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We show that a certain simple and well-known measure-preserving transformation due to Chacón has the property of commuting only with its powers. We also state a theorem concerning the centralizer of certain rank one transformations. In addition we state without proof the fact that Chacón's example has no factors.

In [5] Ornstein showed that there exists an invertible measure-preserving transformation (automorphism) which is mixing and rank one and, moreover, that any such automorphism has trivial centralizer, i.e., commutes only with its powers. The purpose of this note is to show that a simple and well-known example of a weakly-mixing automorphism, which is due to Chacón [3], also has trivial centralizer (Theorem 1). It is worth observing that Theorem 1 is false for an arbitrary weakly-mixing rank one automorphism. In fact, it is possible to construct a weakly-mixing rank one automorphism S which splits, $S = S_1 \times S_2$.

Although the mixing property of Ornstein's example makes it more remarkable there are at least two reasons why Chacón's example is of some interest in this context. First Ornstein's construction is complicated and nonexplicit in nature, as opposed to Chacón's which could hardly be simpler or more explicit. Secondly, even the examples of weakly-mixing transformations having no roots constructed in [2, 1] are quite difficult compared with the present one which actually does much more.

Chacón's transformation can be most efficiently described as follows. Let T be the shift on $\{0, 1\}^Z$ and define by recursion the finite sequences $\sigma(n)$ of 0's and 1's:

$$\sigma(0) = 0$$

$$\sigma(n) = \sigma(n-1)\sigma(n-1)1\sigma(n-1)\,.$$

$$(\sigma(1) = 0010,\ \sigma(2) = 0010001010010,\ \text{etc.})$$

Let $\bar{\mathcal{O}} \subset \{0, 1\}^Z$ consist of those sequences x such that each finite segment of x is a segment of $\sigma(n)$ for some n. Then it is easy to show that $(\bar{\mathcal{O}}, T)$ is a minimal uniquely ergodic system so there is a unique T-invariant Borel measure μ on $\{0, 1\}^Z$ which is supported on $\bar{\mathcal{O}}$. μ can be defined explicitly on finite sequences x (that is, on cylinder sets) by

$$(1) \qquad \mu(x) = \lim_{n\to\infty} fr(x, \sigma(n))$$

where $fr(x, \sigma(n))$ denotes the frequency of appearance of x as a segment of $\sigma(n)$.

To see that T is (isomorphic to) Chacón's example one takes in his geometric construction the partition $\{P_0, P_1\}$ where P_0 consists of the initial single level of the construction and P_1 all levels added later on. P is a generator (this follows from Lemma 2 below) and hence sets up an isomorphism of the geometric construction with T. It is easily seen that the image measure is supported on $\bar{\mathcal{O}}$ and satisfies (1).

We can now state the main result.

THEOREM 1. *If φ is a (not necessarily invertible) measure-preserving transformation of $(\{0, 1\}^{\mathbf{Z}}, \mu)$ and $\varphi T = T\varphi$ then $\varphi = T^k$ for some integer k.*

For the proof we need a bit of terminology and three lemmas. If α is a sequence of 0's and 1's of length n we write $|\alpha| = n$. For $i, j \in \mathbf{Z}[i, j] = \{k \in \mathbf{Z}: i \leqq k \leqq j\}$ and if $x \in \{0, 1\}^{\mathbf{Z}}$ we write $x[i, j]$ for the segment of x of length $j - i + 1$ beginning at i and ending at j. If $x[i, j] = \alpha$ we say α occurs at i in x. For x and y finite sequences of length n, $\bar{d}(x, y) = 1/n \,\#\{i: x(i) \neq y(i)\}$. (We think of a finite sequence as a string of symbols with no specified index set but for definitions like the above x and y must be indexed in an obvious way.) For $(x, y) \in \{0, 1\}^{\mathbf{Z}}$, $\bar{d}(x, y) = \lim_{i,j\to\infty} \bar{d}(x[-i, j], y[-i, j])$ if the limit exists.

