with

$$(3.8) \qquad \begin{cases} \psi(\pi_\omega^G(h)) = \pi_\omega^{\bar{\alpha}}(\lambda_\omega^H(h) \otimes 1) \\ \psi(\pi_\omega^G(g_s)) = \lambda(s)\,. \end{cases}$$

But from the duality theorem, $\mathscr{R}(\mathscr{M}(H)(\omega) \otimes \mathscr{B}(L^2(\boldsymbol{R})), \bar{\alpha}_{t,\omega} \otimes i)$ is isomorphic with $\mathscr{M}(G)(\omega) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. We compute the images of $\pi_\omega^{\bar{\alpha}}(\lambda_\omega^H(h) \otimes 1)$ and $\lambda(s)$ in $\mathscr{M}(G)(\omega) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. For this, it is easiest to note that from [21], there is a normal isomorphism $\psi^1$, of $\mathscr{R}(\mathscr{M}(H)(\omega)\otimes\mathscr{B}(L^2(\boldsymbol{R})); \bar{\alpha}_{t,\omega}\otimes i)$ with $\mathscr{R}(\mathscr{M}(H)(\omega)\otimes\mathscr{B}(L^2(\boldsymbol{R})); \theta_{t,\omega})$ carrying $\pi_\omega^{\bar{\alpha}}(x)$ to $\pi_\omega^\theta(x)$ $(x \in \mathscr{M}(H)(\omega) \otimes \mathscr{B}(L^2(\boldsymbol{R})))$ and $\lambda(s)$ to $\lambda(s)$. Notice also that the unitary operator $W$ defined in the proof of Theorem 3.1 is decomposable over $\mathscr{Z}(G) \otimes 1$, and examination of equations (3.4) and the form of $W$ show that the images of $\pi_\omega^\theta(\lambda_\omega^H(h) \otimes 1)$ and $\lambda(s)$ in $\mathscr{M}(G)(\omega) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ may be taken as being defined by

$$(3.9) \qquad \begin{cases} (\bar{\lambda}_\omega^H(h)\xi)(p) = \lambda_\omega^G(g_p^{-1} h g_p)\xi(p) \\ (\bar{\lambda}(s)\xi)(p) = \xi(p - s)\,. \end{cases}$$

Finally, we may define a unitary operator $V_\omega$ on $\mathscr{G}(\omega) \otimes L^2(\boldsymbol{R})$, where $\mathscr{G}(\omega)$ is the Hilbert space of the representation $\lambda_\omega^G$, by $(V_\omega\xi)(p) = \lambda_\omega^G(g_p)\xi(p)$. Thus, from (3.9) we have

$$(3.10) \qquad \begin{cases} (V_\omega\bar{\lambda}_\omega^H(h) V_\omega^*\xi)(p) = \lambda_\omega^G(h)\xi(p) \\ (V_\omega\bar{\lambda}(s) V_\omega^*\xi)(p) = (\lambda_\omega^G(g_s)\xi(p - s))\,. \end{cases}$$

However, from the proof of 3.1, we see there is an isomorphism $\delta_\omega$ of $\mathscr{M}(G)(\omega)$ into $\mathscr{M}(G)(\omega) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ with $\delta_\omega(\lambda_\omega^G(g)) = \lambda_\omega^G(g) \otimes v(\chi(g))$; it is necessary to take the Fourier transform in equations (3.5). Thus the representation of $G$ determined by (3.10) is quasi-equivalent with $\lambda_\omega^G$.

Before proceeding with our main theorem we recall the following criteria, proved in [21], for the type of a crossed product. Let $\mathscr{M}_0$ be a property infinite semfinite von Neumann algebra, and $\{\theta_t: t \in \boldsymbol{R}\}$ a continuous one-parameter automorphism group on $\mathscr{M}_0$. Let $\mathscr{M} =$

$\mathscr{R}(\mathscr{M}_0; \theta)$. We also suppose that $\mathscr{M}_0$ admits a faithful normal semifinite trace $\tau$ with $\tau \circ \theta_t = e^{-t}\tau$. Then

( i ) $\mathscr{M}$ is a factor if and only if $\{\theta_t\}$ is ergodic on the centre $\mathscr{Z}_0$ of $\mathscr{M}_0$.

(ii) if $\mathscr{M}$ is a factor and $t_0 \in \boldsymbol{R}$, then $e^{t_0} \in S(\mathscr{M})$ if and only if $\theta_{t_0}$ is the identity on $\mathscr{Z}_0$.

(iii) if $\mathscr{M}$ is a factor, then it is semifinite if and only if the action of $\{\theta_t\}$ on $\mathscr{Z}_0$ is equivalent to translation on $L^\infty(\boldsymbol{R})$.

Conversely, if $\mathscr{M}$ is a type III factor and $\{\sigma_t\}$ is any modular automorphism group on $\mathscr{M}$, then $\mathscr{R}(\mathscr{M}; \sigma_t)$ is properly infinite, semifinite, and possess a faithful normal trace $\tau$ with $\tau \circ \theta_t = e^{-t}\tau$ ($\{\theta_t\}$ is the dual automorphism group.) We also note that if $\mathscr{M}$ is not necessarily of type III, but is a factor, then the restriction of the dual automorphism to the centre of $\mathscr{R}(\mathscr{M}; \sigma_t)$ is still ergodic.

Let $G$ be a locally compact separable group acting on a standard measure space $(X, \mu)$ so that the map $(g, x) \in G \times X \to \beta_g(x)$ is Borel. (Here $\beta: G \to \operatorname{Aut} X$ is a homomorphism.) An invariant Borel set $B \subset X$ will be said to be *smooth for* $\mu$ if there is an invariant $\mu$-null Borel set $B_1 \subset X$ such that $(B - B_1)/G$ is countably separated in the quotient Borel structure (see [11]). We will have need of the following

LEMMA 3.3. *Let $G$, $(X, \mu)$ be as above. Then there is an invariant Borel set $B \subset X$ which is smooth for $\mu$, and such that if $B_1$ is any other invariant Borel set which is smooth for $\mu$, then $\mu(B_1 - B) = 0$.*

*Proof.* Follows trivially from the observation that any family of disjoint invariant Borel sets of positive measure in $X$ is countable, and a simple exhaustion argument.

In the particular situation where we consider the action $\{\hat{\alpha}_t : t \in \boldsymbol{R}\}$ of $\boldsymbol{R}$ (or $G/H$) on $(\Gamma_H, \mu_H)$, we let $B_H$ be an invariant Borel set in $\Gamma_H$ satisfying the conclusion of Lemma 3.3. We refer to $B_H$ as the smooth part of $(\Gamma_H, \mu_H, \hat{\alpha}_t)$ (or just of $\Gamma_H$).

Finally if $\mathscr{M}$ is an arbitrary von Neumann algebra with separable predual, and $p$ is a central projection in $\mathscr{M}$, we will say that $p$ is of *pure type III$_\lambda$* ($\lambda \in [0, 1]$) if $\mu\{\gamma: \mathscr{M}(\gamma)$ is not of type III$_\lambda\} = 0$ where $\mathscr{M}p = \int_\Gamma^\oplus \mathscr{M}(\gamma) d\mu(\gamma)$ is the central decomposition of $\mathscr{M}p$. We note that the set $\{\gamma: \mathscr{M}(\gamma)$ not of type III$_\lambda\}$ is in fact Borel by the results of [19], and the definition makes sense.

We keep the notations $G$, $H$, $L$, $(\Gamma_H, \mu_H)$, $\{\hat{\alpha}_t : t \in \boldsymbol{R}\}$, $\mathscr{X}(H)$, $(\Gamma_G, \mu_G)$ etc. as developed above. We recall that to each $\hat{\alpha}_t$-invariant

Borel set in $\Gamma_H$ (or $\tilde{\alpha}_t$-invariant projection in $\mathscr{Z}(H)$) there corresponds a projection in $\mathscr{Z}(G)$, and vice versa, via the isomorphism $\kappa$.

THEOREM 3.4. *With $\delta_G(G) = \boldsymbol{R}_+$, we have:*

(i) *The maximal central projection $p_I$ of type I in $\mathscr{Z}(H)$, and the maximal central projection $q_I$ of type I in $\mathscr{Z}(G)$ satisfy $\kappa(q_I) = p_I$. Thus $\mathscr{M}(G)$ is of type I if and only if $\mathscr{M}(H)$ is of type I.*

(ii) *The maximal central semifinite projection $q_{II}$ in $\mathscr{Z}(G)$ corresponds to the set*

*$P_{II} = \{\gamma \in B_H$: the isotropy subgroup of $\gamma$ uder $\{\hat{\alpha}_t: t \in \boldsymbol{R}\}$ is trivial$\}$.*

(iii) *For $\lambda \in (0, 1]$ the maximal central projection $q_\lambda$ in $\mathscr{Z}(G)$ of pure type $III_\lambda$ corresponds to the set*

*$P_\lambda = \{\gamma \in B_H$: $\gamma$ has period precisely* $\log \lambda$ *under $\{\hat{\alpha}_t: t \in \boldsymbol{R}\}\}$.*

(iv) *The maximal central projection $q_0$ of pure type $III_0$ in $\mathscr{Z}(G)$ corresponds to the set*

$$P_0 = T_H - B_H .$$

REMARK. (i) Considering the action $\{\hat{\alpha}_g: g \in G\}$ of $G$ on $(\Gamma_H, \mu_H)$ we may redefine the sets $P_{II}$, $P_\lambda$ by

$$P_{II} = \{\gamma \in B_H: G_\gamma = H\}$$
$$P_\lambda = \{\gamma \in B_H; \delta_G(G_\gamma) = \{\lambda^n: n \in \boldsymbol{Z}\}\} \qquad \lambda \in (0, 1] ,$$

where $G_\gamma$ is the isotropy subgroup of $\gamma$.