Any measurable transformation φ of $(\{0, 1\}^{\mathbf{Z}}, \mu)$ which commutes with the shift is called a code. φ is said to be finite if there is an n such that $(\varphi x)(0)$ depends only on x_{-n}^{n} for a.a.x. In this case we write $|\varphi| \leqq n$. Any code φ can be approximated by a finite code in the sense that for each $\varepsilon > 0$ there is a finite code φ' such that $\bar{d}(\varphi(x), \varphi'(x)) < \varepsilon$ for a.a.x. (Approximate $\varphi^{-1}(P_0)$ by cylinder sets and apply the erodic theorem.) In this case we write $\varphi \overset{\varepsilon}{\sim} \varphi'$.

For purpose of the following lemma only, we adopt $[0, |\sigma(n)| - 1]$ as the index set of $\sigma(n)$ and let $I_{n,m} \subset [0, |\sigma(n + m)| - 1]$ be the set of indices corresponding to initial entries of the component σ_n's of σ_{n+m}. The following lemmas say that $\sigma(n)$ occurs in $\sigma(n + m)$ only where we expect it.

LEMMA 2. *If $\sigma(n)$ occurs at i in $\sigma(n + m)$ then $i \in I_{n,m}$.*

Proof. This is easy to see by induction on n.

REMARK. It follows from Lemma 2 that if $x \in \bar{\mathcal{O}}$ and $\sigma(n)$ and $\sigma(n+m)$ occur in x on nondisjoint sets then $\sigma(n)$ is contained entirely in $\sigma(n+m)$.

The copies of $\sigma(n)$ which appear in $\sigma(n+1)$ (there are only the three obvious ones, by Lemma 2) will be said to have orders 1, 2, and 3 respectively, counting from the left. Similarily if $x \in \bar{\mathcal{O}}$ any $\sigma(n)$ appearing in x has a unique order determined by its position in the $\sigma(n+1)$ containing it.

LEMMA 3. *Suppose α occurs at i in $\sigma(n)(n \geqq 5)$, $|\alpha| > |\sigma_{n-3}|$ and β is the string of length $|\alpha|$ which occurs at $i+1$. Then $\bar{d}(\alpha, \beta) > 10^{-3}$.*

Proof. Let $\sigma(n)^*$ (respectively $\sigma(n)_*$) denote $\sigma(n)$ with the final (respectively initial zero) deleted. Then it is easy to see that for all n, $\bar{d}(\sigma(n)^*, \sigma(n)_*) > \bar{d}(\sigma(1)^*, \sigma(1)_*) = 2/3$. Since $|\alpha| > |\sigma(n-3)|$ there must be a $\sigma(n-4)^*$ and $\sigma(n-4)_*$ occuring in the same relative positions in α and β respectively. This forces

$$\bar{d}(\alpha, \beta) > \frac{|\sigma(n-4)^*|}{|\sigma(n)|}\frac{2}{3} > \frac{1}{4^4}\frac{2}{3} > 10^{-3}.$$

LEMMA 4. *Suppose φ is a code and φ' is a finite code such that $|\varphi'| < 10^{-8}|\sigma(N-2)|$, $N \geqq 5$. Consider an $x \in \{0,1\}^z$ such that $\varphi(x) \in \bar{\mathcal{O}}$ and suppose that for some $n \geqq N$, $\sigma(n)$ occurs at i in x and codes under φ' with error less than 10^{-8}, that is*

$$\bar{d}(\varphi'(x)[i, i+|\sigma(n)|-1], \varphi(x)[i, i+|\sigma(n)|-1]) < 10^{-8}.$$

Then there is a k such that $|k| < |\sigma(n-2)|$ and $\sigma(n)$ occurs at $i+k$ in $\varphi(x)$.

Proof. Since $\varphi(x) \in \bar{\mathcal{O}}$, $\sigma(n)$ must occur at some j in x with

$$i - \frac{|\sigma(n)|}{2} - 1 \leqq j \leqq i + \frac{|\sigma(n)|}{2} + 1.$$

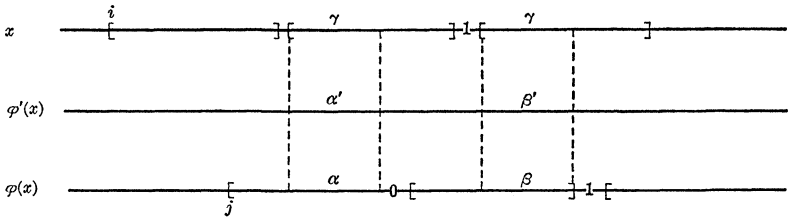
(Since $\varphi(x) \in \bar{\mathcal{O}}$ any $|\sigma(n)|+1$ consecutive indices must contain the beginning of a $\sigma(n)$.) If the lemma is false then either