(ii) According to the theorem, $\Gamma_H$ is split in four distinct parts. First there is the nonsmooth part ($\mathrm{III}_0$). There is also the fixed points under $\{\hat{\alpha}_t: t \in \boldsymbol{R}\}$ ($\mathrm{III}_1$). Within $B_H$, there are those points whose orbits are "loops" (type $\mathrm{III}_\lambda$, with the "size" of the "loop" determining $\lambda$). Finally there are the points with orbits which are copies of the line, corresponding to the semifinite part of $\mathscr{M}(G)$.

*Proof of* 3.4. We let $E$ be a Borel cross-section for the action of $\{\hat{\alpha}_t: t \in \boldsymbol{R}\}$ on $B_H$. As usual, we let $\lambda^G = \int_{\Gamma_G}^{\oplus} \lambda_\omega^G d\mu_G(\omega)$ and $\lambda^H = \int_{\Gamma_H}^{\oplus} \lambda_\gamma^H d\mu_H(\gamma)$ be the central decompositions of $\lambda^G$ and $\lambda^H$ respectively.

(i) Let $p_I$, $q_I$ be as described in the statement of the theorem and let $\bar{\alpha}_t^I$, $\sigma_t^I$ be the restrictions of $\bar{\alpha}_t$, $\sigma_t$ to $\mathscr{M}(H)_I = \mathscr{M}(H)p_I$ and $\mathscr{M}(G)_I = \mathscr{M}(G)q_I$ respectively.

The crossed product $\mathscr{R}(\mathscr{M}(G)_I, \sigma_t^I)$ is isomorphic with $\mathscr{M}(G)_I \otimes L^\infty(\boldsymbol{R})$, since $\sigma_t^I$ is inner; on the other hand it is also isomorphic with $\mathscr{M}(H)_{\kappa(q_I)} \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. Thus $\mathscr{M}(H)_{\kappa(q_I)}$ is of type $I$ and $\kappa(q_I) \leqq p_I$.

Conversely, the crossed product $\mathscr{R}(\mathscr{M}(H)_I; \bar{\alpha}_t^I)$ is isomorphic with $\mathscr{M}(H)_I \otimes L^\infty(\boldsymbol{R})$ ($\{\bar{\alpha}_t^I: t \in \boldsymbol{R}\}$ being inner, as $\mathscr{M}(H)_I$ is of type $I$). On the other hand this crossed product is isomorphic with $\mathscr{M}(G)_{\kappa^{-1}(p_I)}$ by Proposition 3.2. Thus $\kappa^{-1}(p_I) \leqq q_I$ and $\kappa(q_I) = p_I$.

(ii) Let $P_{II}$ be as defined in the theorem, and $p_{II}$ be the corresponding projection in $\mathscr{Z}(H)$. (Note that $P_{II}$ is Borel as the map $\gamma \in \Gamma_H \to$ isotropy subgroup of $\gamma$ under $\{\hat{\alpha}_t: t \in \boldsymbol{R}\}$ is a Borel map from $\Gamma_H$ to the closed subgroups of $\boldsymbol{R}$, in the sense of [13].)

We identify $P_{II}$ with $(P_{II} \cap E) \times \boldsymbol{R}$; the action $\{\hat{\alpha}_t: t \in \boldsymbol{R}\}$ on $P_{II}$ is given by $\hat{\alpha}_s(\gamma, t) = (\gamma, t - s)$ for $\gamma \in P_{II} \cap E$ and $s, t \in \boldsymbol{R}$. Since $\{\hat{\alpha}_t: t \in \boldsymbol{R}\}$ is a point realization of the restriction of $\{\bar{\alpha}_t: t \in \boldsymbol{R}\}$ on $p_{II}$, and $\{\bar{\alpha}_t: t \in \boldsymbol{R}\}$ is equivalent to translation on $\boldsymbol{R}$, the crossed product of $\mathscr{M}(H)p_{II}$ with respect to the restrictions of $\{\bar{\alpha}_t: t \in \boldsymbol{R}\}$ to $\mathscr{M}(H)p_{II}$ is a direct integral of semi- finite factors (by Takesaki's criterion). On the other hand, this crossed product is $\mathscr{M}(G)\kappa^{-1}(p_{II})$. Thus $\kappa^{-1}(p_{II}) \leqq q_{II}$.

Conversely, if $\sigma_t^{II}$ denotes the restriction of $\sigma_t$ to $\mathscr{M}(G)q_{II}$, then $\mathscr{R}(\mathscr{M}(G)_{II}; \sigma_t^{II}) \simeq \mathscr{M}(G)q_{II} \otimes L^\infty(\boldsymbol{R})$, since $\{\sigma_t^{II}: t \in \boldsymbol{R}\}$ is inner. Furthermore the restriction of the dual automorphism to the centre of the crossed product is given by translation on $L^\infty(\boldsymbol{R})$. This crossed product may also be thought of as $\mathscr{M}(H)_{\kappa(q_{II})} \otimes \mathscr{B}(L^2(\boldsymbol{R}))$. If $P'_{II}$ is a Borel set realizing $\kappa(q_{II})$ in $\Gamma_H$, then it is clear that $\mu_H(P'_{II} - P_{II}) = 0$, and that almost every point in $P'_{II}$ has trivial isotropy subgroup under $\{\hat{\alpha}_t: t \in \boldsymbol{R}\}$. Thus $\kappa(q_{II}) \leqq p_{II}$ as required.

(iii) Let $P_\lambda$, $q_\lambda$ be as in the statement of the theorem, and $p_\lambda$ be the projection corresponding to $P_\lambda$ in $\mathscr{Z}(H)$.

The representation $\lambda_\lambda^H$ of $H$ defined by $\lambda_\lambda^H(h) = \lambda^H(h)p_\lambda$ evidently is a direct integral $\lambda_\lambda^H = \int^\oplus \lambda_\omega^H d\mu_G(\omega)$ of representations having the properties

(a) $\lambda_\omega^H$ is $\{\hat{\alpha}_t: t \in \boldsymbol{R}\}$-invariant, and the corresponding action of $\boldsymbol{R}$ on the centre of $\{\lambda_\omega^H(H)\}''$ is periodic of period $\log \lambda$.

(b) the representations $\mathrm{Ind}_H^G \lambda_\omega^H$ furnish the central decomposition of the restriction of $\lambda^G$ to $\kappa^{-1}(p_\lambda)$. Using Takesaki's criterion, we conclude that $\kappa^{-1}(p_\lambda) \leqq q_\lambda$.

So far the proof has been little more than an interpretation of [21] in the group-algebra context. The converse of (iii) is a little more delicate and will require an auxilliary result, which may have independent interest. Let $q_0$, $q_{III}$ be the maximal central projections in $\mathscr{Z}(G)$ of pure type $III_0$, and type $III$ respectively. Choose invariant Borel sets $Q_0$, $Q_{III}$ realizing the projections $\kappa(q_0)$ and $\kappa(q_{III})$ respectively. Suppose for the moment we have been able to proof that $Q_{III} - Q_0 \subset B_H$ (this will be the content of our auxilliary result). Then we will have $P_0 \subseteqq Q_0$. On the other hand, the first part of the proof of (iii) shows that $\Gamma_H - P_0 \subseteqq \Gamma_H - Q_0$ so that we will have $P_0 = Q_0$ (or more precisely $\kappa(q_0) = p_0$). But then we will also have $\kappa(q_\lambda) = p_\lambda$ ($\lambda \in (0, 1]$); to see this note that we already have $\kappa(q_\lambda) \geqq p_\lambda$, and the projections $q_\lambda$ ($\lambda \in (0, 1]$) are disjoint. Thus if for some $\lambda_0 \in (0, 1]$ $\kappa(q_{\lambda_0}) > p_{\lambda_0}$, then $\kappa(q_{\lambda_0}) \cap p_0 \neq 0$. But then from above $\kappa(q_{\lambda_0}) \cap \kappa(q_0) \neq 0$ and so $q_{\lambda_0} \cap q_0 \neq 0 (\lambda_0 \in (0, 1])$. This is a contradiction, and $\kappa(q_\lambda) = p_\lambda$ for all $\lambda \in [0, 1]$.

Before giving our auxilliary result, we note some other facts which are necessary for its application. Consider the action $\{\hat{\alpha}_t : t \in \boldsymbol{R}\}$ of $\boldsymbol{R}$ on $Q_{III} - Q_0$. Let $\pi^H$ be defined by $\pi^H(h) = \lambda^H(h)\kappa(q_{III} - q_0)$. Let $\bar{\mu} = \int_X^{\oplus} \mu_\omega d\mu(\omega)$ be the ergodic decomposition of the restriction $\bar{\mu}$ of $\mu_H$ to $Q_{III} - Q_0$, and $\pi^H = \int_X^{\oplus} \pi_\omega^H d\mu(\omega)$ the corresponding decomposition of $\pi^H$. For each $\omega \in X$ the representation $\operatorname{Ind}_H^G \pi_\omega^H$ of $G$ generates a factor of type $III_\lambda$ for $\lambda = \lambda(w) \neq 0$. Furthermore, the map $\omega \in X \rightarrow \lambda(\omega) \in (0, 1]$ may be assumed Borel, from the results of [19]. Also the action of $\{\hat{\alpha}_t : t \in \boldsymbol{R}\}$ on the centre of $\{\lambda_\omega^H(H)\}''$ is equivalent to the canonical action of $\boldsymbol{R}$ on $L^\infty(\boldsymbol{R}/\lambda(\omega)\boldsymbol{Z})$.

PROPOSITION 3.5. *Let $(Y, \bar{\mu})$ be a standard measure space, and $\{\hat{\alpha}_t : t \in \boldsymbol{R}\}$ be an action of $\boldsymbol{R}$ on $Y$ with $(y, t) \rightarrow \hat{\alpha}_t(y)$ Borel, and $\hat{\alpha}_t \circ \bar{\mu} \sim \bar{\mu}$. Let $\bar{\mu} = \int_X^{\oplus} \mu_\omega d\mu(\omega)$ be the ergodic decomposion of $\bar{\mu}$ with respect to $\hat{\alpha}$. Suppose there is a Borel function $\lambda : X \rightarrow (0, \infty)$ such that for almost all $\omega \in X$, the action $\{\hat{\alpha}_t : t \in \boldsymbol{R}\}$ on $(Y, \mu_\omega)$ is equivalent to the canonical action of $\boldsymbol{R}$ on $\boldsymbol{R}/\lambda(\omega)\boldsymbol{Z}$. Then $Y$ is smooth for $\bar{\mu}$ under $\hat{\alpha}_t$.*

*Proof.* Consider $A = \{(y, \omega) \in Y \times X : \mu_\omega(\mathcal{O}(y)) > 0\}$ where $\mathcal{O}(y)$ is the orbit of $y$ under $\{\hat{\alpha}_t, t \in \boldsymbol{R}\}$. Note that for fixed $y \in Y$, $\omega \rightarrow \mu_\omega(\mathcal{O}(y))$ is Borel, and that for fixed $\omega \in X$, $y \rightarrow \mu_\omega(\mathcal{O}(y))$ is also Borel to see this, note that if we normalize the $\mu_\omega$ to be probability measures, and choose (arbitrarily) $y(\omega) \in Y$ with $\mu_\omega(\mathcal{O}(y(\omega))) = 1$ (such points exist by hypothesis), then $\{y : \mu_\omega(\mathcal{O}(y)) > 0\} = \mathcal{O}(y(\omega))$. Thus $A$ is Borel as $(y, \omega) \rightarrow \mu_\omega(\mathcal{O}(y))$ is Borel.

Also, the projection of $A$ on $X$ is (to a $\mu$-null set) all of $X$. Thus by the cross-section theorem [2], there is a Borel map $y : X \rightarrow Y$ with $(y(\omega), \omega) \in A$ for almost all $\omega \in X$.

We claim that $y$ is injective on the complement of some null set in $X$. For if $y(\omega_1) = y(\omega_2) = y$, then $\mu_{\omega_1}(\mathcal{O}(y)) = \mu_{\omega_2}(\mathcal{O}(y)) = 1$, and so $\mu_{\omega_1}$ and $\mu_{\omega_2}$ are mutually absolutely continuous. But this is excluded by the construction of the ergodic decomposition. Thus the range of $y$, say $E$, may be assumed Borel and meets each orbit in at most one point. The saturation $S$ of $E$ is also Borel ([23]); since $\mu_\omega(Y - S) = 0$ for almost all $\omega$, $\bar{\mu}(Y - S) = \int_X \mu_\omega(X - S) d\mu(\omega) = 0$, so that $Y$ is smooth for $\bar{\mu}$.

**4. The structure theorem; the case $\delta_G(G) = \{e^{nT}: n \in \boldsymbol{Z}\}$.** We turn briefly to the case where $\delta_G(G)$ is a singly generated subgroup of $\boldsymbol{R}$; $\delta_G(G) = \{e^{nT}: n \in \boldsymbol{Z}\}$, where we suppose for convenience that $T > 0$. Again, our method is to identify the automorphism group of $\mathscr{R}(\mathscr{M}(G); \sigma_t)$ dual to $\{\sigma_t: t \in \boldsymbol{R}\}$, and then to use Takesaki's type criterion. In this case however, $\mathscr{R}(\mathscr{M}(G); \sigma_t)$ and $\mathscr{M}(H) \otimes \mathscr{B}(L^2(\boldsymbol{R}))$ are not necessarily isomorphic, and we need instead to utilize the notion of "induced covariant system" as expounded in [21].

We choose once and for all an element $g_0 \in G$ with $\delta_G(g_0) = e^T$, and denote by $\alpha_0$ the automorphism of $H$ determined by $\alpha_0(h) = g_0 h g_0^{-1}$. Of course in this situation $G$ is the semi-direct product of $H$ by $\boldsymbol{Z}$ with respect to $\alpha_0$, and $\mathscr{M}(G) = \mathscr{R}(\mathscr{M}(H); \bar{\alpha}_0^n: n \in \boldsymbol{Z})$.

Recall that $\sigma_t(\lambda^G(g)) = \delta_G(g)^{it}\lambda^G(g)$, so that $\{\sigma_t: t \in \boldsymbol{R}\}$ is periodic of period $2\pi/T$. Thus the modular automorphism should be considered as an action of $\boldsymbol{R}/(2\pi/T)\boldsymbol{Z}$ on $\mathscr{M}(G)$. Let $\varepsilon: \boldsymbol{R} \to \boldsymbol{R}/(2\pi/T)\boldsymbol{Z}$ be the canonical homomorphism, and define an action of $\boldsymbol{R}/(2\pi/T)\boldsymbol{Z}$ on $\mathscr{M}(G)$ by

$$\psi_{\varepsilon(s)}(x) = \sigma_s(x)\,; \qquad x \in \mathscr{M}(G)\,.$$

We identify the dual of $\boldsymbol{R}/(2\pi/T)\boldsymbol{Z}$ with $T\boldsymbol{Z}$ via the duality $\langle \varepsilon(s), nT \rangle = e^{isnT}$. For convenience we introduce the following unitary operators; on $L^2(\boldsymbol{R}/(2\pi/T)\boldsymbol{Z})$ define operators $p(\varepsilon(s))$ $(s \in \boldsymbol{R})$ and $q(nT)$, $(n \in \boldsymbol{Z})$ by

$$\text{(4.1)} \qquad \begin{cases} (p(\varepsilon(s))\xi)(\varepsilon(t)) = \xi(\varepsilon(t-s)) \\ (q(nT)\xi)(\varepsilon(t)) = e^{intT}\xi(\varepsilon(t)) \end{cases}$$

and on $L^2(T\boldsymbol{Z})$ define unitaries $\hat{p}(\varepsilon(s))$, $\hat{q}(nT)$ $(s \in \boldsymbol{R}, n \in \boldsymbol{Z})$ by

$$\text{(4.2)} \qquad \begin{cases} (\hat{p}(\varepsilon(s))\xi)(mT) = e^{-ismT}\xi(mT) \\ (\hat{q}(nT)\xi)(mT) = \xi((m-n)T)\,. \end{cases}$$

Recall that the Fourier transform $\mathscr{F}: L^2(\boldsymbol{R}/(2\pi/T)\boldsymbol{Z}) \to L^2(T\boldsymbol{Z})$ defined by $(\mathscr{F}\xi)(nT) = \int e^{-inTs}\xi(s)ds$ carries $p$ to $\hat{p}$ and $q$ to $\hat{q}$.