$$(2) \qquad i + |\sigma(n-2)| \leqq j \leqq i + \frac{|\sigma(n)|}{2} + 1$$

or

$$(3) \qquad i - \frac{|\sigma(n)|}{2} - 1 \leqq j \leqq i - |\sigma(n-2)| \,.$$

We now consider these cases separately and arrive at a contradiction in each. In case (2) we have the following picture:



Here square brackets denote beginnings and ends of $\sigma(n-1)$'s and the indicated zero is a final zero in a $\sigma(n-1)$. We have depicted $j < i + |\sigma(n)|$ but the picture is intended to cover the case $j \geqq i + |\sigma(n)|$ as well. In either case the leftmost γ shown in x is $x[m, n]$ where $[m, n] = [i + |\sigma(n)|, i + 2|\sigma(n)| - 1] \cap [j, j + |\sigma(n)| - 2]$. Note that $\sharp\,[m, n] \geqq |\sigma(n-2)| - 1 \geqq |\sigma(n-3)|$ since we are in case (2).

Now since α' and β' are both coded from γ they will agree except for end-effects which are in number less than

$$2|\varphi'| < 2 \cdot 10^{-8} |\sigma(N-2)| < 10^{-7} |\sigma(n-2)| \,.$$

Thus

$$\bar{d}(\alpha', \beta') < 10^{-7} \frac{|\sigma(n-2)|}{|\alpha'|} < 10^{-7} \frac{|\sigma(n-2)|}{|\sigma(n-3)|} < 10^{-6} \,.$$

Moreover since $x[i, i + |\sigma(n)| - 1]$ codes with error less than 10^{-8}

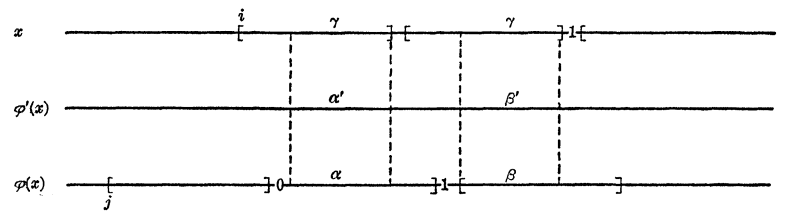
$$\bar{d}(\alpha, \alpha') < \frac{|\sigma(n)|}{|\alpha|} 10^{-8} < 4^4 10^{-8} < 10^{-5} \,,$$

and the same for $\bar{d}(\beta, \beta')$. Thus

$$\bar{d}(\alpha, \beta) < 10^{-6} + 10^{-5} + 10^{-5} < 10^{-3}$$

which contradicts Lemma 3, since the relative positions of α and β in $\sigma(n)$ differ by one.

In case (3) the picture is



where we have indicated an initial zero of a $\sigma(n-1)$ and we arrive at a contradiction in a similar way.

Proof of Theorem 1. Choose a finite code φ' such that $\varphi' \overset{10^{-8}}{\sim} \varphi$. Fix an $x \in \bar{\mathcal{O}}$ such that $\varphi(x) \in \bar{\mathcal{O}}$ (this is true for a.a.x since φ is measure-preserving) and $\bar{d}(\varphi(x), \varphi'(x)) < 10^{-8}$ (which is also true for a.a.x). Since $x \in \bar{\mathcal{O}}$ we may choose sequences $i(n) \to -\infty$ and $j(n) \to \infty$ such that $x[i(n), j(n)] = \sigma(n)$. Fix $N \geqq 5$ such that $|\varphi'| \leqq 10^{-8}|\sigma(N-2)|$ and

$$\bar{d}(\varphi(x)[i(n), j(n)], \varphi'(x)[i(n), j(n)]) < 10^{-8} \quad \text{for all} \quad n \geqq N .$$