PROPOSITION 4.1. *There is an algebraic isomorphism of $\mathscr{R}(\mathscr{M}(G); \psi_{\varepsilon(s)}; \boldsymbol{R}/(2\pi/T)\boldsymbol{Z})$ with $\mathscr{M}(H) \otimes \mathscr{B}(L^2(T\boldsymbol{Z}))$ which carries the automorphism group dual to $\{\psi_{\varepsilon(s)}\}$ to $\{\bar{\alpha}_0^n \otimes \operatorname{Ad} q(nT)\}$. Thus the restriction of the dual automorphism to the centre of the crossed product is equivalent to the action $\{\bar{\alpha}_0^n: n \in \boldsymbol{Z}\}$ of $\boldsymbol{Z}$ on $\mathscr{Z}(H)$.*

*Proof.* As in the proof of Theorem 3.1, the indicated crossed product in the fixed point subalgebra of $\mathscr{M}(G)\otimes\mathscr{B}(L^2(\mathscr{R}/(2\pi/T)\boldsymbol{Z}))$ under the automorphism group $\{\psi_{\varepsilon(s)}\otimes \mathrm{Ad}\, p(\varepsilon(s))^*: s\in \boldsymbol{R}\}$. Further, the generators of this fixed point subalgebra are

$$\begin{cases}\lambda^G(g)\otimes q(\chi(g))^*\ ; & g\in G\\ \quad 1\otimes p(\varepsilon(s))\ ; & s\in \boldsymbol{R}\end{cases}\tag{4.3}$$

where $\chi(g)=\log\delta_G(g)\in T\boldsymbol{Z}$.

The dual automorphism group is given by $\{\mathrm{Ad}\,(1\otimes q(nT));\ n\in\boldsymbol{Z}\}$, and the second dual (bidual) automorphism group by $\{\psi_{\varepsilon(s)}\otimes\mathrm{Ad}\, p(\varepsilon(s))^*\}$. Performing the Fourier transform in the second variable we obtain:

$$\begin{cases}\text{Generators:} & \lambda^G(g)\otimes \hat{q}(\chi(g))^*\ ; & g\in G\\ \qquad\qquad : & 1\otimes\hat{p}(\varepsilon(s))\ ; & s\in\boldsymbol{R}\\ \text{Dual automorphism:} & \mathrm{Ad}\,(1\otimes\hat{q}(nT))\ ; & n\in\boldsymbol{Z}\\ \text{Bidual automorphism:} & \psi_{\varepsilon(s)}\otimes\mathrm{Ad}\,\hat{p}(\varepsilon(s))^*\ ; & s\in\boldsymbol{R}\,.\end{cases}\tag{4.4}$$

We define a unitary $W$ on $L^2(G\times T\boldsymbol{Z})$ by

$$(W\xi)(g,nT)=\xi(g_0^{-n}g,nT)\,.$$

As in Theorem 3.1 we compute

$$\begin{aligned}&(W(\Delta^{it}\otimes\hat{p}(\varepsilon(t)^*)W^*\xi))(g,nT)\\ &\quad=\delta_G(g_0^{-n}g)^{it}e^{itnT}\xi(g,nT)\\ &\quad=((\Delta^{it}\otimes 1)\xi)(g,nT)\end{aligned}$$

and also

$$\begin{aligned}&(W(1\otimes\hat{q}(nT))W^*\xi)(g,mT)\\ &\quad=\xi(g_0^{m-n}g_0^{-m}g,(m-n)T)\\ &\quad=((\lambda^G(g_0^n)\otimes\hat{q}(nT))\xi)(g,mT)\,.\end{aligned}$$

Since $W\in\mathscr{M}(G)\otimes\mathscr{B}(L^2(T\boldsymbol{Z}))$, the fixed point subalgebra of $\mathscr{M}(G)\otimes\mathscr{B}(L^2(T\boldsymbol{Z}))$ under the bidual automorphism group (4.4) is isomorphic (via $x\to WxW^*$) with $\mathscr{M}(H)\otimes\mathscr{B}(L^2(T\boldsymbol{Z}))$, and the dual automorphism is given by $\{\bar{\alpha}_0^n\otimes\mathrm{Ad}\,\hat{q}(nT): n\in\boldsymbol{Z}\}$.

We let $\mathscr{N}_0=\mathscr{R}(\mathscr{M}(G);\psi_{\varepsilon(s)})$ and $\{\beta_{nT}: n\in\boldsymbol{Z}\}$ denote the dual automorphism group on $\mathscr{N}_0$. Our next object is to relate the covariant system $\{\mathscr{N}_0;\beta_{nT};n\in\boldsymbol{Z}\}$ and the covariant system $\{\mathscr{R}(\mathscr{M}(G);\sigma_t;\boldsymbol{R});\theta_t: t\in\boldsymbol{R}\}$, where $\theta_t$ is dual to $\sigma_t$ on $\mathscr{N}_1=\mathscr{R}(\mathscr{M}(G);\sigma_t;\boldsymbol{R})$.

PROPOSITION 4.2.

(i) *The covariant systems* $\{\mathscr{N}_1;\theta_t: t\in\boldsymbol{R}\}$ *and* $\mathrm{Ind}_{T\boldsymbol{Z}}^{\boldsymbol{R}}\{\mathscr{N}_0;\beta\}$ *are equivalent.*

(ii) *The restriction* $\tilde{\theta}_t$ *of* $\theta_t$ *to the centre of* $\mathscr{N}_1$ *is the identity* *if and only if* $t = nT$ *for some* $n \in \boldsymbol{Z}$, *and, for this* $n$, $\beta_{nT}$ *is the identity on the centre of* $\mathscr{N}_0$.

*Proof.* (i) We consider the algebra $L^\infty(\boldsymbol{R}) \otimes \mathscr{N}_0$ as the algebra of all essentially bounded, $\sigma$-strong*-measurable operator fields from $\boldsymbol{R}$ to $\mathscr{N}_0$. Following [21] we define $\mathscr{M}_0$ as the subalgebra of $L^\infty(\boldsymbol{R}) \otimes \mathscr{N}_0$ consisting of operator fields $x$ with

$$x(t) = \beta_{nT}(x(t + nT)) \qquad \text{a.e.}$$

On $\mathscr{M}_0$ define an action of $\boldsymbol{R}$ by

$$(\chi_s(x))(t) = x(t - s) .$$

The covariant system $\{\mathscr{M}_0; \chi_s\}$ is by definition $\mathrm{Ind}_{TZ}^{R} \{\mathscr{N}_0; \beta\}$.

Evidently $\mathscr{M}_0$ may be identified with $L^\infty([0, T)) \otimes \mathscr{N}_0$ (with the same measurability requirements as above). Under this identification, the action $\chi_s$ becomes

$$(4.5) \qquad (\chi_s(x))(t) = \begin{cases} \beta_{(n+1)T}(x(t - r + T)) ; & 0 \leqq t < r \\ \beta_{nT}(x(t - r)) ; & r \leqq t < T \end{cases}$$

where $s = nT + r$, $0 \leqq r < T$.

Alternatively, $\mathscr{N}_0$ may be considered as acting on $L^2(G \times T\boldsymbol{Z})$, with generators as given in (4.4). It is readily verified that if we identify $L^2([0, T) \times G \times T\boldsymbol{Z})$ with $L^2(G \times \boldsymbol{R})$ by the unitary $U$,

$$(U\xi)(g, nT + r) = \xi(r, g, nT) \qquad (0 \leqq r < T) ,$$

then the image of $L^\infty([0, T)) \otimes \mathscr{N}_0$ is generated on $L^2(G \times \boldsymbol{R})$ by the operators

$$(4.6) \qquad \begin{cases} \lambda^G(g) \otimes \lambda^R(\chi(g)) , & g \in G \\ 1 \otimes u(s) , & s \in \boldsymbol{R} . \end{cases}$$

But these operators are nothing but (the Fourier transforms of) the generators of the crossed product $\mathscr{R}(\mathscr{M}(G); \sigma_t, t \in \boldsymbol{R})$.

A direct computation using (4.5), the form of $\beta_{nT}$ as given in 4.4, and the above identification shows that the action of $\chi_s$ on the generators of $\mathscr{R}(\mathscr{M}(G); \sigma_t)$ (4.6) is given by

$$\chi_s(\lambda^G(g)) \otimes v(\chi(g)) = \lambda^G(g) \otimes v(\chi(g))$$

and

$$\chi_s(1 \otimes u(t)) = e^{ist} 1 \otimes u(t) .$$

But this is precisely the action of the dual automorphism.

(ii) Consider the restriction $\tilde{\chi}_s$ of $\chi_s$ to the centre $L^\infty([0, T)) \otimes \mathscr{Z}(\mathscr{N}_0)$ of $L^\infty([0, T)) \otimes \mathscr{N}_0$. From (4.5) it is trivial that $\tilde{\chi}_s$ being the identity implies that $s = nT$ for some $n \in \boldsymbol{Z}$, and that the resriction $\tilde{\beta}_{nT}$ of $\beta_{nT}$ to $\mathscr{X}(\mathscr{N}_0)$ is the identity. The converse is trivial.

We remark that the discussion immediately preceeding Proposition 3.2, and the proposition itself persists in this situation with obvious modifications (the proof being based on the covariant system $\{\mathscr{M}(G); \psi_{\varepsilon(s)}\}$ rather than $\{\mathscr{M}(G); \sigma_t\}$). We omit details, but will appeal to the result when necessary.

Let $(\Gamma_H, \mu_H)$ be the reduced quasi-dual of $H$, and $\hat{\alpha}_0$ the automorphism of $(\Gamma_H, \mu_H)$ corresponding to the automorphism $\alpha_0$ of $H$. We note that the action $\{\hat{\theta}_t: t \in \boldsymbol{R}\}$ of $\boldsymbol{R}$ on $\Gamma_H \times [0, T)$ induced by the action $\{\tilde{\theta}_t: t \in \boldsymbol{R}\}$ of $\boldsymbol{R}$ on $L^\infty([0, T)) \otimes \mathscr{X}(\mathscr{M}(H))$ is nothing but the flow built under the constant function $\Phi(\gamma) = T(\gamma \in \Gamma_H)$ from $\hat{\alpha}_0$ (see [1], [12]). Here we are identifying $\mathscr{N}_0$ with $\mathscr{M}(H) \otimes \mathscr{B}(L^2(T\boldsymbol{Z}))$, from Proposition 4.1.

We let $n$ denote (normalized) Lebesgue measure on $[0, T)$; to continue our analysis we need to compare the ergodic decomposition of $\mu_H$ (with respect to $\hat{\alpha}_0$) and $\mu_H \times n$ (with respect to $\theta_t$) as well as relating the "smooth parts" of these actions (see Lemma 3.3 and the discussion preceeding it). Throughout the following lemma, $\pi_H$ will denote the projection of $\Gamma_H \times [0, T)$ on $\Gamma_H$.

LEMMA 4.3. (i) *Let $B_{\boldsymbol{Z}}$, $B_{\boldsymbol{R}}$ denote the smooth parts of the actions $\{\Gamma_H, \hat{\alpha}_0\}$ and $\{\Gamma_H \times [0, T); \hat{\theta}_t\}$ respectively. Then $\mu_H(\pi_H(B_{\boldsymbol{R}}) \Delta B_{\boldsymbol{Z}}) = 0$.*

(ii) *If $\mu_H = \int_X^\oplus \mu_x dm(x)$ is the ergodic decomposition of $\mu_H$ (with respect to $\hat{\alpha}_0$), then $\mu_H \times n = \int_X^\oplus (\mu_x \times n) dm(x)$ is the ergodic decomposition of $\mu_H \times n$ with respect to $\{\hat{\theta}_t\}$.*

(iii) *Let $\hat{\alpha}_0 = \int_X^\oplus \hat{\alpha}_{0,x} dm(x)$ be the decomposition of $\hat{\alpha}_0$ corresponding to the decomposition in* (ii), *and $\hat{\theta}_t = \int_X^\oplus \hat{\theta}_{t,x} dm(x)$ the decomposition of $\{\hat{\theta}_t: t \in \boldsymbol{R}\}$ corresponding to ergodic decomposition of $\mu_H \times n$. Then, for almost all $x$, $\{\hat{\theta}_{t,x}: t \in \boldsymbol{R}\}$ on $\{\Gamma_H \times [0, T); \mu_x \times n\}$ is the flow built under the constant function $\Phi(\gamma) = T$ from the automorphism $\hat{\alpha}_{0,x}$ on $(\Gamma_H, \mu_x)$.*

*Proof.* The only nontrivial observation needed is that if $B \subseteq \Gamma_H \times [0, T)$ is smooth for $\mu_H \times n$, and $E$ is a Borel cross-section for the action of $\{\hat{\theta}_t: t \in \boldsymbol{R}\}$ then by deletion of a null set we may assume $\pi_H$ is one-to-one on $E$. Thus $\pi_H(E) \subseteq \Gamma_H$ is Borel and is a cross-section for the action of $\{\tilde{\alpha}_0^n: n \in \boldsymbol{Z}\}$ on the saturation of $\pi_H(E)$. Parts (ii) and (iii) are trivial and left to the reader.

We turn now to the anologue of Theorem 3.4. We keep the notations as already developed.

THEOREM 4.4. *Suppose $G$ is separable, and $\delta_G(G) = \{e^{nT}: n \in \boldsymbol{Z}\}$.*

(i) *Let $p_I$ and $q_I$ denote the maximal central projections of type I in $\mathscr{M}(H)$, $\mathscr{M}(G)$ respectively; then $\kappa(q_I) = p_I$.*

(ii) *Let $q_{II}$ be the maximal central semifinite projection in $\mathscr{M}(G)$ and $p_{II}$ the central projection in $\mathscr{M}(H)$ corresponding to $P_{II} = \{\gamma \in B_Z$: the isotropy subgroup of $\gamma$ under $\{\hat{\alpha}_0^n: n \in \boldsymbol{Z}\}$ is trivial$\}$. Then $\kappa(q_{II}) = p_{II}$.*

(iii) *For each $n$, let $p_n$ be the central projection in $\mathscr{M}(H)$ corresponding to*

$$P_n = \{\gamma \in B_Z: \text{isotropy subgroup of } \gamma \text{ under } \{\hat{\alpha}_0^m: m \in \boldsymbol{Z}\} \text{ is } \{kn: k \in \boldsymbol{Z}\}\}\,.$$

*Then $\kappa^{-1}(p_n)$ is the maximal central projection of $\mathscr{M}(G)$ of pure type $III_{\lambda^n}$ where $\lambda = e^{-T}$.*

(iv) *If $P_0 = \Gamma_H - B_Z$, and $p_0$ is the corresponding central projection of $\mathscr{M}(H)$, then $\kappa^{-1}(p_0)$ is the maximal central projection in $\mathscr{M}(G)$ of pure type $III_0$.*

The proof of the theorem is identical in essence to that of Theorem 3.4; the analogue of Proposition 3.5 for actions of $\boldsymbol{Z}$ is needed also; we leave the details to the reader.

One consequence of the theorem is that if $\delta_G(G) = \{e^{nT}: n \in \boldsymbol{Z}\}$ with $T > 0$, then any central summand of pure type $III_\lambda$ in $\mathscr{M}(G)$ is actually of pure type $III_{e^{-nT}}$ for some $n \in \boldsymbol{Z}^+$. This of course may seen be somewhat more easily by observing that the spectrum of the canonical modular operator on $\mathscr{M}(G)$ is $\{e^{nT}: n \in \boldsymbol{Z}\}$, and thus, in the central decomposition any "component" of the modular automorphism also has spectrum in $\{e^{nT}: n \in \boldsymbol{Z}\}$ (see [18]).

5. **Examples.** In Theorems 3.4 and 4.4, we have derived necessary and sufficient conditions for the regular representation of a locally compact group to generate a von Neumann algebra with a central summand of pure type $III_\lambda$ ($\lambda \in [0, 1]$). To the authors knowledge only one example of a group with type *III* regular representation exists in the literature; this example is due to Godement, and it turns out that the associated von Neumann algebra is a factor of type $III_1$. Slight modification of this construction yields a family of groups $G_\lambda$, $G_{0,\lambda}$ $\lambda \in (0, 1]$ with $\mathscr{M}(G_\lambda)$ a factor of type $III_\lambda$, $\mathscr{M}(G_{0,\lambda})$ a factor of type $III_0$, and $T(\mathscr{M}(G_{0,\lambda})) = 2\pi/\log \lambda\boldsymbol{Z}$. The von Neumann algebras associated with these groups are unfortunately not hyperfinite; to remedy the situation another construction is needed. We construct groups $P_\lambda$, $P_{0,\lambda}$ ($\lambda \in (0, 1]$) such that $S(\mathscr{M}(P_\lambda)) = \{\lambda^n: n \in \boldsymbol{Z}\}$, $S(\mathscr{M}(P_{0,\lambda})) = \{0, 1\}$ and $T(\mathscr{M}(P_{0,\lambda})) =$

$2\pi/\log\lambda\boldsymbol{Z}$. Furthermore $\mathscr{M}(P_\lambda)$, $\mathscr{M}(P_{0,\lambda})$ are hyperfinite. The author is indebted to A. Connes for this second family of examples (private communication).

Before giving the examples we need the following remarks. Suppose $H$ is a unimodular group such that $\mathscr{M}(H)$ is a factor; let $d_H$ be a Haar measure on $H$, and $\alpha_\lambda$ a continuous automorphism of $H$ such that $d_H\alpha_\lambda^{-1}/d_H = \lambda$. With $\bar{\alpha}_\lambda$ the corresponding automorphism of $\mathscr{M}(H)$ and $G_\lambda = H \times_{\alpha_\lambda} \boldsymbol{Z}$, we have by Lemma 2.3 $\phi^H \circ \bar{\alpha}_\lambda = \lambda\phi^H$; so that $\mathscr{M}(G_\lambda) \simeq \mathscr{R}(\mathscr{M}(H); \bar{\alpha}_\lambda)$ is a factor of type $III_\lambda$ by Proposition 2.2 and [3]. The difficulty of course is to produce examples of such groups and automorphisms as $H$ and $\alpha_\lambda$.