Now by Lemma 4, $\sigma(N)$ occurs at $i(N)+k$ in $\varphi(x)$ for some $|k| < |\sigma(N-2)|$. Applying Lemma 4 again we find that $\sigma(N+1)$ occurs at $i(N+1)+l$ in $\varphi(x)$ for some $|l| < |\sigma(N-1)|$ and this $\sigma(N+1)$ must contain the $\sigma(N)$ appearing at $i(N)+k$ in $\varphi(x)$, since $|\sigma(N-2)| + |\sigma(N-1)| < |\sigma(N)|$ (see the remark following Lemma 1). We claim that the order of the $\sigma(N)$ at $i(N)+k$ in $\varphi(x)$ is the same as that of the $\sigma(N)$ at $i(N)$ in x. This is so because otherwise we would have

$$|l| \geqq |\sigma(N)| - |k| \geqq |\sigma(N)| - |\sigma(N-2)| \geqq 2|\sigma(N-1)| ,$$

a contradiction. It follows that the orders are the same, as claimed, which obviously implies that $l = k$.

Now we apply Lemma 4 a third time and conclude in the same manner that $\sigma(N+2)$ occurs at $i(N+2)+k$. Continuing in this way we find that $\sigma(N+m)$ occurs at $i(N+m)+k$ in $\varphi(x)$ for all $m \geqq 0$. Since $i(n) \to -\infty$ and $j(n) \to \infty$ this implies that $\varphi(x) = T^k x$. Of course k depends on x but since there is such a k for a.a.x, there is a fixed k such that $\varphi(x) = T^k x$ for $x \in E$, $\mu(E) > 0$. It then follows from the ergodicity of T that $\varphi(x) = T^k x$ for a.a.x.

We could also have obtained Theorem 1 as a corollary of the following theorem. To state it we first describe (somewhat informally) a certain class of automorphisms to which the T of Theorem 1 belongs. A rank one automorphism (this notion has gone by many names in the past- in [5] it is called class 1) is an automorphism S such that for any partition P, the (P, S) names have the following structure: Given ε almost have every name ξ is within ε (in $\bar{d}$) of a sequence η which is a concatenation of copies of a single finite string η_0 and arbitrary "joining" strings such that the joining strings occupy a proportion less than ε of the places in η. If η can be chosen so that the joining strings have only two lengths a and b and if the number of successive joiners of length a is bounded then we'll say that S is rank one with two return times. For the T of Theorem 1, the two return times are just 0 and 1.

THEOREM 5. *If S is rank one with two return times then the set of measure-preserving transformations which commute with S is the weak closure* [4, p. 61] *of the group generated by S.*

Theorem 1 follows from Theorem 5 as follows. If $\varphi T = T\varphi$ and $T^{k_i} \to \varphi$ weakly for some nondecreasing sequence k_i then $T^{k_i - k_{i-1}}$ converges weakly to the identity. One then shows, by arguments as in Lemmas 2 and 3, that for $x \in \bar{\mathcal{O}}$, $\bar{d}(x, T^k x) > 10^{-100}$ for all $k \neq 0$, so that $k_i - k_{i-1} = 0$ for large i, forcing $\varphi = T^k$.

We do not know whether Theorem 5 is valid for any rank one S.

We conclude by stating without proof another property of T. It is known [6] that a mixing rank one automorphism has only the trivial factor algebras. The same holds for the T of Theorem 1:

THEOREM 6. *Up to null sets the only σ-algebras invariant under T are the full σ-algebra and the trivial σ-algebra $\{\phi, \bar{\mathcal{O}}\}$.*

Theorem 6 is proved by coding arguments similar to those in Theorem 1. The idea is that if φ is a homomorphism of T which is not $1-1$ we can find $x, y \in \bar{\mathcal{O}}$ such that $x \neq T^n y$ for all $n \in \boldsymbol{Z}$ and $\varphi(x) = \varphi(y)$. Approximating φ by a finite code φ' one then shows that $\bar{d}(\varphi'(\sigma(n)^*), \varphi'(\sigma(n)_*))$ (notation as in Lemma 3) is small for large n. One concludes that $\bar{d}(\varphi(x), \varphi(Tx)) = 0$ for $x \in \bar{\mathcal{O}}$, which implies that φ is constant by the ergodicity of T.

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