We will also need the following rather easy,

LEMMA 5.1. *Let $H$, $\alpha_\lambda$ be as above. Let $K = \{0, 1\}^{\boldsymbol{Z}}$, regarded as a compact group, and $s$ the coordinate shift on $K$. Let $\hat{K}$ be the dual group of $K$, and $\hat{s}$ the autmorphism of $\hat{K}$ dual to $s$ on $K$. Define the automorphism $\beta_\lambda$ of $H \times \hat{K}$ by $\beta_\lambda(h, \hat{x}) = (\alpha_\lambda(h), \hat{s}(\hat{x}))$, and put $G_{0,\lambda} = (H \times \hat{K}) \times_{\beta_\lambda} \boldsymbol{Z}$. Then $S(\mathscr{M}(G_{0,\lambda})) = \{0, 1\}$, $T(\mathscr{M}(G_{0,\lambda})) = 2\pi/\log\lambda\boldsymbol{Z}$.*

*Proof.* The centre of $\mathscr{M}(H \times K)$ is isomorphic with $L^\infty(K)$ and so is nonatomic. The action on $L^\infty(K)$ corresponding to the automorphism $\bar{\beta}_\lambda$ of $\mathscr{M}(H \times \hat{K})$ is precisely $\bar{s}$, and so is ergodic. Furthermore the canonical trace $\tau$ on $\mathscr{M}(H) \otimes \mathscr{M}(\hat{K})$ satisfies $\tau \cdot \bar{\beta}_t = \lambda\tau \leqq \tau$ so that [3] applies, to tell us that $S(\mathscr{M}(G_{0,\lambda})) = \{0, 1\}$.

To compute $T(\mathscr{M}(G_{0,\lambda}))$ we note that if elements of $G_{0,\lambda}$ are denoted by $(h, x, n) \in H \times \hat{K} \times \boldsymbol{Z}$, then the canonical modular automorphism group of $\mathscr{M}(G_{0,\lambda})$ satisfies $\sigma_t(\pi(h, \hat{x}, n)) = \lambda^{int}\pi(h, \hat{x}, n)$, where $\pi(h, \hat{x}, n) = \lambda^{G_{0,\lambda}}(h, \hat{x}, n)$. Thus with $T_\lambda = 2\pi/\log\lambda$,

$$
\begin{aligned}
\sigma_{T_\lambda}(\pi(h, x, n)) &= \lambda^{inT_\lambda}\pi(h, \hat{x}, n) \\
&= \pi(h, \hat{x}, n) .
\end{aligned}
$$

Thus $\sigma_{T_\lambda} = i$, and $T_\lambda\boldsymbol{Z} \subseteqq T(\mathscr{M}(G_{0,\lambda}))$.

Conversely suppose that $\sigma_t$ is inner for some value $T_0$ of $t$. By [3], $\sigma_{T_0} = \operatorname{Ad} u$ for some unitary $u$ in the centre of the fixed point subalgebra of $\mathscr{M}(G_{0,\lambda})$ under $\{\sigma_t\}$. But the fixed point subalgebra is $\{\pi(h, \hat{x}, 0): (h, \hat{x}) \in H \times \hat{K}\}''$, so that $u \in \{\pi(e, \hat{x}, 0): \hat{x} \in \hat{K}\}''$. Furthermore, $u$ satisfies the equation

$$
\begin{aligned}
&u\pi(e, 0, n)u^* = \lambda^{inT_0}\pi(e, 0, n) , \quad \text{or alternatively} \\
&\pi(e, 0, n)^*u\pi(e, 0, n) = \lambda^{inT_0}u .
\end{aligned}
$$

But, by definition of the crossed product, $\operatorname{Ad}\pi(e, 0, n)$ implements the automorphism $\bar{\beta}_\lambda^n$ of $\mathscr{M}(H \times \hat{K})$; thus identifying $\mathscr{M}(\hat{K})$

with $L^\infty(K)$, and letting $w$ be the image of $u$ under this identification, we obtain

$$w(s^{-n}(x)) = \lambda^{inT_0}w(x) \quad \text{a.e. on } K\,.$$

It is well known however that $s$ is weakly mixing, so that (see [9]) $s$ has pure point spectrum. Thus $\lambda^{iT_0} = 1$ and $T_0 \in T_\lambda \boldsymbol{Z}$. Hence we obtain $T(\mathscr{M}(G_{0,\lambda})) = T_\lambda \boldsymbol{Z}$.

We now proceed to the examples.

*The hyperfinite examples.*

Let $U(2, Q)$ denote the upper triangular $2 \times 2$ matrices with rational entries, and nonzero determinant. Let $N(2, Q)$ be the normal subgroup consisting of elements of $U(2, Q)$ having determinant one. Let $N_\lambda(2, Q)$ be the subgroup of $GL(2, \boldsymbol{R})$ generated by $N(2, Q)$ and the matrices $\begin{pmatrix}\lambda^{n/2}, & 0\\ 0, & \lambda^{n/2}\end{pmatrix}$, $\lambda \in (0, 1)$. Note that $N_\lambda(2, Q)$ may be regarded as the direct product of $N(2, Q)$ and $\boldsymbol{Z}$. We consider the groups $P_\lambda$ $(\lambda \in (0, 1])$ defined by

$$P_1 = \boldsymbol{R}^2 \times_s U(2, Q)$$
$$P_\lambda = \boldsymbol{R}^2 \times_s N_\lambda(2, Q)$$

where the actions of $U$ and $N_\lambda$ on $\boldsymbol{R}^2$ are the usual ones. We choose to regard $P_\lambda$ as $(\boldsymbol{R}^2 \times_s N(2, Q)) \times_s \boldsymbol{Z}$ where the action of $N(2, Q)$ is the usual one, and the action of $\boldsymbol{Z}$ on $\boldsymbol{R}^2 \times_s N(2, Q)$ is given by

$$n(x, T) = (\lambda^n x, T)$$
$$(x, T) \in \boldsymbol{R}^2 \times N(2, Q)\ .$$

LEMMA 5.2. *$\mathscr{M}(P_\lambda)$ is a factor of type $III_\lambda$ for each $\lambda \in (0, 1]$. Further, $\mathscr{M}(P_\lambda)$ is hyperfinite.*

*Proof.* We first ascertain the type of $\mathscr{M}(P_\lambda)$.

(1) With $H = \boldsymbol{R}^2 \times_s N(2, Q)$, $\mathscr{M}(H)$ is a hyperfinite factor of type $II_\infty$. For this we note that the discrete group $N(2, Q)$ acts ergodically on $\boldsymbol{R}^2$, for if $A \subset \boldsymbol{R}^2$ is measurable and of positive Lebesgue measure, $m(A) > 0$, with $m(A \varDelta TA) = 0$ for all $T \in N(2, Q)$, then $m(A \varDelta SA) = 0$ for all unimodular upper triangular matrices $S$. On the other hand, these matrices act transitively on the complement of a null set, and hence ergodically. Thus $A$ is the complement of a null set, and $N(2, Q)$ acts ergodically. By the classical criteria for the type of factors arizing in the group measure space construction $\mathscr{M}(H)$ is a factor of type $II_\infty$.

The quickest proof that $\mathscr{M}(H)$ is hyperfinite is to observe that $N(2, Q)$ is solvable, and hence $\mathscr{M}(H)$ is cohyperfinite; on the other hand, $\mathscr{M}(H)$ is properly infinite, and thus is hyperfinite, (see [4]).

To see that $\mathscr{M}(P_\lambda)$ is of type $III_\lambda$ is now easy for $\lambda \in (0, 1)$. For $P_\lambda = H \times_s Z$, and the action of the generator of $Z$ on $H$ clearly scales the Haar measure by a factor of $\lambda$. The observations at the beginning of the section show that $\mathscr{M}(P_\lambda)$ is of type $III_\lambda$ ($\lambda \in (0, 1)$). Again, since $N_\lambda(2, Q)$ is solvable $\mathscr{M}(P_\lambda)$ is cohyperfinite, and properly infinite, thus hyperfinite.

Finally we check the type of $\mathscr{M}(P_1)$. For this it is easiest to calculate Kreiger's ratio set for the group measure space construction (see [3], [10]). In order to apply the results of [3], we must check that the action of $N(2, Q)$ is almost free. So let $A \subset \boldsymbol{R}^2$ be measurable with $m(A) > 0$, and let $T \in N(2, Q)$. We must show that if, for every measurable set $B \subset A$ with $m(B) > 0$, we have $TB \cap B \neq \emptyset$, then $T =$ identity. Clearly we may suppose that $m(TB \Delta B) = 0$ for every measurable $B \subset A$ of positive measure (else the condition $TB \cap B \neq \emptyset$ for all measurable $B \subset A$ is violated). But then $A_1 = \{x \in A: Tx \neq x\}$ is of measure zero. Since $T$ is linear, it is the identity on the vector space difference of $A - A_1$ with itself. But this difference contains a neighborhood of $0 \in \boldsymbol{R}^2$, since $m(A - A_1) > 0$, so that $T =$ identity. Note that our argument in fact shows that any nontrivial subgroup of $GL(2, \boldsymbol{R})$ acts almost freely on $\boldsymbol{R}^2$.

We let $r = r(\boldsymbol{R}^2; U(2, Q))$ be Kreiger's ratio set; for a given $p \in Q_+$, we wish to show that $p \in r$. So let $A \subset \boldsymbol{R}^2$ be measurable of positive measure, and $T \in U(2, Q)$ be any transformation with $\det T = p$. By ergodicity of $N(2, Q)$, there is a transformation $S \in N(2, Q)$ with $m(A \cap STA) > 0$. Let $B = A \cap T^{-1}S^{-1}A$; then $m(B) > 0$, $B \subset A$, $STB \subseteq A$, and $\det (ST) = p$, so that $(dm \circ (ST)^{-1})/dm = p$ on $B$, and $p \in r$. Since the ratio set is closed, it is all of $\boldsymbol{R}_+$, and $\mathscr{M}(G_1)$ is of type $III_1$. The hyperfiniteness of $\mathscr{M}(G_1)$ follows from the solvability of $U(2, Q)$ and [4].

Based on Lemma 5.1 we may now produce examples of groups $P_{0,\lambda}$ with $\mathscr{M}(P_{0,\lambda})$ a type $III_0$ hyperfinite factor, and $T(\mathscr{M}(P_{0,\lambda})) = 2\pi/\log \lambda \boldsymbol{Z}$, $\lambda \in (0, 1)$.

REMARK. There is an alternative development of the examples, which has some technical advantages and is obtained as follows. Let $P_1$ be as above, and $R_1 = P_1 \times S$ where $S = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} a, b \in \boldsymbol{R}, a \neq 0\right\}$. Recall that $\mathscr{M}(S)$ is a factor of type $I_\infty$ so that $\mathscr{M}(R_1)$ is again a type $III_1$ hyperfinite factor. Let $H_1 = \ker \delta_{R_1}$; since $R_1$ contains a closed subgroup on which $\delta_{S_1}$ is one-to-one and onto $\boldsymbol{R}_+$, we may write $R_1 = H_1 \times_s \boldsymbol{R}_+$. For $\lambda \in \boldsymbol{R}_+$, let $\alpha_\lambda$ be the corresponding automorphism of $H_1$; from [22] and Lemma 2.3, $\phi^{H_1} \circ \bar{\alpha}_\lambda = \lambda \phi^{H_1}$.

The semidirect product $R_\lambda = H_1 \times_{\alpha_\lambda} Z$ then has $\mathcal{M}(R_\lambda)$ a factor of type $III_\lambda$, (we note that by Theorem 3.4 $\mathcal{M}(H_1)$ is a factor); since $\mathcal{M}(R_1)$ is hyperfinite, $\mathcal{M}(H_1)$ is cohyperfinite and properly infinite, thus hyperfinite, and $\mathcal{M}(R_\lambda)$ is hyperfinite. The advantage of this construction is that the automorphisms $\bar{\alpha}_\lambda$ of $\mathcal{M}(H_1)$ satisfying $\phi^{H_1} \circ \bar{\alpha}_\lambda = \lambda\phi^{H_1}$ may be chosen to form a one parameter group.

*The nonhyperfinite examples.*

We give a brief description of the construction. Consider Godements example $G_1 = \boldsymbol{R}^2 \times_s GL(2, Q)$. For $\lambda \in (0, 1)$, let $SL_\lambda(2, Q)$ denote the group generated by $SL(2, Q)$ and $\lambda^{1/2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Put $G_\lambda = \boldsymbol{R}^2 \times_s SL_\lambda(2, Q)$; as in the previous examples it is readily shown that $\mathcal{M}(G_\lambda)$ is a factor of type $III_\lambda$ ($\lambda \in (0, 1]$). Alternatively we may consider $G_1 \times S = H_1 \times_s \boldsymbol{R}$ as in the remark above, and then the groups $G_\lambda^1 = H_1 \times_s \log\lambda\boldsymbol{Z}$; again $\mathcal{M}(G_\lambda^1)$ is a factor of type $III_\lambda$. The construction of Lemma 5.1 may be performed, producing groups $G_{0,\lambda}$ with $S(\mathcal{M}(G_{0,\lambda})) = \{0, 1\}$ and $T(\mathcal{M}(G_{0,\lambda})) = 2\pi/\log\lambda\boldsymbol{Z}$. The fact that $\mathcal{M}(G_\lambda)$ and $\mathcal{M}(G_{0,\lambda})$ are nonhyper-finite follows from the observation that $G_\lambda$, $G_{0,\lambda}$ contain copies of $SL(2, Q)$, and that in turn $SL(2, Q)$ contains the free group on two generators. It is now sufficient to appeal to the results of [16].

REMARK. It should be noted that at this point there are no known examples of groups $G$ for which $\mathcal{M}(G)$ has nontrivial central summands of pure types $III_\lambda$ and $III_\mu$ for $\lambda \neq \mu$. The author has failed in all attempts to construct such an example.

6. **Plancherel theory.** In [18], the author has given a Plancherel formula valid for arbitrary separable locally compact groups. Let $G$ be such a group (we assume implicitly that it is not unimodular), and $H = \ker\delta_G$. The important part of the results of [18] is that the Plancherel formula for $G$ may be expressed in terms of that of $H$.

Specifically, let $G$, $H$ be as above and $\phi(\tau$ resp.) be the canonical weight (trace resp.) on $\mathcal{M}(G)$ ($\mathcal{M}(H)$ resp.). Let $\bar{\phi}(\bar{\tau})$ be the associated weight (trace) on the reduced group $C^*$-algebra $C_r^*(G)$ ($C_r^*(H)$). Consider the central decomposition $\bar{\tau} = \int_{\Gamma_H}^{\oplus} \bar{\tau}_\gamma d\mu_H(\gamma)$ of $\bar{\tau}$, and the ergodic decomposition $\mu_H = \int_{\Gamma_G}^{\oplus} \mu_\omega^H d\mu_G(\omega)$ of $\mu_H$ (cf. Theorem 2.4). Let $\bar{\tau} = \int_{\Gamma_G}^{\oplus} \bar{\tau}_\omega d\mu_G(\omega)$ be the decomposition of $\bar{\tau}$ over $\kappa^{-1}(\mathcal{Z}(G))$, where we have $\bar{\tau}_\omega = \int_{\Gamma_H}^{\oplus} \bar{\tau}_\gamma d\mu_\omega^H(\gamma)$. For $x, y \in C_r^*(H)$, define $\chi_\omega^H(x, y) = \bar{\tau}_\omega(y^*x)$; we refer to $\chi_\omega^H$ as the bitrace defined by $\bar{\tau}_\omega$ on $C_r^*(H)$.

Suppose now $\xi \in \mathcal{K}(G)$, the continuous function of compact support on $G$; we define $\tilde{\xi}_g \in \mathscr{K}(H)$ by the formula $\tilde{\xi}_g(h) = \xi(hg)$. It is known ([18]) that that $\chi^H_\omega(\pi_l(\tilde{\xi}_g), \pi_l(\tilde{\xi}_g))$ is constant on cosets of $H$ in $G$, where $\pi_l$ is the "regular representation" of $\mathscr{K}(H)$ in $C^*_r(H)$ (i.e., $\pi_l(\eta)$ is convolution on the left by $\eta$ on $L^2(H)$). The Plancherel formula for $G$ takes the form;

$$||\xi||_2^2 = \int_{\Gamma_G}\int_{G/H} \chi^H_\omega(\pi_l(\tilde{\xi}_g), \pi_l(\tilde{\xi}_g))d\dot{g}d\mu_G(\omega)$$

where $d\dot{g}$ is Haar measure in $G/H$, chosen so that

$$\int_G \xi(g)d_G(g) = \int_{G/H}\int_H \xi(gh)d_H(h)d\dot{g}$$

($d_G$ and $d_H$ denote left Haar measures in $G$ and $H$ respectively). Also if $\bar{\phi} = \int^{\oplus}_{G^\frown} \bar{\phi}_\omega d\mu_G(\omega)$ is the central decomposition of $\bar{\phi}$, then

$$\bar{\phi}_\omega(\pi_l(\xi)^*\pi_l(\xi)) = \int_{G/H} \chi^H_\omega(\pi_l(\tilde{\xi}_g), \pi_l(\tilde{\xi}_g))d\dot{g}\ .$$

We wish to specialize these results to the case when $\delta_G(G) = \boldsymbol{R}_+$ and $\mathscr{M}(G)$ is semifinite. The class of groups satisfying these conditions embraces all connected, nonunimodular separable groups by virtue of [5] and [15]. Our viewpoint throughout the discussion is that the quasi-dual and Plancherel formula for $H$ are known precisely; and we seek to give a computable Plancherel formula for $G$ in terms of that of $H$.

Suppose then $\delta_G(G) = \boldsymbol{R}_+$, and $\mathscr{M}(G)$ is semifinite. According to Theorem 3.4, the measure spaces $(\Gamma_H, \mu_H)$ and $(\Gamma_G \times \boldsymbol{R}; \mu_G \times m)$ are equivalent, where $m$ denotes Lebesgue measure on $\boldsymbol{R}$. So let

$$\{\lambda^H, \tau, \bar{\tau}\} = \int^{\oplus}_{\Gamma_G\times\boldsymbol{R}} \{\lambda^H_{(\omega,t)}, \tau_{(\omega,t)}, \bar{\tau}_{(\omega,t)}\}d\mu_G(\omega)dm(t)$$

be the central decomposition of $\lambda^H, \tau, \bar{\tau}$; we regard $\lambda^H$ and $\lambda^H_{(\omega,t)}$ as representations of both $H$ and $C^*_r(H)$. Note that $\lambda^H_{(\omega,t)}(x) = \lambda^H_{(\omega,0)}(\alpha_t(x))$ $(x \in C^*_r(H))$, and that $\tau_{(\omega,t)}$ and $\bar{\tau}_{(\omega,t)}$ are related by $\bar{\tau}_{(\omega,t)}(x) = \tau_{(\omega,t)}(\lambda^H_{(\omega,t)}(x))$ for $x \in C^*_r(H)$. The algebras $\mathscr{M}^H(\omega, 0)$ and $\mathscr{M}^H(\omega, t)$ generated by $\{\lambda^H_{(\omega,0)}(h)\colon h \in H\}$ and $\{\lambda^H_{(\omega,t)}(h)\colon h \in H\}$ are identical, and thus we must have $\tau_{(\omega,t)}(x) = K(\omega, t)\tau_{(\omega,0)}(x)$, where $K(\omega, t)$ is some positive constant, and $x \in \mathscr{M}^H(\omega, 0)$. On the other hand $\bar{\tau}(\alpha_t(x)) = e^{-t}\bar{\tau}(x)$ for $x \in C^*_r(H)$ so that we may compute

$$\begin{aligned}\bar{\tau}(\alpha_s(x)) &= \int_{\Gamma_G\times\boldsymbol{R}} \tau_{(\omega,t)}(\lambda^H_{(\omega,t)}(\alpha_s(x)))d\mu_G(\omega)dm(s)\\ &= \int_{\Gamma_G\times\boldsymbol{R}} \tau_{(\omega,t)}\lambda^H_{(\omega,t-s)}(x)d\mu_G(\omega)dm(s)\end{aligned}$$

$$= \int_{\Gamma_G \times R} \tau_{(\omega, s+t)} \lambda^H_{(\omega,t)}(x) d\mu_G(\omega) dm(s) .$$
$$= \int_{\Gamma_G \times R} K(\omega, s+t) \tau_{(\omega,0)}(\lambda^H_{(\omega,t)}(x)) d\mu_G(\omega) dm(s)$$

and

$$e^{-s}\bar{\tau}(x) = \int_{\Gamma_G \times R} e^{-s} \tau_{(\omega,t)}(\lambda^H_{(\omega,t)}(x)) d\mu_G(\omega) dm(s)$$
$$= \int_{\Gamma_G \times R} e^{-s} K(\omega, t) \tau_{(\omega,0)}(\lambda^H_{(\omega,t)}(x)) d\mu_G(\omega) dm(t) .$$

Thus, we obtain $K(\omega, s+t) = e^{-s}K(\omega, t)$ almost everywhere on $\Gamma_G \times \boldsymbol{R} \times \boldsymbol{R}$. But $K(\omega, 0) = 1$, so that $K(\omega, s) = e^{-s}$ almost everywhere. Evidently we may assume then that $K(\omega, s) = e^{-s}$ for all $(\omega, s) \in \Gamma_G \times \boldsymbol{R}$. Maintaining the previous notations then we obtain

THEOREM 6.1. *Suppose $\delta_G(G) = \boldsymbol{R}_+$ and $\mathscr{M}(G)$ is semifinite. Then, for $\xi \in \mathscr{K}(G)$ we have*

$$\|\xi\|_2^2 = \int_{\Gamma_G} \int_{G/H} \int_{R} e^{-s} \chi^H_{(\omega,0)}(\pi_l(\tilde{\xi}_g), \pi_l(\tilde{\xi}_g)) dm(s) d\dot{g} d\mu_G(\omega) ,$$

*where $\chi^H_{(\omega,0)}$ is the bitrace on $C^*_r(H)$ defined by the trace $\bar{\tau}_{(\omega,0)}$ appearing in the central decomposition $\bar{\tau} = \int^{\oplus}_{\Gamma_G \times R} \bar{\tau}_{(\omega,s)} d\mu_G(\omega) dm(s)$ of $\bar{\tau}$. Furthermore,*

$$\phi_\omega(\pi_l(\xi)^* \pi_l(\xi)) = \int_{G/H} \int_{R} e^{-s} \chi^H_{(\omega,0)}(\pi_l(\tilde{\xi}_g), \pi_l(\tilde{\xi}_g)) dm(s) dg .$$

REMARK. It is well known that $\mathscr{M}(G)$ is semifinite if and only if $\{\sigma_t : t \in \boldsymbol{R}\}$ is inner (see [20]); indeed we must have $\sigma_t(x) = e^{itH} x e^{-itH}$ for some selfadjoint operator $H$ affiliated with the centre of the fixed point subalgebra of $\{\sigma_t : t \in \boldsymbol{R}\}$. In case $\delta_G(G) = \boldsymbol{R}$ and $\mathscr{M}(G)$ is semifinite such an operator $H$ may be described as follows. Regard $\Gamma_H$ as $\Gamma_G \times \boldsymbol{R}$, and decompose the Hilbert space $L^2(G)$ over $\mathscr{Z}(H)$; $L^2(G) = \int^{\oplus}_{\Gamma_G \times R} \mathscr{G}_{(\omega,t)} d\mu_G(\omega) dt$. With respect to this decomposition $H$ may be defined by $(H\xi)(\omega, t) = t\xi(\omega, t)$ where $\xi$ $\xi(\omega, t)$ and $\int_R t^2 \|\xi(\omega, t)\|^2 d\mu_G(\omega) dt < \infty$. In case $\delta_G(G) = \{e^{nT} : n \in \boldsymbol{Z}\}$ and $\mathscr{M}(G)$ is semifinite an explicit generator for $\{\sigma_t : t \in \boldsymbol{R}\}$ may also be found in the same way.

REMARK. The recent work of A. Connes (unpublished) show that for each $\lambda \in (0, 1)$, the von Neumann algebra $\mathscr{M}(P_\lambda)$ constructed in the above examples is in fact the Powers' factor of type $III_\lambda$. This then gives yet another construction for this family of factors.

## References

1. W. Ambrose, *Representation of ergodic flows*, Ann. Math., **42** (1941), 723–739.
2. Aumann, *Measurable utility and the measurable choice Theorem* (Research in Game Theory and Mathematical Economics; No. 30, Hebrew University, Jersalem, 1967.
3. A. Connes, *Une classification des facteurs de type III*, Annales de l'Ecole Normale Supériure, 4$^e$ serie, t. 6. fasc 2, 1973.
4. A. Cones, P. Ghez, R. Lima, D. Testard, and E. J. Woods, *Review of crossed products of von Neumann algebras*, unpublished preprint.
5. J. Dixmier, *Sur la representation régulière d'un groupe localement compact convexe*, Ann. Scient. Ec. Norm. Sup., 4$^e$ serie t. 2 1969, 423–436.
6. ———, *Les C*-algebras et leurs representations*, Gauthier Villars, 1964.
7. ———, *Les Algebres d'operateurs dan l'espace Hilbertien*, Gautheir Villars, Paris, 1964.
8. R. Godement, *Memoite sur la theorie des characteres dan les groupes localement compacts unimodulaires*, J. Math. Pures et Appl., t **30** (1951), 1–110.
9. P. R. Halmos, *Lectures on ergodic theory*; Chelsea, 1960.
10. W. Kreiger, *On non singular transformations of a measure space I*, (Z-Wahrscheinlickeitstheorie verw-Gel Bd II, (1969), 83–99.
11. G. W. Mackey, *Borel structures in groups and their duals*, Trans. Amer. Math. Soc., **85** (1957), 134–165.
12. ———, *Ergodic theory and virtual groups*, Math. Ann., **166**, 187–207.
13. L. Auslander and C. C. Moore, *Unitary representations of solvable Lie groups*, Memoirs of the Amer. Math. Soc., No. **62**, 1966.
14. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience tracts in pure and applied mathematics, Vol. 1, Interscience.
15. L. Pukanský, *Sur la representation régulière des groupes de Lie résolubles connexes I & II*, (C. R. Acad. Sc. t 268 serie A, (1969), 1077–1079 and 1172–1173).
16. S. Sakai, *C*-algebras and W*-algebras*, (Ergebnisse der Mathematik und iher grenzbiete, Bd 60).
17. I. E. Segal, *An extension of Plancherel's formula to separable unimodular groups*, Ann. Math., t **52** (1950), 272–292.
18. C. E. Sutherland, *Direct integral theory for weights and the Plancherel formula*, Queen's Mathematical preprint, Math. Scand. **40** (1977), 209–214.
19. ———, *Crossed products, direct integrals and connes classification of factors*, Queen's Mathematical preprint, (1974–8).
20. M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications*, Lecture notes in Mathematics 128, Springer-Verlag.
21. ———, *Duality for crossed products and the structure of von Neumann algebras of type III*, Acta Math., **131** (1973).
22. N. Tatsuuma, *Plancherel formula for non-unimodular locally compact groups*, J. Math. Kyoto University, **12**, No. 1, 179–261.
23. V. S. Varadanajan, *Geometrey of Quantum Theory*, Vol. II, Van Nostrand, 1970.

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